A Hopf-power Markov chain on compositions: descent sets under riffle-shuffling

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Hopf-power Markov chains

(generalisation of P. Diaconis, C. Y. A. Pang, and A. Ram. Hopf algebras and Markov chains: two examples and a theory, to appear in *J. Alg. Combi.*.)

What: A Markov chain modelling breaking-and-recombining of combinatorial objects.

Why: Can use Hopf-algebra structure theory (Eulerian idempotent, Poincare-Birkhoff-Witt) to diagonalise matrix of transition probabilities and get convergence rate.

How: A combinatorial Hopf algebra has basis $\amalg \mathcal{B}_n$ indexed by combinatorial objects, graded by "size".
The ath Hopf-power map $\Psi^a := m^{[a]}\Delta^{[a]}$ represents breaking into a parts and recombining.
For $x, y \in \mathcal{B}_n$, set

$$\textbf{Prob}(x \to y) = \textbf{coefficient of } y \textbf{ in } a^{-n}\Psi^a(x).$$

(In most cases, $\mathcal{B}$ can be reweighted so that these coefficients sum to 1.)

$\mathcal{S}$: the shuffle algebra

- $\mathcal{B}_n$ = words of length n;
- product = sum of all interleavings: $m(13 \otimes 52) = 1352 + 1532 + 1523 + 5132 + 5123 + 5213$;
- coproduct = sum of all deconcatentations: $\Delta(316) = \emptyset \otimes 316 + 3 \otimes 16 + 31 \otimes 6 + 316 \otimes \emptyset$.

Associated Hopf-power Markov chain is a-shuffle of Bayer-Diaconis:

- cut the deck into a piles symmetrically;
- drop cards one-by-one from the piles with probability proportional to pile size.

$QSym$: the algebra of quasisymmetric functions

Subalgebra of $\mathbb{R}[x_1, x_2, \dots]$ spanned by *monomial quasisymmetric functions*: for I a composition,

$$M_I = \sum_{j_1 < \dots < j_{l(I)}} x_{j_1}^{i_1} \dots x_{j_{l(I)}}^{i_{l(I)}}.$$

- product = product as polynomials;
- coproduct = sum of all deconcatentations: $\Delta(M_I) = \sum_{j=0}^{l(I)} M_{(i_1, i_2, \dots, i_j)} \otimes M_{(i_{j+1}, \dots, i_{l(I)})}$.
- Here, take $\mathcal{B} = \{F_I\}$, the *fundamental quasisymmetric functions*:
$$F_I = \sum_{J \geq I} M_J$$
where the sum runs over all compositions J refining I.

Descent set under riffle-shuffling

The *descent composition* $DC(w)$ is the lengths of the rising sequences in the word w: $DC(4261) = (1, 2, 1)$.

Theorem: There is a morphism of Hopf algebras $\theta : \mathcal{S} \to QSym$ such that, if w is a word with distinct letters, then $\theta(w) = F_{DC(w)}$.

Proof: Apply the universal construction of Aguiar-Bergeron-Sottile to the character $\zeta : \mathcal{S} \to \mathbb{R}$,

$$\zeta(w) = \begin{cases} 1 & \text{if } w_1 < w_2 < \dots < w_n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem: **The descent set process of a deck of n distinct cards under a-shuffling is the Hopf-power Markov chain on $QSym$ with respect to $\{F_I\}$.**

So we can study the descent set process using Hopf-algebraic techniques.

Theorem: Eigenvalues are: $1, a^{-1}, a^{-2}, \dots, a^{-n+1}$; multiplicity of a^{-n+k} is coefficient of $x^n y^k$ in $\prod_i \left(1 - yx^i\right)^{-d_i}$, where d_i= number of Lyndon compositions I with $|I| = i$.

Using the eigenfunction formulae below:

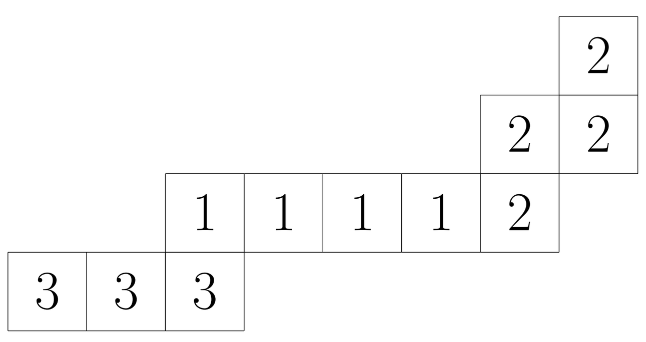
Corollary:

$$\text{Prob}(\emptyset \to J \text{ in } m \text{ steps}) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} a^{m(-n+\#\text{ cycles}(\sigma))} \chi^J(\sigma).$$

Left eigenfunctions g_λ

Theorem: $g_\lambda(J) = \chi^J(\lambda)$ = ribbon character with skew-shape J evaluated at cycle type λ. Eigenvalue $= a^{-n+l(\lambda)}$.

Example: Fillings of skew-shape of $J = (3, 5, 2, 1)$ with $\lambda = (4, 4, 3)$ is



$$g_{(4,4,3)}((3, 5, 2, 1)) = (-1)^{(0+2+0)} = 1.$$

Corollary: Stationary distribution $= g_{(1,1,\dots,1)}(J)$
= proportion of permutations with descent composition J

Right eigenfunctions f_λ

$$f(J) := \frac{1}{|J|} \frac{(-1)^{l(J)-1}}{\binom{|J|-1}{l(J)-1}}, \quad f_\lambda(J) := \frac{1}{l(\lambda)!} \sum_{I' \sim \lambda} \prod_{r=1}^{l(I')} f\left(J_r^{I'}\right)$$

Theorem: $f_\lambda(J)$ = coefficient of any permutation with descent composition J in Garsia-Reutanauer idempotent E_λ. Eigenvalue $= a^{-n+l(\lambda)}$.

Example: Compositions I' with underlying set partition $\lambda = (4, 4, 3)$ are $(4, 4, 3)$, $(4, 3, 4)$, $(3, 4, 4)$.
Decompositions $J_r^{I'}$ of $J = (3, 5, 2, 1)$ with respect to I' are

$(\cdot\cdot\cdot|\cdot,\ \cdot\cdot\cdot\cdot,\ \cdot\cdot|\cdot)$ $(\cdot\cdot\cdot|\cdot,\ \cdot\cdot\cdot,\ \cdot|\cdot\cdot|\cdot)$ $(\cdot\cdot\cdot,\ \cdot\cdot\cdot\cdot,\ \cdot|\cdot\cdot|\cdot)$.

$$f_{(4,4,3)}((3, 5, 2, 1)) = \frac{1}{3!} \left(\frac{-1}{4\binom{3}{1}} \frac{1}{4} \frac{-1}{3\binom{2}{1}} + \frac{-1}{4\binom{3}{1}} \frac{1}{3} \frac{1}{4\binom{3}{2}} + \frac{1}{3} \frac{1}{4} \frac{1}{4\binom{3}{2}} \right) = \frac{7}{5184}.$$

Corollary: Normalised number of descents $= f_{(2,1,1,\dots,1)}(J)$. So expected number of descents after shuffling l times $= (1 - a^{-l})\frac{n-1}{2} + a^{-l}(\#\text{ descents at start})$.

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