Representation Theory

Some groups that we shall work with:

S_n = group of permutations of n objects — order $n!$

A_n = group of even permutations of n objects — order $n!/2$

$C_n = \mathbb{Z}/n\mathbb{Z}$ = cyclic group of order n = rotations of n-gon — order n

D_{2n} = symmetries (rotations and reflections) of n-gon — order $2n$.

$SO(2)$ = all rotations of the plane about the origin

$O(2)$ = all rotations and reflections of the plane fixing the origin

$SO(3)$ = all rotations of $\mathbb{R}^3$ about the origin

Unless stated otherwise, all groups are finite and all representations act on finite-dimensional vector spaces. Sometimes we will not distinguish between an element and its representation.

Let F be a field, and V a vector space over F. Then a (linear) representation of a group G on V is an action of G on V such that, for all g, the function $g: V \to V$ is linear. That is,

$g(x+y) = gx + gy$, $g(\lambda x) = \lambda g(x)$ $\forall g \in G$, $x, y \in V$, $\lambda \in F$.

e.g. C_n, D_{2n} can be thought of as isometries of $\mathbb{R}^2$; A_4, S_4 can be viewed as isometries of $\mathbb{R}^3$ (via Platonic solids).

Observe that the set of linear maps: $V \to V$, $End(V)$, is a ring under pointwise addition and composition:

$(f+g)x = f(x) + g(x)$, $(fg)(x) = f(g(x))$ $\forall x \in V$, $f, g \in End(V)$.

The set of invertible elements of this ring (ie bijective linear maps: $V \to V$) form a group under multiplication (= composition). This is the general linear group, $GL(V)$, or $Aut(V)$.

Hence a linear representation of G on V is precisely a homomorphism: $G \to GL(V)$.

Remarkably, it is often extremely easy to find all the representations over $\mathbb{C}$ of a group.

If V_1, V_2 are representations of the same group G, then they are isomorphic if $\exists f: V_1 \to V_2$ linear with $f g_{V_1} f^{-1} = g_{V_2}$ $\forall g \in G$ (where g_{V_i} = the action of g on V_i). f is then a G-isomorphism.

ie V_1, V_2 can be identified via f, and G acts on them in the same way.

If $V = F^n = \{(z_1, z_2, \dots z_n) : z_i \in F\}$ for some $n \geq 1$, then $GL(F^n)$ is written $GL(n, F)$, and $End(F^n)$

$M_n(F) = \{n \times n$ matrices with coefficients in $F\}$. Then $GL(n,F) = \{A : A \in M_n(F), \det(A) \neq 0\}$.

Proposition: For any group G, the set of isomorphic classes of n-dimensional complex representations of G can be identified with $Hom(G, GL(n,\mathbb{C}))/GL(n,\mathbb{C})$ where $GL(n,\mathbb{C})$ acts on the set $Hom(G, GL(n,\mathbb{C}))$ by conjugation — ie if $\rho \in Hom(G, GL(n,\mathbb{C}))$ and $A \in GL(n,\mathbb{C})$, $(A\rho)(g) = A^{-1}\rho(g)A$ (we are quotienting a set by a group)

Proof: given a representation ρ of G on an n-dimensional vector space V, choose a basis for V.

With respect to this basis, $\rho(g)$ is an element of $GL(n,\mathbb{C})$ $\forall g \in G$.

Changing the basis for V corresponds to conjugating by the change-of-basis matrix.

e.g. A representation of the group $\mathbb{Z}$ in $GL(n,\mathbb{C})$ is uniquely determined by $\rho(1)$, which can be any element of $GL(n,\mathbb{C})$. (then $\rho(r) = [\rho(1)]^r$ $\forall r \in \mathbb{Z}$). So classifying n-dimensional representations of $\mathbb{Z}$ over $\mathbb{C}$ is equivalent to classifying invertible $n \times n$ matrices up to conjugation, which we can do by considering Jordan normal form.

$\therefore$ 1-dimensional representations of $\mathbb{Z}$ are specified by $a \in \mathbb{C}^*$

a 2-dimensional representation of $\mathbb{Z}$ is isomorphic to one where $\rho(1) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ for some $a, b \in \mathbb{C}$.

The trivial representation is given by $\rho: G \to GL(1, \mathbb{C})$, $\rho(g) = I \ \forall g$.

A representation is faithful if the corresponding homomorphism is injective

ie $\rho(g) =$ identity map $\Rightarrow g =$ identity element of G.

e.g. the representation of $\mathbb{Z}$ given by $\rho(1) = a$ is faithful $\Leftrightarrow a \neq$ root of unity

Let ρ_V, ρ_W be two representations of G, acting on vector spaces V and W. Then we can define their direct sum, acting on $V \oplus W$, by $\rho g(v, w) = (\rho_V g, \rho_W g)$ $\therefore$ in an appropriate basis, the matrix $\rho(g)$ has the form $\begin{pmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{pmatrix}$.

If ρ is a representation acting on V and W is a subspace on V invariant under ρ, then a subrepresentation of ρ acting on W is given by $g \to \rho(g)$ restricted to W.

A representation acting on V is irreducible if V is non-empty and has no proper invariant subspace. V is reducible if it has a subrepresentation.

A representation is completely reducible if it isomorphic to a direct sum of irreducible representations.

Finite abelian groups

First consider a finite cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ with generator g $\therefore \rho(g)$ completely determines the representation.

$\rho(g)$ can be any matrix A with $A^n = I$. By conjugation, we may assume A is in Jordan normal form. The powers of a Jordan block have off-diagonal entries (expand $(\lambda I + B)^n$ by the binomial series, where $B = \begin{pmatrix} \ddots & 1 \\ & \ddots \end{pmatrix}$) unless it has size $1 \Rightarrow A$ is diagonal, and the diagonal entries are n^{th} roots of unity. Two such representations are isomorphic $\Leftrightarrow$ the diagonal entries are equal up to ordering.

Hence all representations of cyclic groups are completely reducible, and all irreducible representations are 1-dimensional, classified by n^{th} roots of unity

$\therefore$ there are n distinct irreducible representations.

By primary decomposition, any abelian group can be written uniquely as a product of $\mathbb{Z}/p^r\mathbb{Z}$ for various primes p.

$\therefore$ given G an abelian group, $G = \mathbb{Z}/a_1\mathbb{Z} \times \mathbb{Z}/a_2\mathbb{Z} \times \cdots \mathbb{Z}/a_r\mathbb{Z}$, and let $A_1, A_2, \ldots A_r \in GL(n, \mathbb{C})$ be generators of n-dimensional representations of $\mathbb{Z}/a_1\mathbb{Z}, \mathbb{Z}/a_2\mathbb{Z}, \ldots \mathbb{Z}/a_r\mathbb{Z}$. For these to generate a representation of G, we require A_i, A_j to commute $\forall i, j$.

We showed above that each A_i is diagonalisable $\Rightarrow$ the A_is are simultaneously diagonalisable. $\Rightarrow$ any product of the A_is is diagonal

$\therefore$ every complex representation of a finite abelian group is completely reducible, with all irreducible representations 1-dimensional. They are classified by $(\xi_{a_1}, \xi_{a_2}, \ldots \xi_{a_r})$, where ξ_n is an n^{th} root of unity $\therefore$ the number of non-isomorphic irreducible representations is the order of the group (though there is no natural correspondence between G and its representations)

The regular representation and complete reducibility

Let G be any finite group. The regular representation of G is the vector space (of dimension $|G|$)
$\mathbb{C}G = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \right\}$. ie its basis is the elements of G

G acts on this vector space by: $h(\sum a_g g) = \sum a_g (hg)$.
The vector e is mapped to different vectors by distinct elements of G $\therefore$ this representation is faithful.

Let G be a finite group acting on X, a finite set. The associated permutation representation of G is the vector space $\mathbb{C}X = \left\{ \sum_{x \in X} a_x x : a_x \in \mathbb{C} \right\}$ on which $g(\sum a_x x) = \sum a_x (gx)$.

Recall that a hermitian form (or inner product) on a $\mathbb{C}$-vector space V is a function $\langle , \rangle : V \times V \to \mathbb{C}$ with $\langle ax + bx, y \rangle = a\langle x, y\rangle + b\langle x, y\rangle$, $\langle x, y\rangle = \overline{\langle y, x\rangle}$, and $\langle x, x\rangle > 0 \ \forall x \neq 0$.
Given a $\mathbb{C}$-vector space V with an inner product, a representation of G acting on V is unitary if it preserves the inner product: $\forall g \in G,\ x, y \in V$: $\langle gx, gy\rangle = \langle x, y\rangle$.
A representation of G over $\mathbb{C}$ is unitarisable if G preserves some inner product.
e.g. the permutation representation is unitarisable by choosing the set $x \in X$ to be orthonormal.

Proposition: a finite-dimensional unitary representation of any group G acting on V is completely reducible.
Proof: if this representation is irreducible, we are done.
Otherwise proceed by induction on the dimension of the representation. All 1-dimensional representations are irreducible.
Let U be an invariant subspace under G, and define $U^\perp = \{ v \in V : \langle u, v\rangle = 0 \ \forall u \in U\}$ which is a subspace by linear algebra, and $U \oplus U^\perp = V$.
$U^\perp$ is G-invariant as, $\forall v \in U^\perp$, $\langle gv, u\rangle = \langle v, g^{-1}u\rangle = 0 \ \forall g \in G$, as $g^{-1}u \in U \ \forall u \in U$.
By inductive hypothesis, $U, U^\perp$ are completely reducible, and the direct sum of their irreducible subrepresentations is V.

Weyl's unitary trick: all finite-dimensional $\mathbb{C}$-representations of a finite group G are unitarisable
Proof: Choose any inner product on V $\langle , \rangle$.
Define a new inner product by averaging: $(x, y) = \frac{1}{|G|} \sum_{g \in G} \langle gx, gy\rangle$
then $(hx, hy) = \frac{1}{|G|} \sum_{g \in G} \langle ghx, ghy\rangle$
$= \frac{1}{|G|} \sum_{gh \in G} \langle ghx, ghy\rangle = (x, y)$ this holds for any inner prod sp

Hence all finite-dimensional $\mathbb{C}$-representations of a finite group is completely reducible. In fact, this is also true of representations over any field with $\mathrm{char}(F) = 0$ or $\mathrm{char}(F)$ not dividing $|G|$.
Proof: given a G-invariant subspace W of V, define some surjective projection $\pi : V \to W$.
Define a new projection $\sigma : V \to W$, $\sigma(x) = \frac{1}{|G|} \sum_{g \in G} g(\pi(g^{-1}x))$
$\forall x \in W$, $g^{-1}x \in W$ by invariance $\Rightarrow \pi(g^{-1}x) = g^{-1}x$, $\forall g \in G$ $\therefore \sigma(x) = x. \Rightarrow \sigma$ is indeed a projection.
Let $y \in \ker\sigma. \Rightarrow \sum g\pi(g^{-1}y) = 0 \Rightarrow \forall h \in G, \sum g\pi(g^{-1}hy) = h\sum h^{-1}g\,\pi(g^{-1}hy) = h(0) = 0$
$\therefore hy \in \ker\sigma \Rightarrow \ker\sigma$ is an invariant subspace and $V = W \oplus \ker\sigma$ (the existence of an invariant complement of a given invariant subspace is Maschke's theorem).
Now apply induction to the lower-dimensional representations on W and $\ker\sigma$.

Given two representations acting on V, W, denote by $\mathrm{Hom}^G(V,W)$ the vector space of linear maps $\phi: V \to W$ with $\phi(gv) = g\phi(v)$. Then ϕ is a G-homomorphism

Schur's lemma: let V, W be irreducible representations of a group G over a field F. then any G-homomorphism $: V \to W$ is either 0 or an isomorphism. Furthermore, if $F = \mathbb{C}$, $W = V$, these isomorphisms are scalar endomorphisms

Proof: let $\phi: V \to W$ be a G-homomorphism.

$\forall v \in \mathrm{Ker}\,\phi$, $\phi(gv) = g(\phi v) = 0 \Rightarrow gv \in \mathrm{Ker}\,\phi \Rightarrow \mathrm{Ker}\,\phi$ is a G-invariant subspace.

V is irreducible $\Rightarrow \mathrm{Ker}\,\phi = \{0\}$ or $\mathrm{Ker}\,\phi = V$

$\forall w \in \mathrm{Im}\,\phi$, $g(w) = g(\phi v) = \phi(gv) \in \mathrm{Im}\,\phi$ for some $v \Rightarrow \mathrm{Im}\,\phi$ is a G-invariant subspace $\Rightarrow \mathrm{Im}\,\phi = \{0\}$ or $\mathrm{Im}\,\phi = W$.

$\therefore$ if $\phi \neq \{0\}$, it is injective and surjective.

If $F = \mathbb{C}$, $\phi: V \to V$ has an eigenvalue - let this be λ, and its eigenspace be E_λ.

$\forall v \in E_\lambda$, $\phi(gv) = g(\phi v) = g(\lambda v) = \lambda g(v) \Rightarrow gv \in E_\lambda \Rightarrow E_\lambda$ is a G-invariant subspace.

As it is non-empty, $V = E_\lambda \Rightarrow \phi v = \lambda v \;\; \forall v \in V$.

Corollary: let V, W be irreducible complex representations of G. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}^G(V,W) = \begin{cases} 1 & \text{if } V, W \text{ are isomorphic} \\ 0 & \text{if } V, W \text{ are non-isomorphic} \end{cases}$$

Proof: By Schur, if V, W are not isomorphic, the only element of $\mathrm{Hom}^G(V,W)$ is the trivial map.

If ϕ, ψ are non-trivial elements of $\mathrm{Hom}^G(V,W)$, then, by Schur, they are isomorphisms $\Rightarrow \phi^{-1}\psi: V \to V$ is an isomorphism $\Rightarrow \phi^{-1}\psi = \lambda\iota \Rightarrow \psi = \lambda\phi$.

Schur's lemma over $\mathbb{C}$ can also be deduced by observing that $\mathrm{Hom}^G(V,V)$ is a division algebra: a ring with scalar multiplication where every element is invertible (by general Schur). $\dim \mathrm{Hom}^G(V,V) \leq \dim \mathrm{Hom}(V,V) = (\dim V)^2$

Lemma: the only finite-dimensional division algebra over $\mathbb{C}$ is $\mathbb{C}$ itself.

Proof: let A be a finite dimensional division algebra over $\mathbb{C}$, and choose any $\alpha \in A$.

The elements $1, \alpha, \alpha^2, \cdots$ are linearly dependent $\therefore \exists p(x) \in \mathbb{C}(x)$ with $p(\alpha) = 0$.

Write $p(x)$ as $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ where $\alpha_i \in \mathbb{C}$ are the roots of p.

Then $0 = p(\alpha) = (\alpha-\alpha_1)(\alpha-\alpha_2)\cdots(\alpha-\alpha_n)$.

As division is possible in A, $(\alpha - \alpha_i) = 0$ for some $i \Rightarrow \alpha = \alpha_i \in \mathbb{C}$.

This shows why the second statement of Schur's lemma fails for $\mathbb{R}$ - $\mathbb{C}$, $\mathbb{H}$ (the quaternions) are both finite-dimensional division algebras over $\mathbb{R}$.

Schur's lemma gives another proof that the irreducible representations of abelian groups are 1-dimensional. For each g, define $\theta_g: V \to V$, $\theta_g(v) = gv$. θ_g is a G-endomorphism since G is abelian. By Schur, each 1-dimensional subspace is invariant under $\theta_g \; \forall g$ $\therefore$ if V is irreducible, $\dim V = 1$.

e.g. let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle g, h : g^2 = h^2 = e \rangle$. g, h must be sent to a square-root of 1 in an irreducible representation $\therefore$ the four possible representations are:

$\rho(g) = 1$	$\rho(g) = -1$	$\rho(g) = 1$	$\rho(g) = -1$
$\rho(h) = 1$	$\rho(h) = 1$	$\rho(h) = -1$	$\rho(h) = -1$
$\mathrm{Ker}\,\rho = G$	$\mathrm{Ker}\,\rho = \{e, h\}$	$\mathrm{Ker}\,\rho = \{e, g\}$	$\mathrm{Ker}\,\rho = \{e, gh\}$

Observe that none of these are faithful - G has no faithful irreducible representation.

Isotypical decomposition

Let V be a representation (over $\mathbb{C}$) of a finite cyclic group G.
Let g be a generator of G. We know its representation is diagonalisable with eigenvalues $\subseteq \{1, \xi, \xi^2, \cdots \xi^{n-1}\}$ where $\xi = e^{2\pi i/n}$
Then V has a unique decomposition into eigenspaces of g: $V = \bigoplus_{i=0}^{n-1} V(i)$ with $V(i) = \{x \in V : g(x) = \xi^i x\}$ and each $V(i)$ is invariant under G.
We know that any finite complex representation of a finite group G is a direct sum of irreducible representations. Is there a way of generalising the above unique decomposition?

Lemma: Let V, V' be any representations of a finite group G, and let $f: V \to V'$ be a *G-linear map* ie f is linear and, $\forall g \in G, x \in V$, $gf(x) = f(g(x))$. Write $V = \oplus m_k W_k$, $V' = \oplus m'_k W_k$ with W_i *non-isomorphic* and *irreducible* (ie W_k occurs m_k times in the sum). Then $f(m_k W_k) \subseteq m'_k W_k$

Proof: for each $W_k \subseteq V$, $W_l \subseteq V'$, consider the composition of G-linear maps
$W_k \xrightarrow{\text{inclusion}} V \xrightarrow{f} V' \xrightarrow{\text{projection}} W_l$ which is G-linear
W_k, W_l are irreducible $\therefore$ by Schur, this map is 0 if $k \neq l$.
So f is given by a block matrix.

Theorem: Let V be a representation of a finite group G, with $V = \oplus m_k W_k$ as above. Let $V_k = m_k W_k$.
Then: the decomposition $V = \oplus V_k$ is *unique*, independent of the choice of W_k.
every sub-representation of V *isomorphic to W_k* is *contained in V_k*
if V is over $\mathbb{C}$, the endomorphism algebra $End^G(V_k)$ (of G-linear maps) is isomorphic to the matrix algebra $M_{m_k}(\mathbb{C})$.
$End^G(V) \cong \prod_k M_{m_k}(\mathbb{C})$

Proof: Suppose W is a subrepresentation of V isomorphic to W_k.
The inverse of this isomorphism is a G-linear map from W_k to $W \subseteq V$.
By above lemma, the image of W_k is in V_k. $\therefore W \subseteq V_k$.
$\therefore$ we can define V_k as the (non-direct) sum of <u>all</u> subrepresentations of V isomorphic to the irreducible representation W_k. Then it is clear V_k is independent of W_k.
By Schur, any G-linear map $W_k \to W_k$ is a scalar map
$\therefore$ a G-linear map $V_k \to V_k$ is specified by the scalar maps sending each copy of W_k to another copy of W_k
$\therefore$ if we take a union of bases of the copies of W_k to be our basis of V_k, a G-linear map $V_k \to V_k$ corresponds to a matrix of blocks of size $\dim W_k * \dim W_k$, and each block is λI for some $\lambda \in \mathbb{C}$. We can replace each block by the single entry λ to give an element of $M_{m_k}(\mathbb{C})$, which behaves "correctly" under composition.
By lemma, any G-linear map acts on the V_k's separately $\therefore$ it is the direct product of maps $f_k: V_k \to V_k$.

The dual representation, tensor products (new representations from old)

Let V be a representation of G over $\mathbb{C}$. The *dual space* $V^* = Hom(V, \mathbb{C})$
V^* is also a representation of G: $\forall g \in G, x \in V, f \in V^*$: $[g(f)](x) = f(g^{-1}x)$ (check that this is a group action). If g is represented as $A \in M_n(\mathbb{C})$, then, in the dual basis, the dual representation of g is $(A^T)^{-1}$.

Proposition: let V be a complex representation of a group G. V is irreducible $\Leftrightarrow V^*$ is irreducible

Proof: assume V is reducible $\Rightarrow \exists$ a G-invariant proper subspace $U \subseteq V$.

Let $S^\circ = \{f \in V^*, f(S)=0\}$. From linear algebra, $\dim(S^\circ) = \dim V - \dim S \Rightarrow S^\circ$ is a proper subspace.

$\forall f \in S^\circ, v \in S$, $gf(v) = fg^{-1}(v) = 0 \Rightarrow gf \in S^\circ \therefore S^\circ$ is G-invariant $\Rightarrow V^*$ is reducible.

$\therefore V^*$ reducible $\Rightarrow V^{**} \simeq V$ reducible since the natural identification $\phi(v)(f) = f(v)$ is a G-isomorphism.

More generally, for any representations V, W of G, we can define a representation acting on $\mathrm{Hom}(V,W)$: $g(f)(x) = gfg^{-1}(x)$ $\forall g \in G$, $f \in \mathrm{Hom}(V,W)$, $x \in V$. (again, many things to check)

Observe that, if $gf = f$ $\forall g \in G$, then $f(x) = gfg^{-1}(x)$ $\forall x \in V \Rightarrow g^{-1}f(x) = fg^{-1}(x)$ $\forall g \in G$ $\Rightarrow f$ is G-linear, and the converse is easy $\therefore \mathrm{Hom}^G(V,W) =$ set of $\mathrm{Hom}(V,W)$ on which G acts trivially.

The tensor product of vector spaces V and W, $V \otimes W$, is the vector space spanned by a basis consisting of all pairs $v \in V, w \in W$, written $v \otimes w$, quotiented out by the subspace spanned by the relations:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$
$$(av) \otimes w = v \otimes (aw) = a(v \otimes w)$$

Lemma: this is the same as the vector space with basis $e_i \otimes f_j$, where $\{e_i\}$ is a basis of V and $\{f_j\}$ is a basis of W.

Proof: the span of $\{e_i \otimes f_j\} \subseteq V \otimes W$, as each $e_i \otimes f_j$ is a basis element of $V \otimes W$.

given any basis element $v \otimes w$ of $V \otimes W$, we can write $v = \sum a_i e_i$, $w = \sum b_j f_j$ whence $v \otimes w = \sum a_i b_j (e_i \otimes f_j)$ $\therefore \{e_i \otimes f_j\}$ span $V \otimes W$.

consider $h_i : V \to \mathbb{C}$, $g_j : W \to \mathbb{C}$ linear with $h_i(e_{i'}) = \delta_{ii'}$, $g_j(f_{j'}) = \delta_{jj'}$ $\forall i', j'$.

now let $F_{ij}(v \otimes w) = h_i(v) g_j(w)$, which is linear and satisfy the relations.

if $\sum \lambda_{ij} (e_i \otimes f_j) = 0$, then applying F_{ij} to this shows that $\lambda_{ij} = 0$.

$\therefore \{e_i \otimes f_j\}$ are linearly independent, and hence a basis.

This gives a more concrete and practical definition of $V \otimes W$, and shows that its dimension is $\dim U \times \dim V$, but a basis-independent definition is more pleasing.

Note that not every element of $V \otimes W$ can be expressed as $v \otimes w$ for $v \in V, w \in W$.

It can be shown that there are natural isomorphisms $U \otimes V \simeq V \otimes U$, $(U \otimes V) \otimes W = U \otimes (V \otimes W)$, $(U \oplus V) \otimes W \simeq U \otimes W \oplus V \otimes W$.

Given V, W representations of G, we can define a representation of G on $V \otimes W$: $g(v \otimes w) = gv \otimes gw$ and extend by linearity (check that this is well-defined – ie it satisfies the relations – and linear).

Lemma: if V, W are finite-dimensional complex vector spaces, then there is a natural isomorphism $\mathrm{Hom}(V,W) \cong V^* \otimes W$. (so $\mathrm{Hom}(V^*, W)$ is an alternative definition of $V \otimes W$)

Proof: Define a map $V^* \otimes W \to \mathrm{Hom}(V,W)$, $f \otimes w \to f(\cdot) w$ and extend linearly.

Check that this satisfies the relations and is linear and has trivial kernel.

As these bases have the same dimension this map must be an isomorphism.

Lemma: let V, W, X be vector spaces over a field F. Then there is a bijection between the linear maps: $V \otimes W \to X$ and bilinear maps: $V \times W \to X$.

Proof: let $f : V \otimes W \to X$ be linear, and define $\tilde{f} : V \times W \to X$, $\tilde{f}(v,w) = f(v \otimes w)$.

As $v \otimes w$ is a basis for $V \otimes W$, $\tilde{f}$ determines f uniquely, by linear extension.
Bilinearity of $\tilde{f}$ follows from the tensor product relations.
Conversely, given any bilinear $\tilde{f}: V \times W \to X$, we can define f as above, and bilinearity of $\tilde{f}$ ensures that f is well-defined.

Lemma: Let V, W be finite dimensional complex vector spaces.
Let $A: V \to V$, $B: W \to W$ be linear maps. Then $A \otimes B: V \otimes W \to V \otimes W$, $A \otimes B\,(v \otimes w) = Av \otimes Bw$ extended linearly is well-defined and $tr\,(A \otimes B) = tr A\; tr B$, $det\,(A \otimes B) = det A^{\dim W} det B^{\dim V}$

Proof: well-defined-ness stems from linearity of A and B.
suppose A, B are diagonalisable $\Rightarrow \exists$ bases e_i of V and f_j of W with $A(e_i) = \lambda_i e_i$, $B(f_j) = \mu_j f_j$ for some $\lambda_i, \mu_j \in \mathbb{C}$.
Then $A \otimes B\,(e_i \otimes f_j) = \lambda_i \mu_j\,(e_i \otimes f_j)$ $\therefore$ in the basis $e_i \otimes f_j$, $A \times B$ is diagonal with entries $\lambda_i \mu_j$.
$\therefore tr\,(A \otimes B) = \sum_{i,j} \lambda_i \mu_j = (\sum_i \lambda_i)(\sum_j \mu_j) = tr A\, tr B$.
$det\,(A \otimes B) = \prod_{i,j} (\lambda_i \mu_j) = (\prod_i \lambda_i)^{\dim W} (\prod_j \mu_j)^{\dim V} = det A^{\dim W} det B^{\dim V}$.
The space of diagonalisable matrices are dense in the space of all matrices (of fixed dimension), so, by continuity, these equations hold $\forall$ matrices.

Let $V^{\otimes n}$ denote the tensor product of n copies of V. There is a natural representation of S_n acting on this space: $\sigma\,(v_1 \otimes v_2 \otimes \dots \otimes v_n) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)}$. (and extend linearly)
The n^{th} symmetric power $S^n V$ is defined to be the subspace of $V^{\otimes n}$ where S_n acts trivially.
The sign representation of S_n is the homomorphism $S_n \to GL(1, \mathbb{C})$, $\sigma \mapsto \begin{cases} 1 & \text{if } \sigma \in A_n \\ -1 & \text{if } \sigma \notin A_n \end{cases}$
The n^{th} exterior power $\Lambda^n V$ is the isotypic subspace of $V^{\otimes n}$ for the sign representation — ie $\Lambda^n V = \{u \in V^{\otimes n},\ \sigma(u) = sgn(\sigma) u\ \ \forall \sigma \in S_n\}$
e.g. Let e_i be a basis of V. Then $e_i \otimes e_j$ is a basis for $V^{\otimes n}$
then S_2 acts of $V^{\otimes n}$, with the non-identity element sending $e_i \otimes e_j$ to $e_j \otimes e_i$.
A basis for $S^2 V$ is $e_i \otimes e_i\ (\forall i)$ and $\frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)\ (\forall i<j)$.
A basis for $\Lambda^2 V$ is $\frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)\ (\forall i<j)$.

If V is a representation of G, then G can be made to act on $V^{\otimes n}$: $g(v_1 \otimes \dots \otimes v_n) = g(v_1) \otimes \dots \otimes g(v_n)$
and this action commutes with S_n $\Rightarrow G$ preserves the S_n-isotypical decomposition
$\therefore S^n V$, $\Lambda^n V$ are representations of G.

Observe that, if V has a bilinear inner product, then there is a natural isomorphism from V to V^*.
By Weyl's unitary trick, we can assume this inner product is G-invariant, so the isomorphism it induces is a G-isomorphism $\Rightarrow V, V^*$ are isomorphic representations. Hence any real representation is self-dual.
$\mathbb{C}$ has a sesquilinear inner product, so V, V^* are in general non-isomorphic.

Characters

Let ρ be a finite-dimensional representation of a group G over $\mathbb{C}$, acting on a vector space V.
The character of ρ is the function $\chi: G \to \mathbb{C}$, $\chi(g) = trace\,(\rho(g))$. Observe that conjugate matrices have the same trace, χ is independent of basis, and isomorphic representations give rise to the

same character function.

Theorem: for any representation ρ of a finite group G acting on V:
- χ is a class function – ie $\chi(hgh^{-1}) = \chi(g)$ $\forall g, h \in G$
- $\chi(e) = \dim V$
- $\chi(g^{-1}) = \overline{\chi(g)}$
- $\chi_{V\oplus W} = \chi_V + \chi_W$, $\chi_{V\otimes W} = \chi_V \chi_W$, $\chi_{V^*} = \overline{\chi_V}$

Proof: $\chi(hgh^{-1}) = tr(\rho(h)\rho(g)\rho(h)^{-1}) = tr(\rho(g)) = \chi(g)$.
$\chi(e) = tr(\text{identity matrix})$
$\langle g \rangle$ is a finite cyclic group $\leq G$. $\therefore$ consider ρ as a representation of $\langle g \rangle$ over V. Then ρ is a direct sum of 1-dimensional subrepresentations $\Rightarrow$ $\rho(g)$ is diagonalisable, with n^{th} roots of unity on the diagonal (where n = order of g).
$\therefore \chi(g) = \sum a_i$ where $a_i^n = 1 \Rightarrow a_i \overline{a_i} = 1$
$\chi(g^{-1}) = \sum a_i^{-1} = \sum \overline{a_i} = \overline{\chi(g)}$ $\therefore$ if g, g^{-1} are conjugate, $\chi(g) \in \mathbb{R}$.

$$\chi_{V\oplus W}(g) = tr\begin{pmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{pmatrix} = tr(\rho_V(g)) + tr(\rho_W(g)) = \chi_V(g) + \chi_W(g)$$

$$\chi_{V\otimes W}(g) = tr(\rho_V(g) \otimes \rho_W(g)) = tr(\rho_V(g))\,tr(\rho_W(g)) = \chi_V(g)\chi_W(g)$$

$$\chi_{V^*}(g) = tr(\rho_V(g)^{T^{-1}}) = tr(\rho_V(g)^{-1}) = \overline{\chi_V(g)} \text{ by diagonalising } \rho_V(g).$$

Lemma: let ρ be a finite-dimensional representation of a finite group G, acting on V.
$\dim V^G = \frac{1}{|G|} \sum_{g\in G} \chi_V(g)$ where V^G is the subspace of V fixed by G
ie $\dim V^G$ = number of times the trivial representation occurs.

Proof: Observe that, for any projection $\pi: V \to W \leq V$, $\dim W = tr(\pi)$.
Consider $\pi(x) = \frac{1}{|G|} \sum_{g\in G} gx$. $Im(\pi)$ is G-invariant, and π is the identity on any x fixed by G $\therefore \pi$ is a projection onto V^G. The result follows from linearity of the trace function.

Lemma: let V, W be representations of a finite group G. Then $\dim Hom^G(V,W) = \langle \chi_V, \chi_W \rangle$
where $\langle f, h \rangle = \frac{1}{|G|} \sum_{g\in G} \overline{f(g)}\, h(g)$ for all $f, h: G \to \mathbb{C}$.

Proof: By previous lemma,
$$\begin{aligned} \dim Hom^G(V,W) &= \tfrac{1}{|G|} \textstyle\sum_{g\in G} \chi_{Hom(V,W)}(g) \\ &= \tfrac{1}{|G|} \textstyle\sum_{g\in G} \chi_{V^*\otimes W}(g) \\ &= \tfrac{1}{|G|} \textstyle\sum_{g\in G} \chi_{V^*}(g)\chi_W(g) \\ &= \langle \chi_W, \chi_V \rangle \\ &= \overline{\langle \chi_V, \chi_W \rangle} = \langle \chi_V, \chi_W \rangle \text{ since this is an integer.} \end{aligned}$$

Theorem: for any irreducible representation V of a finite group over $\mathbb{C}$, $\langle \chi_V, \chi_V \rangle = 1$
for any non-isomorphic irreducible representations V and W of a finite group over $\mathbb{C}$, $\langle \chi_V, \chi_W \rangle = 0$
These follow from the above lemma and Schur's lemma.

Corollary: the number of times an irreducible representation V occurs in a decomposition of a representation W into irreducibles is $\langle \chi_V, \chi_W \rangle$ (by linearity of $\langle , \rangle$)
Since complex representations of G are completely irreducible, by taking the inner product of any character with characters of irreducible representations, we can completely determine the representation which gave rise to the character. In particular, representations with the same character

are isomorphic. This is a little less mysterious if we observe that, by knowing χ, we know $\chi(g^r) = \lambda_1^r + \lambda_2^r + \dots \lambda_n^r$ for each $g \in G, r \in \mathbb{Z}$, where λ_i are the eigenvalues of g. We can solve these to find λ_i.

Lemma: V is an irreducible representation $\Leftrightarrow \langle \chi_V, \chi_V \rangle = 1$. Then we call χ_V an irreducible character.
Proof: Let $V = \oplus m_i W_i$ with W_i irreducible and non-isomorphic.
$\langle \chi_V, \chi_V \rangle = \sum_{i,j} m_i m_j \langle \chi_{W_i}, \chi_{W_j} \rangle = \sum_i m_i^2$ which can only equal 1 if $m_j = 1, m_i = 0 \ \forall i \neq j$.

Example: Consider $\mathbb{Z}/n\mathbb{Z}$ represented as rotations in $\mathbb{R}^2$: if g is a generator of $\mathbb{Z}/n\mathbb{Z}$, then g^p is represented by $\begin{pmatrix} \cos\frac{2\pi p}{n} & -\sin\frac{2\pi p}{n} \\ \sin\frac{2\pi p}{n} & \cos\frac{2\pi p}{n} \end{pmatrix}$

This is irreducible over $\mathbb{R}$ (for $n \geq 3$), but reducible over $\mathbb{C}$ since all irreducible representations over $\mathbb{C}$ of a cyclic group is 1-dimensional. By finding eigenvalues, we know that this is the direct sum of the 1-dimensional representations sending g to $e^{2\pi i/n}$ and to $e^{-2\pi i/n}$.

Similarly, we can represent D_{2n} as the symmetries of an n-gon in $\mathbb{R}^2$; represent g as before, and represent the generating reflection r by: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (again take $n \geq 3$).
This is irreducible over $\mathbb{R}$ since no proper subspace is invariant under the subgroup of rotations.
This is also irreducible over $\mathbb{C}$ since, any invariant subspace must be invariant under the subgroup of rotations, so it is the complex line on which g acts by $e^{2\pi i/n}$ or $e^{-2\pi i/n}$.
But the relation $grg = r$ implies that r interchanges these two lines.
Assuming $e^{2\pi i p/n} \neq e^{-2\pi i p/n}$, this argument also shows that $g \to \begin{pmatrix} \cos\frac{2\pi p}{n} & -\sin\frac{2\pi p}{n} \\ \sin\frac{2\pi p}{n} & \cos\frac{2\pi p}{n} \end{pmatrix}$ $r \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is irreducible.

Example: Take the permutation representation of S_3 on $\mathbb{C}^3$.
S_3 acts as the identity on $(1,1,1)$. By Maschke, $\exists W \leq \mathbb{C}^3$ invariant under S_3 with $W \oplus \langle (1,1,1) \rangle = \mathbb{C}^3$. We already know a 2-dimensional representation of $S_3 = D_6$, to know if this is isomorphic to W, we compute the characters:

	identity	permutation	three cycle
trivial:	1	1	1
permutation:	3	1	0
dihedral:	2	0	-1

$\chi_W = \chi_{permutation} - \chi_{trivial}$, which, by above, is the dihedral character $\therefore W$ is indeed isomorphic to the dihedral representation.

Proposition: the multiplicity of any irreducible representation V in the regular representation $\mathbb{C}G$ is $\dim V$. In particular, every irreducible representation occurs in $\mathbb{C}G$, so there are only finitely many of these.
Proof: $\forall g \neq e$ the regular representation of g does not fix any basis vector, so $\chi_{\mathbb{C}G}(g) = 0$.
$\therefore$ multiplicity of $V = \langle \chi_V, \chi_{\mathbb{C}G} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_{\mathbb{C}G}(g)$
$= \frac{1}{|G|} \chi_V(e) \chi_{\mathbb{C}G}(e) = \chi_V(e) = \dim V$.
Observe that $|G| = \langle \chi_{\mathbb{C}G}, \chi_{\mathbb{C}G} \rangle = \sum_i \dim W_i \langle W_i, \chi_{\mathbb{C}G} \rangle = \sum_i \dim W_i^2$ where W_i are the irreducible representations of G.

Completeness of characters: for any finite group G, the irreducible characters form an orthonormal basis for the space of class functions on G. In particular, the number of irreducible complex representations of G, up to isomorphism, is the number of conjugacy classes in G.

Proof: View $\mathbb{C}G$ as the non-commutative group ring of G: $\sum a_g g \sum b_h h = \sum a_g b_h (gh)$.

$\mathbb{C}G$ is also a commutative ring of functions: $G \to \mathbb{C}$: $(\sum a_g g)(h) = a_h$ (ie send group element to its coefficient).

$\sum a_g g \in$ centre of $\mathbb{C}G \Leftrightarrow \forall h \in G, \sum a_g g = \sum a_g h^{-1}gh$
$\Leftrightarrow \forall g, h \in G, (\sum a_g g)(g) = (\sum a_g g)(h^{-1}gh)$
$\Leftrightarrow (\sum a_g g)$ is a class function.

Define θ: group ring $\to \mathrm{End}(V)$, $\theta(\sum a_g g)(v) = \sum a_g (gv)$ for $v \in V$, a representation
$\therefore \theta(e)v = ev = v \ \forall v$, $\theta(\sum a_g g)[\theta(\sum b_h h)(v)] = \theta(\sum a_g g)(\sum b_h (hv)) = \sum a_g b_h (ghv)$ by linearity of g (V is a module over $\mathbb{C}G$, and conversely, all modules over $\mathbb{C}G$ are representations).

Now let ϕ be a class function, and V be irreducible.

$\mathrm{Trace}(\theta(\phi)) = \mathrm{trace}(\sum_g \phi(g) g)$
$= \sum_g \phi(g)\, \mathrm{trace}(g)$
$= |G| \langle \phi, \chi_{V^*} \rangle$ $\therefore$ if ϕ is orthogonal to all irreducible characters, $\mathrm{trace}\, \theta(\phi) = 0$. But $\sum_g \phi(g) g \in$ centre of $\mathbb{C}G \Rightarrow \theta(\phi)$ is G-linear since θ is a ring homomorphism $\Rightarrow$ by Schur, $\theta(\phi)$ is a scalar multiple of the identity map.
$\therefore \theta(\phi) =$ zero map $\therefore$ by considering the restriction of $\theta(\phi)$ on each irreducible subspace, $\theta(\phi) =$ zero map on $\mathbb{C}G \Rightarrow \therefore 0 = \theta(\phi)g = \phi(g) \ \forall g \in G \subseteq \mathbb{C}G$.

$\therefore$ we can represent all the irreducible characters in a square matrix (since we only need to specify the image of each conjugacy class under the characters rather than the image of each element) – this gives a character table

Example: character table of S_3:

elements	identity	3-cycle	transposition
sizes of conjugacy classes	1	2	3
trivial character	1	1	1
sign character	1	1	-1
dihedral character	2	-1	0

observe that the rows are orthogonal: $\frac{1}{6}(1(1)(\overline{1}) + 2(1)(\overline{1}) + 3(1)(\overline{-1})) = 0$
$\frac{1}{6}(1(1)(\overline{2}) + 2(1)(\overline{-1}) + 3(1)(\overline{0})) = 0$ etc.

and rows are normalised: $\frac{1}{6}(1(2)(\overline{2}) + 2(-1)(\overline{-1}) + 3(0)(\overline{0})) = 1$ etc.

and $|G| = 6 = 1^2 + 1^2 + 2^2 = \sum (\dim W_i)^2$.

~~where C is the size of that conju~~

Observe that, if we multiply each column by $\sqrt{|C|/|G|}$, where C is the size of that conjugacy class, the character table becomes a unitary matrix $\Rightarrow$ its columns are also then orthonormal ie $\sum_\chi \chi(g)\overline{\chi(h)} = \begin{cases} |G|/|C| & \text{if } g, h \text{ are in the same conjugacy class, of size } C \\ 0 & \text{otherwise.} \end{cases}$

Here we are summing over all irreducible characters.

These column orthogonality relations are often easier to use than row orthogonality relations when trying to complete the character table, since no rescaling is needed.

Example: let G be the cyclic group of order n, generated by g. let $\omega = e^{2\pi i/n}$.
Each element is its own conjugacy class $\therefore$ the character table is an $n \times n$ matrix.
The irreducible representations are 1-dimensional so the characters are the representations
Then the j-k^{th} entry is ω^{jk}.

Define the derived subgroup of $G = \langle aba^{-1}b^{-1} : a,b \in G \rangle$. This is normal and its quotient, the abelianisation G^{ab}, is abelian. It can be shown that all 1-dimensional representations are induced from representations of $|G^{ab}|$

Example: let $n = 2m+1$. $(D_{2n})^{ab} = \langle a,b : a^n = b^2 = 1,\ aba = b,\ ab = ba \rangle$
$= \langle a,b : a^n = b^2 = 1,\ a^2 = 1,\ ab = ba \rangle = \langle b : b^2 = 1 \rangle = \mathbb{Z}/2\mathbb{Z}$.

if $n = 2m$, $(D_{2n})^{ab} = \langle a,b : a^n = b^2 = 1,\ aba = b,\ ab = ba \rangle$
$= \langle a,b : a^2 = b^2 = 1,\ ab = ba \rangle = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

$\therefore$ character table of $D_{2n} = D_{2(2m+1)}$:

	e	a, a^{-1}	$\cdots$	a^k, a^{-k}	$\cdots$	a^m, a^{-m}	$b, ab, a^2b, \cdots a^{2m}b$
trivial	1	1		1		1	1
sign	1	1		1		1	-1
j-dihedral	2	$2\cos\frac{2\pi j}{n}$		$2\cos\frac{2\pi jk}{n}$		$2\cos\frac{2\pi jm}{n}$	0

character table of $D_{2n} = D_{2(2m)}$:

	e	$a, a^{-1} \cdots$	$a^k a^{-k}$	$\cdots$	$a^{(m-1)}, a^{-(m-1)}$	a^m	$b, a^2b, \cdots a^{2m}b$	$ab, a^3b, \cdots a^{2m-1}b$
trivial	1	1	1		1	1	1	1
	1	1	1		1	1	-1	-1
	1	-1	$(-1)^k$		$(-1)^{m-1}$	1	1	-1
	1	-1	$(-1)^k$		$(-1)^{m-1}$	1	-1	1
j-dihedral	2	$2\cos\frac{2\pi j}{n}$	$2\cos\frac{2\pi jk}{n}$		$2\cos\frac{2\pi j(m-1)}{n}$	$2\cos\frac{2\pi jm}{n}$	0	0

The j-dihedral representations are non-isomorphic since $\chi(a) = 2\cos\frac{2\pi j}{n}$ is distinct $\forall j : 0 < j < [\frac{n-1}{2}]$

Proposition: Let G, H be finite groups, with representations V and W.
Then $V \otimes W$ is a representation of $G \times H$,
$V \otimes W$ is irreducible $\Leftrightarrow$ V, W are irreducible. Moreover, every irreducible representation of $G \times H$ arises uniquely this way.

Proof: Let $\rho_V : G \to GL(V)$, $\rho_W : H \to GL(W)$.
define $\rho(gh) = \rho_V(g) \otimes \rho_W(h)$, which is bilinear in $V \times W \Rightarrow$ linear in $V \otimes W$.
$\rho(g_1h_1\, g_2h_2) = \rho(g_1g_2h_1h_2)$ by structure of $G \times H$
$= \rho_V(g_1g_2) \otimes \rho_W(h_1h_2) = \rho_V(g_1)\rho_V(g_2) \otimes \rho_W(h_1)\rho_W(h_2)$
$= [\rho_V(g_1) \otimes \rho_W(h_1)] \cdot [\rho_V(g_2) \otimes \rho_W(h_2)] = \rho(g_1h_1)\rho(g_2h_2)$

$\therefore \rho$ is an homomorphism.
$V \otimes W$ irreducible $\Leftrightarrow \langle \chi_{V\otimes W}, \chi_{V \otimes W} \rangle = 1 \Leftrightarrow \frac{1}{|G|}\frac{1}{|H|}\sum_{g,h} \chi_V(g)\overline{\chi_V(g)}\chi_W(h)\overline{\chi_W(h)} = 1$
$\Leftarrow \langle \chi_V, \chi_V \rangle = 1, \langle \chi_W, \chi_W \rangle = 1$

By considering the restriction of this representation to $V \otimes 0$ and $0 \otimes W$, we recover ρ_V and ρ_W
$\therefore \rho_V, \rho_W$ are uniquely determined by ρ.
To see that every irreducible representation of $G \times H$ arises this way, we count the number of irreducible representations = number of conjugacy classes. Observe that (g,h) conjugate to $(g', h') \Leftrightarrow \exists a \in G, b \in H$ with $(a,b)(g,h)(a,b)^{-1} = (g',h') \Leftrightarrow \exists a \in G, b \in H$ with $aga^{-1} = g'$, $bhb^{-1} = h'$
$\Leftrightarrow g, g'$ are conjugate and h, h' are conjugate $\therefore$ a conjugacy class in $G \times H$ is precisely the product of a conjugacy class in G and a conjugacy class in H.
$\therefore$ number of conjugacy classes in $G \times H$ = number of conjugacy classes in G $\times$ number of conjugacy classes in H.

Lemma: Let χ_V be a character of a finite group G. Then $\chi_{S^2V}(g) = \frac{1}{2}[\chi(g)^2 + \chi(g^2)]$
$\chi_{\Lambda^2 V}(g) = \frac{1}{2}[\chi(g)^2 - \chi(g^2)]$

Proof: fix $g \in G$, and choose a basis for V in which g is diagonal $\therefore g(e_i) = a_i e_i$ for some $a_i \in \mathbb{C}$.
The basis elements of S^2V are $\frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$, $i \leq j$, and these are eigenvectors of g with eigenvalue $a_i a_j$. $(i \leq j)$.
Similarly, g acts on the basis $\frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)$ of $\Lambda^2 V$ $(i<j)$ by $a_i a_j$.
$\therefore \chi_{S^2V}(g) = \sum_{i \leq j} a_i a_j = \frac{1}{2}((\sum a_i)^2 + \sum a_i^2)$, $\chi_{\Lambda^2 V}(g) = \sum_{i<j} a_i a_j = \frac{1}{2}((\sum a_i)^2 - \sum a_i^2)$

Theorem: for every finite group G, the dimension of any complex irreducible representation divides $|G|$.
Recall that, for commutative rings $R \subset S$, an element $s \in S$ is integral over R if x satisfies a monic polynomial in $R[x]$. This is equivalent to $R[s]$ being finitely generated over R.
(if s integral, finite powers of s generate $R[s]$. Conversely, the generators of $R[s]$ involve finitely many powers of s. Let the highest be s^N. Then s^{N+1} = sum of lower powers of s).
An algebraic integer is a complex number which is integral over $\mathbb{Z}$. Recall that a rational algebraic integer is an element of $\mathbb{Z}$, and that the algebraic integers form a ring.

Proof: Let V be an irreducible representation, C_i be the conjugacy classes of G, and ϕ_i be the indicator class functions: $\phi_i(g) = \begin{cases} 1 & \text{if } g \in C_i \\ 0 & \text{if } g \notin C_i \end{cases}$.
The ring $\mathbb{Z}G$ is finitely generated (by the elements of G). It is a free module, so $\mathbb{Z}[\phi_i]$, which is a submodule, is finitely generated also. $\Rightarrow \phi_i$ is integral.
Recall the ring homomorphism θ: group ring $\to \mathrm{End}(V)$, which sends class functions to scalar maps. Define $\tilde{\theta}$: class functions $\to \mathbb{C}$, $\tilde{\theta}(\phi)$ = the scalar of $\theta(\phi)$. This ring homomorphism fixes the coefficients of the monic polynomial satisfied by ϕ_i, so $\tilde{\theta}(\phi_i)$ also satisfies a monic polynomial over $\mathbb{Z}$ $\Rightarrow \tilde{\theta}(\phi_i)$ is integral.
Previously we saw trace $(\theta(\phi)) = |G|\langle \phi_i, \chi_{V^*}\rangle = |C_i|\chi_{V^*}(g_i)$ for some $g_i \in C_i$
$\therefore \tilde{\theta}(\phi_i) = \frac{1}{\dim V}|C_i|\chi_{V^*}(g_i)$ is an algebraic integer, $\forall i$
V is irreducible, so $1 = \frac{1}{|G|}\sum_g |\chi_V(g)|^2 = \frac{1}{|G|}\sum_i \chi_V(g_i)\chi_{V^*}(g_i)|C_i|$
$\therefore |G|/\dim V = \sum_i \frac{|C_i|}{\dim V}\chi_{V^*}(g_i)\chi_V(g_i)$ which is the sum of products of algebraic integers, since the value of any character of G are sums of roots of unity, and hence an algebraic integer.

Induced representations

Let $H \subseteq G$ be finite groups, with W a representation of H. The induced representation of G by W acts on $g_1W \oplus g_2W \oplus \dots \oplus g_rW$, of dimension $|G|/|H| \dim W$, where g_iH are distinct cosets and any coset has the form g_iH. $\therefore$ each element of G can be written as g_ih for some $h \in H$, then $g(g_jw) = g_rh(w)$ where $gg_j = g_rh$ $\therefore$ each g acts linearly from g_jW to g_rW.
We then extend this definition linearly. To show that this is independent of the representatives g_i, we take a different view:

Let $R_1 \subseteq R_2$ be rings (possibly non-commutative), and M an R_1-module. Define $R_2 \otimes M$ to be the group generated by $r_2 \otimes m$ $\forall r_2 \in R_2, m \in M$, with the relations $(r_1 + r_2)\otimes m = r_1 \otimes m + r_2 \otimes m$, $r \otimes (m_1 + m_2) = r \otimes m_1 + r \otimes m_2$, $r \otimes (am) = ra \otimes m$ $\forall r, r_1, r_2 \in R_2, a \in R_1, m_1, m_2, m \in M$.

We can make $R_2 \otimes M$ into an R_2-module by defining $s(r\otimes m) = sr \otimes m$. $\forall r,s \in R_2,\ m\in M$.
Taking R_1 to be the group ring of H, and R_2 the group ring of G, M a representation of H, $R_2 \otimes M$ is precisely the induced representation of G by H, since $\mathbb{C}G$ is a free $\mathbb{C}H$ module generated by any choice of g_i's.

Theorem: let $R_1 \subseteq R_2$ be rings, M, N be modules over R_1, R_2 respectively. Then
$\mathrm{Hom}_{R_2}(R_2 \otimes M, N) = \mathrm{Hom}_{R_1}(M,N)$ viewing N as an R_1-module on the right hand side.
Proof: let $\phi: R_2 \otimes M \to N$ be R_2-linear.
define $\tilde{\Phi}: M \to N$, $\tilde{\Phi}(m) = \phi(1\otimes m)$. From the third relation on $R_2 \otimes M$, $\tilde{\Phi}$ is R_1-linear.
since $\phi(r\otimes m) = r\phi(1\otimes m) = r\tilde{\Phi}(m)$, ϕ is uniquely determined by $\tilde{\Phi}$, and we can use this to define a $\phi: R_2 \otimes M \to N$ given $\tilde{\Phi}: M\to N$. this satisfies the tensor product relations since $\tilde{\Phi}$ is R_1-linear.
In terms of induced representations, this says $\mathrm{Hom}_G(\oplus g_i W, V) = \mathrm{Hom}_H(W,V)$ where V is a representation of G. This is the Frobenius reciprocity theorem (on the right hand side we view V as a representation of H) ie H-linear maps: $W \to V$ extend uniquely to G-linear maps: $\oplus g_i W \to V$.

Example: Given any subgroups H of a finite group G, every irreducible representation V of G is contained in the induced representation of G by H for some irreducible representation W of H.
Proof: Consider the restriction of V to H, and take an irreducible component W.
The inclusion map $W \to V$ is H-linear $\Rightarrow$ this extends to a G-linear map: $\oplus g_i W_i \to V$.
its restriction to W is the inclusion map, so this map is non-zero. As V is irreducible, by Schur, $\exists$ an irreducible subspace of $\oplus g_i W_i$ isomorphic to V.

By the definition of tensor products $\oplus g_i (W_1 \oplus W_2) = (\oplus g_i W_1) \oplus (\oplus g_i W_2)$, also written $\mathrm{Ind}_H^G(W_1 \oplus W_2) = \mathrm{Ind}_H^G(W_1) \oplus \mathrm{Ind}_H^G(W_2)$, and for $K \subseteq H \subseteq G$ and W a representation of K, $\mathrm{Ind}_H^G(\mathrm{Ind}_H^K W) = \mathrm{Ind}_H^G(W)$ (by Fröbenius reciprocity, since restriction is transitive.

Let ϕ be a class function on $H \subseteq G$, and extend it by defining $\tilde{\Phi}(g) = \begin{cases} \phi(g) & \text{if } g\in H \\ 0 & \text{if } g \notin H \end{cases}$
Suppose $x, y \in$ the same coset of $H \Rightarrow y = xh$ for some $h \in H$.
Then $\tilde{\Phi}(y^{-1}gy) = \tilde{\Phi}(h^{-1}x^{-1}gxh) = \begin{cases} \phi(h^{-1}x^{-1}gxh) = \phi(x^{-1}gx) & \text{if } h^{-1}x^{-1}gxh \in H \Leftrightarrow x^{-1}gx \in H \\ 0 = \phi(x^{-1}gx) & \text{if } h^{-1}x^{-1}gxh \notin H. \end{cases}$
$\therefore \frac{1}{|H|}\sum_{a\in G} \tilde{\Phi}(a^{-1}ga) = \sum_{i=1}^{r} \tilde{\Phi}(g_i^{-1}gg_i)$ and is independent of the choice of representatives g_i.
The summation ensures that this is a class function.

Theorem: the character of the induced representation is $\sum_{i=1}^{r} \tilde{\chi}_W(g_i^{-1}gg_i)$
Proof: fix $g \in G$, and choose w_i a basis of W $\therefore \{g_i w_j\}$ is a basis for the induced representation.
If $g(g_iH) \neq g_iH$, then the g_iw_j component of $g(g_iw_j)$ is 0 $\forall j \Rightarrow$ the rows corresponding to g_iw_j do not contribute to the trace.
$\therefore$ the only contributions come from those g_i for which $g_i^{-1}gg_i = h_i \in H$.
Then $g(g_iw) = g_ih_iw \Rightarrow \text{trace}(g) = \sum \text{trace } h_i = \sum \chi_W(g_i^{-1}gg_i)$, summing over all i with $g_i^{-1}gg_i \in H$
$= \sum \tilde{\chi}_W(g_i^{-1}gg_i)$ by definition of the extension $\tilde{\chi}_W$.

Example: $H = \langle(1234)\rangle \subseteq S_4$ has a faithful 1-dimensional representation: $\rho((1234)^r) =$ multiplication by i^r.
The induced representation on S_4 acts on $\mathbb{C}^6$ (dimension $= |S_4|/|H|$) $\Rightarrow$ character of e is 6.

$H = \{e, (1234), (13)(24), (1432)\}$

H has no elements of the cycle shape (123) or (12) $\therefore$ no conjugates of (123) or (12) are in $H \Rightarrow$ their character is 0.

(1234) has 6 conjugates (since there are 6 elements of this shape) $\Rightarrow$ 4 elements of S_4 send (1234) to each of its conjugates (orbit-stabiliser theorem). Since 2 of these conjugates lie in H, character of 4-cycle $= \frac{1}{4}(4 \times \rho(1234) + 4 \times \rho(1432)) = 0$

$(13)(24)$ has 3 conjugates $\Rightarrow$ 8 elements of S_4 send $(13)(24)$ to each of these, precisely one of which lies in $H \Rightarrow$ character is $\frac{1}{4}(8 \times \rho(13)(24)) = -2$.

Recall that $\dim_G \mathrm{Hom}(V,W) = \langle \chi_V, \chi_W \rangle$. Hence, in the language of characters, Frobenius reciprocity says $\langle \chi_{\mathrm{Ind}_H^G(W)}, \chi_V \rangle_G = \langle \chi_W, \chi_V \rangle_H$ where on the right hand side we regard V as a representation restricted to H.

Let K, H be subgroups of G. The **double cosets** of K on G/H is the set of **K-orbits** on the **cosets** gH, or, equivalently, the **orbits** of the action $K \times H$ on G, $(k,h)(g) = kgh^{-1}$. So this is symmetric in H and K.

If $k \in \mathrm{Stab}(gH)$, then $kgH = gH \Leftrightarrow k \in gHg^{-1}$. Hence $\mathrm{Stab}(gH) = K \cap gHg^{-1}$. (under action of K)

Mackey's restriction formula: let $K, H \leq G$, W be a representation of H. Then, considered as a **representation of K**, $\mathrm{Ind}_H^G(W) = \bigoplus_s \mathrm{Ind}_{K \cap sHs^{-1}}^K ({}_sW)$ where we sum over s a set of representatives in G for the double cosets.

Proof: $\mathrm{Ind}_H^G(W) = \bigoplus g_i W$ (g_i = representatives of H-cosets), and K permutes the $g_i W$. $\therefore$ The irreducible components of $\mathrm{Ind}_H^G(W)$, as a representation of K, has the form $\bigoplus g_i W$ where all $g_i \in$ one double coset, and, by orbit-stabiliser, each $k \in K$ acts as $g_i \times$ some element of $\mathrm{Stab}(sW)$. $\therefore$ this is $\mathrm{Ind}_{\mathrm{Stab}(sW)}^K (sW)$.

Mackey's irreducibility criterion: let W be a representation of $H \leq G$. Then $\mathrm{Ind}_H^G(W)$ is **irreducible** $\Leftrightarrow$ W is **irreducible**, the representations sW and W of $H_s = sHs^{-1} \cap H$ are **disjoint** (ie have no irreducible component in common) $\forall s$ representatives of the double cosets.

In particular, if H is **normal**, $H_s = H \; \forall s$, so $\mathrm{Ind}_H^G(W)$ is **irreducible** $\Leftrightarrow$ W is **irreducible** and the representations $g_i W, W$ are **non-isomorphic** $\forall$ representatives g_i of the cosets gH.

Proof:
$$\langle \chi_{\mathrm{Ind}_H^G(W)}, \chi_{\mathrm{Ind}_H^G(W)} \rangle_G = \langle \chi_W, \chi_{\mathrm{Ind}_H^G(W)} \rangle_H \quad \text{by Frobenius reciprocity}$$
$$= \textstyle\sum_s \langle \chi_W, \chi_{\mathrm{Ind}_{H_s}^H (sW)} \rangle_H \quad \text{by Mackey's formula with } K = H.$$
$$= \textstyle\sum_s \langle \chi_{sW}, \chi_W \rangle_{H_s} \quad \text{by Frobenius reciprocity}$$

The right hand side is the sum of non-negative integers, and the term corresponding to $s=1$ contributes at least 1 to the sum $\therefore$ $\chi_{\mathrm{Ind}_H^G(W)}$ irreducible $\Leftrightarrow \langle \chi_{\mathrm{Ind}_H^G(W)}, \chi_{\mathrm{Ind}_H^G(W)} \rangle_G = 1$ $\Leftrightarrow \langle \chi_W, \chi_W \rangle_H = 1$, $\langle \chi_{sW}, \chi_W \rangle_{H_s} = 0 \; \forall s \neq e \Leftrightarrow W$ irreducible, $\dim_{H_s} \mathrm{Hom}(sW, W) = 0 \; \forall s \neq e$ $\Leftrightarrow W$ irreducible, sW, W are disjoint on $sHs^{-1} \cap H \; \forall s \neq e$ (by Schur).

Example: Consider $\mathbb{Z}/n\mathbb{Z} \leq D_{2n}$. As $\mathbb{Z}/n\mathbb{Z}$ is normal, the double cosets are usual cosets, and there are only two of these. Let W be an irreducible representation of $\mathbb{Z}/n\mathbb{Z} \Rightarrow W$ is completely specified by the image of a generating rotation a, which has the form $e^{\frac{2\pi i j}{n}}$ for some j.

Then the representation bW is given by $a \mapsto e^{-2\pi i j/n}$, since $a(bw) = ba^{-1}w$.

The induced 2-dimensional representation of D_{2n} is irreducible $\Leftrightarrow W, bW$ are non-isomorphic $\Leftrightarrow 0 < j < n/2$.

Example: consider $A_4 \subseteq S_4$. Again, A_4 is normal with index 2, so $\mathrm{Ind}_{A_4}^{S_4} W$ is irreducible $\Leftrightarrow$ W is irreducible and not isomorphic to $(12)W \Leftrightarrow W$ is irreducible and $\chi_W \neq \chi_{(12)W}$.

Since $g((12)w) = (12)[(12)g(12)]w$, $\chi_{(12)W}(g) = \chi_W((12)g(12))$, so the induced representation is irreducible if W is one of the irreducible representations of A_4 on which the two 3-cycle groups take different values.

A group of order p^N, for some prime p and $N>0$, is a p-group.

A group is nilpotent if it has a chain of subgroups $G = G^0 \supseteq G^1 \cdots \supseteq G^r = \{e\}$ with $G^i/G^{i+1} \subseteq Z(G/G^{i+1})$ and G^i normal in G $\forall i<r$. All abelian groups are nilpotent – take $r=1$.

Theorem: p-groups are nilpotent

Proof: let $|G| = p^N$. Every conjugacy class of G has size dividing $|G| \Rightarrow$ their sizes are 1 or a power of p.

Every conjugacy class of size 1 contains an element of $Z(G)$

$\therefore |G| =$ sum of |conjugacy class| $= |Z(G)| +$ sum of powers of p. Since p divides $|G|$, p must divide $|Z(G)| \Rightarrow |Z(G)| > 1$. $\therefore |G/Z(G)| < |G|$, and $G/Z(G)$ is also a p-group.

We construct the chain by induction – it is trivial for $N=1$ as then G is abelian.

Given $G/Z(G) \supseteq G^1/Z(G) \supseteq \cdots \supseteq G^r/Z(G) = \{e\}$ a nilpotent chain, $G \supseteq G^1 \supseteq G^2 \cdots \supseteq G^r = Z(G) \supseteq \{e\}$ is also a nilpotent chain (by third isomorphism theorem, and as $Z(G) \subseteq G^i \ \forall i$)

A group is solvable if $\exists$ a chain of subgroups $G = G^0 \supseteq G^1 \cdots \supseteq G^r = \{e\}$ with G^i/G^{i+1} abelian and G^{i+1} normal in G^i. Hence all nilpotent groups are solvable, but dihedral groups are solvable but not nilpotent, since they have trivial centre, so $G^{r-1} \subseteq Z(G)$ cannot be satisfied.

Lemma 1: let G be a finite group with A a normal subgroup. Then any irreducible representation V of G is induced from a representation W of some subgroup $H \subseteq G$, $A \subseteq H$, with W restricted to A isotypical (ie sum of copies of the same irreducible representation)

Proof: let $m_1V_1 \oplus \cdots \oplus m_nV_n$ be the isotypical decomposition of V restricted to A – ie V_i's are irreducible and non-isomorphic. If $n=1$, take $H=G$.

Otherwise, fix $g \in G$ and some i and consider the action of A on $g(V_i)$:

$a(gv) = g(g^{-1}ag)(v)$ and normality of A means $g^{-1}ag \in A$. $(a \in A, v \in V_i)$

Since conjugation action of g permutes the elements of A, this action of A must stay irreducible $\therefore gV_i = V_j$ for j where the actions on V_i and V_j of A are related by $\rho_i(a) = \rho_j(g^{-1}ag)$

For all copies of V_i, A acts on $g(V_i)$ in the same way (since A acts on all copies of V_i in the same way), so these are all copies of $V_j \Rightarrow m_i = m_1 \ \forall i$, and G permutes the m_iV_i's.

V is irreducible, so G permutes the m_iV_i's transitively $\Rightarrow$ for each i, $\exists g_i \in G$ with $g_i(m_1V_1) = m_iV_i$

Let $H \subseteq G$ be the stabiliser of m_1V_1. clearly $A \subseteq H$. By orbit-stabiliser, the g_i's are precisely the representatives of gH, so $V = \mathrm{Ind}_H^G(m_1V_1)$.

Lemma 2: let G be a non-abelian p-group. Then $\exists A$ normal and abelian with $A \subseteq G$, $A \not\subseteq Z(G)$.

Proof: G is nilpotent, so $G/Z(G)$ has non-trivial centre (and $G/Z(G) \neq \{e\}$ since G is not abelian)

Choose g with $gZ(G) \in Z(G/Z(G))$, $g \notin Z(G)$, and set $A = \langle Z(G), g\rangle$.

Then $A \not\subseteq Z(G)$, and A is abelian since g commutes with $Z(G)$.

$\forall x \in G, z \in Z(G)$, we have $x^{-1}zx = z \in Z(G)$ (by definition of $Z(G)$), and $x^{-1}gxZ = gZ$
$\Rightarrow x^{-1}gx = gz \in A$ (for some $z \in Z(G)$) $\Rightarrow$ A is normal.

Theorem: every complex irreducible representation V of a finite p-group G is induced from a 1-dimensional representation of some subgroup

Proof: If G is abelian, V is 1-dimensional, so the theorem is trivial.

If G is not abelian, then, by lemma 2, $\exists A \trianglelefteq G$, A abelian and normal.

By lemma 1, V is induced from $\rho: H \to GL(W)$ with W isotypical on $A \leq H$.

Since A is abelian, each irreducible component of ρ (restricted to A) must be 1-dimensional

These components are isomorphic $\Rightarrow \rho(A)$ acts as scalar multiplication on W.

$\therefore$ in the induced representation $\tilde{\rho}$, A acts as scalar multiplication $\Rightarrow$ it commutes with $\tilde{\rho}(g)$ $\forall g \in G$. As $A \not\leq Z(G)$, $\tilde{\rho}$ is not faithful.

$\therefore V$ is a representation of $G/\ker\tilde{\rho}$, which has strictly smaller order than G.

By induction on $|G|$, $V = \mathrm{Ind}_{H'}^{G'}(W)$ of some 1-dimensional representation W of some $H' \leq G' = G/\ker\tilde{\rho}$.

$H' = H/\ker\tilde{\rho}$ for some $H \leq G$ (take preimage of H' under the natural map $g \to g\ker\tilde{\rho}$).

Then $V = \mathrm{Ind}_H^G(W)$ (lifting W to a representation of H)

Lemma 3: for any $g \in G$ and any character χ, $|\chi(g)| = \chi(1) \Leftrightarrow \frac{\chi(g)}{\chi(1)}$ is an algebraic integer $\neq 0$.

Proof: let c_i $(1 \leq i \leq \chi(1))$ be the eigenvalues of g which, by Lagrange, are $|G|^{th}$ roots of unity

$\therefore c_i \in \mathbb{Q}(\xi_{|G|})$ = splitting field of $x^{|G|} - 1$, so $\mathbb{Q}(\xi_{|G|})/\mathbb{Q}$ is Galois.

$\Rightarrow x = \prod_h \frac{\chi(g)}{\chi(1)}$, producting over all $h \in Gal(\mathbb{Q}(\xi_{|G|})/\mathbb{Q})$, is fixed by $Gal(\mathbb{Q}(\xi_{|G|})/\mathbb{Q})$, so it is rational.

Now suppose $\frac{\chi(g)}{\chi(1)}$ is an algebraic integer $\Rightarrow$ it satisfies some monic polynomial $f \in \mathbb{Z}[x]$.

$h \in Gal(\mathbb{Q}(\xi_{|G|})/\mathbb{Q})$ does not change f, so $h(\frac{\chi(g)}{\chi(1)})$ also satisfies $f \Rightarrow h(\frac{\chi(g)}{\chi(1)})$ is an algebraic integer $\forall h \in Gal(\mathbb{Q}(\xi_{|G|})/\mathbb{Q}) \Rightarrow x$ is an algebraic integer $\Rightarrow x \in \mathbb{Z}$

By the triangle inequality, $|\chi(g)| \leq \chi(1)$ $\Rightarrow |x| \leq 1$ with equality iff $|\chi(g)| = \chi(1)$.

Conversely, if $|\chi(g)| = \chi(1)$, $\chi(g) = c_1 \chi(1)$ (by triangle inequality), and c_i is a non-zero algebraic integer.

Lemma 4: If G has order $p^a q^b$, then $\exists g \in G$ and an irreducible non-trivial character χ with $\chi(g)/\chi(1)$ an algebraic integer $\neq 0$ (assuming $a, b \geq 1$)

Proof: Let H be a q-Sylow subgroup of G. q-groups are nilpotent, and therefore have non-trivial centre. Pick $g \in Z(H)$, $g \neq e$.

Consider conjugation action of G on G. $H \leq \mathrm{Stab}(g) \Rightarrow q^b$ divides $\mathrm{Stab}(g)$.

Then $|\text{conjugacy class of } g| = |G|/|\mathrm{Stab}(g)|$ = power of p.

Apply the column orthogonality relation to e and g: $1 + \sum \chi(g)\chi(1) = 0$ where we sum over all non-trivial irreducible character χ with $\chi(g) \neq 0$.

If p divides $\chi(1)$ $\forall \chi$, then $1/p + \sum \chi(g) a_\chi = 0$ for some integers a_χ.

Since $\chi(g)$ is an algebraic integer, this implies $1/p$ is an algebraic integer, which is a contradiction $\therefore$ we can find χ with $\chi(1)$ not divisible by p.

So $\chi(1)$, $c = |\text{conjugacy class of } g|$ are coprime $\Rightarrow \exists m, n \in \mathbb{Z}$ with $m\chi(1) + nc = 1$

$\Rightarrow m\chi(g) + n\frac{c\,\chi(g)}{\chi(1)} = \frac{\chi(g)}{\chi(1)}$. Recall that $\frac{c\,\chi(g)}{\chi(1)}$ is an algebraic integer, as is $\chi(g)$.

$\Rightarrow$ so is $\frac{\chi(g)}{\chi(1)}$.

Burnside's theorem: let G be a group of order $p^a q^b$ with p, q prime, $a, b \in \mathbb{N}$. Then G is solvable.

Proof: We proceed by induction on $|G|$. Groups of order p, q are abelian $\Rightarrow$ solvable.

Suppose G is non-abelian and simple. Let ρ be any non-trivial representation $\Rightarrow \ker\rho = \{e\}$.

By lemmas 4 and 3, $\exists$ an irreducible representation ρ and $g \in G$ with $|\chi_\rho(g)| = \chi_\rho(1)$

$\Rightarrow \chi_\rho(g) =$ multiple of $\chi_\rho(1) \Rightarrow \rho(g)$ acts by scalar multiplication.

As all ρ are faithful, this implies $g \in Z(G)$, which is a contradiction as $g \neq e$ and $Z(G)$ is normal.

$\therefore \exists N$ normal in G, with $N, G/N$ smaller than G and having orders of the same form

$\therefore$ by induction, $\exists$ solvable chains $N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_r = \{e\}$, $G/N \supseteq G_1/N \supseteq \cdots G_s/N = \{e\}$.

By third isomorphism theorem, $G \supseteq G_1 \supseteq \cdots \supseteq G_s = N \supseteq N_1 \supseteq \cdots N_r = \{e\}$ is a solvable chain.

Compact groups

A **topological group** G is a group which is also a topological space such that the multiplication and inverse functions are continuous (with the product topology)

e.g. any group is a topological group with the discrete topology – then every map is continuous.

e.g. $GL(n, \mathbb{R})$ is a topological group, with topology inherited from $\mathbb{R}^{n^2}$ (and similarly for $\mathbb{C}$).

A topological group is **compact** if it is compact when considered as a topological space

e.g. the circle group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ under multiplication.

the orthogonal group $O(n) = \{A \in GL(n, \mathbb{R}) : A^T A = I\}$ and $SO(n) = \{A \in O(n) : \det A = 1\}$

$\therefore SO(2) \cong S^1$, $O(n)/SO(n) \cong \mathbb{Z}/2\mathbb{Z}$ (ie $\exists f$ an isomorphism of groups and homeomorphism of spaces)

we can view $O(n)$ as $\{(v_1, v_2, \dots v_n) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ n times $: \langle v_i, v_j \rangle = \delta_{ij}\}$ = preimage of a point under a continuous function = closed. $\|v_i\| = 1 \; \forall i$ so this set is also bounded $\therefore$ compact (v_i are columns of an orthogonal matrix)

similarly, $U(n) = \{A \in GL(n, \mathbb{C}) : \bar{A}^T A = I\}$ is compact. $SU(n) = \{A \in U(n) : \det A = 1\}$ is a closed subset of $U(n)$, hence compact also. $U(1) \cong S^1$, $SU(2) \cong S^3 \subseteq \mathbb{R}^4$, $U(n)/SU(n) \cong S^1$.

In fact, S^1 and S^3 are the only spheres with a topological group structure, corresponding to multiplication by complex numbers and quaternions respectively, of norm 1. (check that multiplication of matrices in $SU(2)$ correspond to multiplication of quaternions: $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \leftrightarrow a + bj$)

Previous averaging arguments hold for compact groups by using Haar measure, which has the properties $\int_G 1 = 1$, and $\int_G f(g)\,dg = \int_G f(gh)\,dg = \int_G f(hg)\,dg$.

e.g. If $G = S^1$, $\int_G f(g)\,dg = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,d\theta$ ie usual definition of volume on a 1-manifold, suitably scaled. This also works for $SU(2) \simeq S^3$.

Schur's lemma also remains true in the infinite setting.

To use the topological properties of the group, we will require our representations to be continuous. If the representation is finite-dimensional, its character is well-defined, and is a continuous function since taking the trace is continuous on the space of matrices.

As a result, various previous proofs still hold, and we still have complete reducibility of representations (since they are unitarisable), orthonormality of irreducible characters (with respect to Haar measure) and completeness of characters: every L^2 class function can be expressed as $\sum a_i \chi_{V_i}$ with V_i irreducible, $a_i \in \mathbb{C}$ and $\sum |a_i|^2$ finite (sums are infinite)

Theorem: every complex irreducible representation ρ of S^1 is isomorphic to the 1-dimensional representation $z \to z^n$ for some integer n.

Proof: S^1 is abelian $\Rightarrow \rho$ is 1-dimensional, by Schur's Lemma. $\therefore \rho(S^1) \subseteq GL(1,\mathbb{C}) \simeq \mathbb{C}^*$.

$\rho(1) = 1$. By continuity, $\exists m$ such that $\rho(\{e^{2\pi i\theta} : -1/m \leq \theta \leq 1/m\}) \subseteq \{re^{i\theta} : -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$

$e^{2\pi i/m}$ has order m in S^1 $\therefore \rho(e^{2\pi i/m})$ must also be an m^{th} root of unity.

$\Rightarrow \rho(e^{2\pi i/m}) = e^{2\pi i n/m}$ for some integer $n \in (-m/4, m/4)$

As ρ is a homomorphism, $\rho(e^{2\pi i/m \cdot 1/2}) = \pm e^{2\pi i n/m \cdot 1/2}$ and, by continuity and our choice of m, we must take the positive sign. Applying this inductively, we obtain $\rho(e^{\frac{2\pi i}{2^r m}}) = e^{\frac{2\pi i n}{2^r m}}$

As ρ is a homomorphism, this determines the value of ρ on all $(2^r m)^{th}$ roots of unity — on these values, $\rho(z) = z^n$. Since ρ is continuous and the $(2^r m)^{th}$ roots of unity are dense in S^1, $\rho(z) = z^n \ \forall z \in S^1$.

$\therefore$ All irreducible representations are given by multiplication by z^n $\therefore$ all irreducible characters are functions $f_n(z) = z^n$, $n \in \mathbb{Z}$. S^1 is abelian so all L^2 functions on S^1 are class functions $\therefore f_n$ form an orthonormal basis for the L^2 functions on $S^1 \Rightarrow \forall f$ an L^2 function on S^1,

$f = \sum \langle f, f_n \rangle f_n = \frac{1}{2\pi}\sum \int_0^{2\pi} f(e^{i\theta})\overline{f_n(e^{\theta})}\, d\theta\, f_n =$ Fourier series.

From linear algebra, we know that all unitary matrices can be diagonalised with a unitary change-of-basis matrix. Hence the conjugacy classes of $U(n)$ are indexed by diagonal matrices with diagonal entries $a_{jj} = e^{i\theta_j}$, $\theta_1 \geq \theta_2 \geq \cdots \theta_n$, $\theta_i \in [0, 2\pi)$ — ie they are indexed by their eigenvalues, neglecting ordering.

$\therefore \{$conjugacy classes of $U(n)\} \simeq (S^1)^n / S_n$.

Similarly, $\{$conjugacy classes of $SU(n)\} \simeq \ker\alpha / S_n$ where $\alpha : (S^1)^n \to S^1$ takes the product of the components.

This is especially simple when $n=2$: the conjugacy classes of $SU(2)$ are represented by $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ where $\theta \in [0, \pi]$ $\therefore$ conjugacy classes of $SU(2) \simeq S^1$. Then it is clear that two matrices are conjugate $\Leftrightarrow$ they have the same trace. Identifying $SU(2)$ with S^3, these conjugacy classes are where different hyperplanes intersect S^3 $\therefore$ the conjugacy classes are indexed by the half-circle perpendicular to those planes.

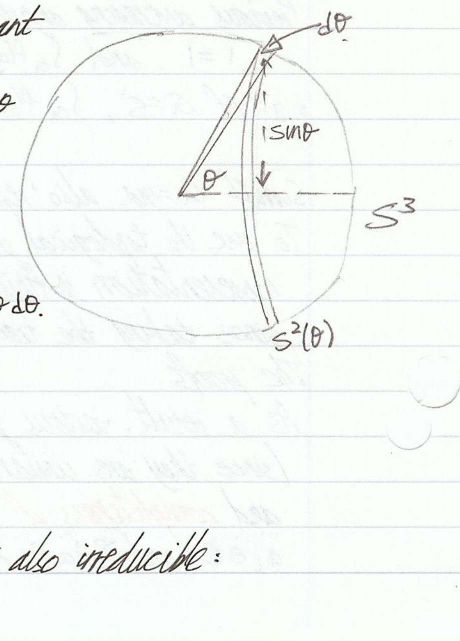
Weyl integration formula: let f be a continuous class function on $SU(2)$. Let $\hat{f}(\theta) = f\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$

Then $\int_{SU(2)} f(g)\, dg = \frac{1}{4\pi}\int_0^{2\pi} \hat{f}(\theta)|\Delta(\theta)|^2\, d\theta = \frac{1}{\pi}\int_0^{2\pi} \hat{f}(\theta)\sin^2\theta\, d\theta$ where $\Delta(\theta) = e^{i\theta} - e^{-i\theta}$, the Weyl denominator.

Proof: $\int_{S^3} f(g)\, dg = \int_0^{\pi}\int_{S^2(\theta)} f(g)\, dx\, d\theta$

$= \int_0^{\pi}$ area of $S^2(\theta)\, \hat{f}(\theta)\, d\theta$ — since $f(g)$ is constant on $S^2(\theta)$ = copy of S^2 making an angle θ

$= \int_0^{\pi} 4\pi \sin^2\theta\, \hat{f}(\theta)\, d\theta = \int_0^{2\pi} 2\pi \sin^2\theta\, \hat{f}(\theta)\, d\theta$



$\therefore \int_{SU^3} f(g)\, dg = \frac{\int_{S^3} f(g)\, dg}{\int_{S^3} 1\, dg} = \frac{\int_0^{2\pi} 2\pi\sin^2\theta\, \hat{f}(\theta)\, d\theta}{2\pi^2} = \frac{1}{\pi}\int_0^{2\pi} \hat{f}(\theta)\sin^2\theta\, d\theta.$

The second equality is trivial.

Now we are in a position to find the irreducible representations of $SU(2)$.

The trivial representation is clearly irreducible.

The "standard" 2-dimensional representation, $SU(2) \to GL(2,\mathbb{C})$ is also irreducible:

any invariant subspace is mapped to itself by all maps of the form $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\therefore$ it must be span e_1 or span e_2 (e_1, e_2 = basis vectors). But $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ exchanges these two subspaces.

Theorem: The irreducible representations of $SU(2)$ are S^nV where V is the 'standard' representation.

Proof: First we find χ_{S^nV}. From previous discussion, it is sufficient to consider its values on $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$.

e_1, e_2 are eigenvectors of $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ acting on V, with eigenvalues $e^{i\theta}$ and $e^{-i\theta}$

$\Rightarrow \sum_{\sigma \in S_n} e_{\sigma i_1} \otimes e_{\sigma i_2} \otimes \cdots \otimes e_{\sigma i_n}$, where $i_1 = i_2 = \cdots = i_r = 1$, $i_{r+1} = i_{r+2} = \cdots i_n = 2$ (for $0 \le r \le n$) is a basis of eigenvectors for $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ acting on S^nV, with eigenvalues $(e^{i\theta})^r (e^{-i\theta})^{n-r}$ $= (e^{i\theta})^{2r-n}$ $\therefore S^nV$ has dimension $n+1$.

$\therefore \chi_{S^nV}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ = sum of eigenvalues $= \sum_{r=0}^{n} z^{2r-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} = z^n + z^{n-2} + z^{n-4} + \cdots + z^{-n}$

$$\langle \chi_{S^nV}, \chi_{S^nV} \rangle = \int_{SU(2)} |\chi_{S^nV}(g)| \, dg \qquad \text{where } z = e^{i\theta} \Rightarrow \bar{z} = 1/z$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \left| \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} \right| |z - z^{-1}| \, d\theta$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \left(z^{n+1} - z^{-(n+1)}\right)\left(z^{-(n+1)} - z^{n+1}\right) d\theta$$

$$= \frac{1}{4\pi} \int_0^{2\pi} -z^{2n+2} + 2 - z^{2n+2} \, d\theta = \frac{1}{4\pi} 2\pi \, 2 = 1 \qquad \therefore \chi_{S^nV} \text{ is irreducible.}$$

Let W be any finite-dimensional representation of $SU(2)$. To split W into the sum of irreducible representations, we express χ_W as the sum of irreducible characters. Since every conjugacy class has a representative in S^1, χ_W is uniquely determined by $\chi_{\tilde{W}}$ where $\tilde{W}$ is W restricted to S^1. $\therefore \tilde{W}$ is the sum of irreducible representations of S^1, each of which has character $z \to z^n$ $\therefore \chi_{\tilde{W}}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \sum a_n z^n$ where $n \in \mathbb{Z}$ ie a Laurent polynomial

$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ and $\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$ are conjugate in $SU(2)$ $\therefore \sum a_n z^n = \sum a_n z^{-n} \Rightarrow a_n = a_{-n}$

$\therefore \chi_{\tilde{W}}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \sum_{n>0}^{\infty} a_n (z^n + z^{-n}) + a_0$ and these are spanned (as a module over $\mathbb{Z}$) by χ_{S^nV}.

The last part of the proof suggests that, to decompose any representation of $SU(2)$ into irreducibles, we should look at $\chi_1 \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ and write it as a linear combination of $\frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}$.

e.g. $\chi_{V \otimes S^2V}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \chi_V\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \chi_{S^2V}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$

$$= (z + z^{-1})(z^2 + 1 + z^{-2}) = z^3 + 2z + 2z^{-1} + z^{-3} = z^3 + z + z^{-1} + z^{-3} + z + z^{-1}$$

$$= \chi_{S^3V}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} + \chi_V\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

$\therefore V \otimes S^2V = S^3V \oplus V$. To see this, let x denote e_1, y denote e_2, and identify the basis elements $\sum_{\sigma \in S_n} e_{\sigma i_1} \otimes e_{\sigma i_2} \otimes \cdots \otimes e_{\sigma i_n}$ with the monomial $x^r y^{n-r}$. Then elements of S^nV are polynomials where each term has degree n. S^3V is the image of the multiplication map on $V \otimes S^2V$. (send $f \otimes g$ to fg)

This can be generalised with the Clebsch-Gordan formula: for any $0 \le p \le q$,

$$S^pV \otimes S^qV \cong S^{p+q}V \oplus S^{p+q-2}V \oplus \cdots \oplus S^{q-p}V$$

Proof: $\chi_{S^pV \otimes S^qV} = \frac{z^{q+1} - z^{-(q+1)}}{z - z^{-1}} \sum_{r=0}^{p} z^{2r-p}$

$$= \frac{1}{z - z^{-1}} \sum_{r=0}^{p} \left(z^{q+2r-p+1} - z^{-q+2r-p-1}\right)$$

$$= \frac{1}{z - z^{-1}} \left(\sum_{r=0}^{p} z^{q+2r-p+1} - \sum_{k=0}^{p} z^{-q-2k+p-1}\right) \qquad \text{set } k = p - r$$

$= \sum_{r=0}^{p} \chi_{S^{q-p+2r}V}$ the formula follows since characters uniquely determine representations.

Note that $SO(3) \cong SU(2)/\pm I$ (written $PSU(2)$). To see this, consider $SU(2)$ as quaternions of length 1 acting on the pure-imaginary quaternions by conjugation.

Also, $SO(4) \cong SU(2) \times SU(2) / \pm(I, I)$: again, consider $(g, h) \in SU(2) \times SU(2)$ as unit quaternions acting on $\mathbb{H}^4$

by $x \to gxh^{-1}$. Since norm is multiplicative, this preserves norm, and $SU(2) \times SU(2)$ is connected so we obtain one connected component of $O(4)$ – namely $SO(4)$.

Finally $U(2) = SU(2) \times S^1 / \pm\{I, 1\}$ since any element of $U(2)$ can be written as AB where $A \in SU(2)$ and $B = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ ie $\det = (e^{i\theta})^2$.

This allows us to compute irreducible representations of $SO(3)$, $SO(4)$ and $U(2)$, since, as for finite groups, all irreducible representations of $G \times H$ are given uniquely by {irreducible representation of $G \otimes$ irreducible representation of H}.

$\therefore$ irreducible representations of $SO(3)$ = irreducible representations of $SU(2)$ in which $-I$ acts as the identity $= S^{2n}V$, since, in S^mV, $-I$ has eigenvalues $(-1)^{2r-m} = (-1)^m$ $(0 \le r \le m)$ $\therefore -I$ acts as $-I$ on S^mV for m odd, and as I on S^mV for m even.

Hence V is not a representation of $SO(3)$ $\therefore$ the representations denoted $S^{2n}V$ are <u>not</u> the symmetric powers of any representation of $SO(3)$.

All these irreducible representations have odd dimension. Since S^2V and S^0V are the only irreducible representations of dimension ≤ 3, and the standard 3-dimensional representation of $SO(3)$ does not act trivially, it corresponds to S^2V.

Similarly, irreducible representations of $SO(4) = S^nV \otimes S^mV$ where $(-I,-I)$ acts as the identity ie $(-1)^n(-1)^m = 1 \Rightarrow$ we require $n \equiv m \pmod 2$

Irreducible representations of $U(2) = S^nV \otimes (z \mapsto z^m)$, where $(-I,-1)$ acts as the identity ie $(-1)^n(-1)^m = 1 \Rightarrow n \equiv m \pmod 2$. $(n \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z})$. In the standard 2-dimensional representation of $U(2)$, $SU(2)$ acts as V and the scalar matrices act by scalar multiplication $\therefore$ this is $V \otimes (z \to z)$. Call this W.

For any n-dimensional representation ρ of any group G, by computing characters, we see that $\Lambda^n\rho(g) = \det \rho(g)$. $\therefore \Lambda^2 W = (z \to z^2)$, and $(z \to z^m)$ is simply $m/2$ copies of this tensored together. Hence a more natural characterisation of the irreducible representations of $U(2)$ is $S^nW \otimes (\Lambda^2 W)^{\otimes m} = S^nV \otimes (z \to z^{n+2m})$ where $n \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z}$, and negative tensor powers are understood to represent division by powers of the determinant.

More generally, each element of $SU(n)$ is diagonalisable, so each conjugacy class has a member in the maximal torus $T = \left\{ \begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{pmatrix} : z_i \in S^1 : z_1 z_2 \cdots z_n = 1 \right\}$. z_n is constrained by the other $n-1$ variables, so $T \cong S^{1(n-1)}$, so its irreducible representations have the form $(z_1, z_2, \cdots, z_{n-1}) \to (z_1^{a_1}, z_2^{a_2}, \cdots z_{n-1}^{a_{n-1}})$ $a_i \in \mathbb{Z}$

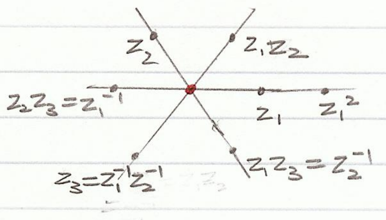
$\therefore$ characters of $SU(n)$, when restricted to T, is a Laurent polynomial in $z_1, z_2, \cdots z_{n-1}$.

Since permuting the z_i's gives a conjugate element in $SU(n)$ (but not in T, which is abelian), these polynomials must be symmetric in $z_1, z_2, \cdots z_n$ – ie if $\sigma \in S_n$, then $p(z_1, z_2, \cdots z_{n-1}) = p(z_{\sigma(1)}, z_{\sigma(2)}, \cdots z_{\sigma(n-1)})$ once we replace all occurences of z^n by $(z_1 z_2 \cdots z_{n-1})^{-1}$.

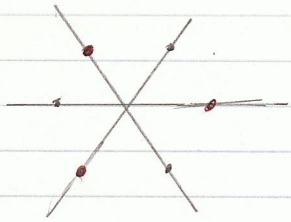
It is clear that the character of the standard representation is $z_1 + z_2 + \cdots + z_{n-1} + z_n$ $= z_1 + z_2 + \cdots + z_{n-1} + z_1^{-1} z_2^{-1} \cdots z_{n-1}^{-1}$.

The characters can be thought of as weighted points in $\mathbb{R}^{n-1}$: identify the z_i's with v_i's, distinct vertices of the unit n-simplex. Then $a z_1^{a_1} \cdots z_{n-1}^{a_{n-1}}$ corresponds to the vector $a_1 z_1 + a_2 z_2 + \cdots a_n z_n$ with weight a (choosing $a_i \ge 0$). The collection of points corresponding to a character must be invariant under any symmetry of the n-simplex.

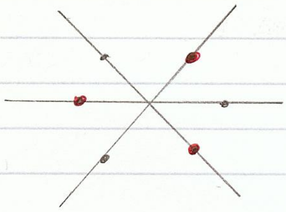
Example : $SU(3)$



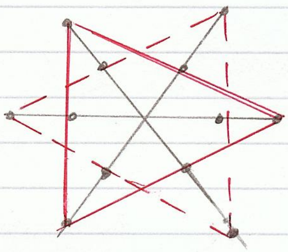
trivial representation



standard 3-dimensional representation $(=V)$



$V^* = \Lambda^2 V$



— S^2V

-- S^2V^*

It turns out that the characters of $S^aV\otimes S^bV^*$ generate all Laurent polynomials with the required symmetry, as a, b varies ∴ This is the source of all representations (though representations of this form may not be irreducible)