Representations of Symmetric Groups

We need different techniques here — Clifford theory is not useful since S_n has few normal subgroups.
Our main ingredient will be the permutation module $\mathbb{C}X$ (where X is a G-set)
If G is transitive, X is isomorphic (as a G-set) to the set of left cosets $g\,\mathrm{Stab}\,x$ for any $x \in X$, and $\chi_{\mathbb{C}X} = (1_{\mathrm{Stab}\,x})^G$

A partition of n is a sequence of non-increasing positive integers summing to n
$\lambda = (\lambda_1, \cdots \lambda_l)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$, $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$.
Partitions parametrise the conjugacy classes of S_n: the conjugacy class corresponding to λ contains the product of disjoint cycles of lengths $\lambda_1 \lambda_2 \cdots \lambda_l$.
The Young diagram of a partition λ has l rows, the i^{th} row being of length λ_i.
A λ-tableau is a way of filling the Young diagram with the numbers $1, 2, \cdots n$, each integer appearing exactly once.
Two λ-tableau are row equivalent if their corresponding rows contain the same integers.
The equivalent class of tableaux (denoted $[t]$) are λ-tabloids.

For each partition λ, S_n acts on the set of λ-tableaux and the set of λ-tabloids:
e.g. $\lambda = (3,2)$, $t = \begin{array}{c}123\\ \hline 45\end{array}$ is a λ-tableaux

$t' = \begin{array}{c}132\\ \hline 54\end{array}$ is a row-equivalent λ-tableaux

$(12)(345)$ sends t to $\begin{array}{c}214\\ \hline 53\end{array}$

Denote by M^λ the permutation module on the set of λ-tabloids (ie λ-tabloids is a basis)
$\therefore M^\lambda = \mathbb{C}(S/H)$ where H is the stabiliser of any λ-tabloid
one possibility for H is elements preserving each of the sets $\{1, 2, \cdots \lambda_1\}, \{\lambda_1+1, \lambda_1+2, \cdots \lambda_1+\lambda_2\}, \cdots$
$\{\lambda_1+\lambda_2+\cdots\lambda_{l-1}+1, \lambda_1+\lambda_2+\cdots\lambda_{l-1}+2, \cdots \lambda_1+\lambda_2+\cdots\lambda_l\}$ ie $H = S_{\lambda_1} \times S_{\lambda_2} \times \cdots S_{\lambda_l}$
e.g. $M^{(n)}$ = trivial module
$M^{(1,1,\cdots 1)}$ = regular module ($\mathbb{C}S^n$)
$M^{(1,n-1)}$ = usual permutation module (on n objects)

For subgroups H of S_n, write $H^+ = \sum_{h \in H} h \in \mathbb{C}S_n$, $H^- = \sum_{h \in H} \mathrm{sgn}(h) h \in \mathbb{C}S_n$.
Given a λ-tableau t, define C_t to be the subgroup of S_n consisting of reordering of columns of t
K_t to be C_t^-
e_t to be $K_t[t] \in M^\lambda$ (orbit of t under C_t, with suitable sign)
Observe that $C_{gt} = gC_tg^{-1}$ since gC_tg^{-1} fixes every column of gt (orbit-stabiliser)
$K_{gt} = gK_tg^{-1}$ since conjugation preserves sign.
$e_{gt} = ge_t$ $\because$ $e_{gt} = K_{gt}[gt] = gK_tg^{-1}g[t] = ge_t$ So S_n permutes $\{e_t : t \text{ a } \lambda\text{-tableau}\}$

Define an inner product on M^λ by setting $\langle [t], [s] \rangle = \delta_{[s][t]}$ and extending linearly.
Observe that this is S_n-invariant: $\langle gu, gv \rangle = \langle u, v \rangle$ $\forall u, v \in M^\lambda$, $g \in S_n$. So $\langle gu, v \rangle = \langle u, g^{-1}v \rangle$
$\therefore$ given any subgroup H of S_n, $\langle H^+u, v \rangle = \langle u, H^+v \rangle$, $\langle H^-u, v \rangle = \langle u, H^-v \rangle$

For any partition λ, the specht module S^λ is the submodule of M^λ spanned by $\{e_t : t \text{ a } \lambda\text{-tableau}\}$. We show that S^λ are all the distinct irreducible S_n representations.

Let λ, μ be partitions of n, **t a λ-tableau, s a μ-tableau**

λ dominates μ if, $\forall i \geq 1$, $\lambda_1 + \lambda_2 + \cdots \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$

(ie more elements in the first i rows of λ than of μ)

This defines a partial order on partitions of n (e.g. $(4,1,1)$ and $(3,3)$ not comparable.

1. Let H be a subgroup of S_n containing a transposition $(a\,b)$.
   If a, b lie in the same row in s, then $H^-([s]) = 0$:
   Let X be the even permutations in $H \Rightarrow H^- = X - X(ab) = X(1-(ab))$
   (ab) fixes $[s]$ $\therefore (1-(ab))[s] = 0$.

2. If no two distinct a, b lie in the same row of s and the same column of t, then λ dominates μ. If, in addition, $\lambda = \mu$, then $\exists g \in C_t$ with $[s] = g[t]$:
   Take any column of t, all its entries lie in different rows of s
   $\therefore \exists g \in C_t$ reordering each column of t so that $\text{row}_s(i_1) < \text{row}_s(i_2) < \cdots$ $\quad \begin{bmatrix} i_1 \\ i_2 \\ \vdots \end{bmatrix}$ a column of gt.
   Then, $\forall r$, $i_r \in$ row r or below of $s \Rightarrow$ more elements in first r rows of t than of s.
   If $\lambda = \mu$, then we must have $i_r \in$ row r of s $\forall i, r \Rightarrow [s] = [gt]$

3. If $K_t[s] \neq 0$, then λ dominates μ
   If $\lambda = \mu$, then $K_t[s] = \pm e_t$ or 0:
   if hypothesis of 2 is not true, $\exists$ transposition $(ab) \in C_t$, and a, b lie in the same row of s
   $\Rightarrow$ by 1, $K_t[s] = 0$. So the conditions in 2 must hold $\Rightarrow \lambda$ dominates μ.
   if $\lambda = \mu$ and $K_t[s] \neq 0$, then by 2, $\exists g \in C_t$ with $[s] = g[t] \Rightarrow K_t[s] = K_t g[t] = \text{sgn}(g)\, K_t[t] = \pm e_t$

4. If U is any submodule of M^λ, $U \supseteq S^\lambda$ or $U \subseteq (S^\lambda)^\perp$:
   if $K_t[u] = 0$ $\forall u \in U$, $\forall \lambda$-tableau t, then, $\forall u, t$: $\langle u, e_t \rangle = \langle u, K_t[t] \rangle = \langle K_t u, [t] \rangle = 0$ $\;U \subseteq S^{\lambda\perp}$
   otherwise, $K_t[u] \neq 0$ for some u and some t. $\Rightarrow K_t[u] = \pm e_t$ by 3.
   for any λ-tableau s, $\exists g \in S_n$ with $s = gt \Rightarrow e_s = ge_t = \pm g K_t[u] \in U$ $\therefore S^\lambda \subseteq U$.
   Since $S^\lambda \cap (S^\lambda)^\perp = \{0\}$, any submodule U of S^λ must be all of S^λ. ie S^λ simple.

5. If λ does not dominate μ, then $\text{Hom}_{\mathbb{C}S_n}(S^\lambda, M^\mu) = 0$
   If $\lambda = \mu$, then $\text{Hom}_{\mathbb{C}S_n}(S^\lambda, M^\mu)$ has dimension 1 ie $\text{Hom}_{\mathbb{C}S_n}(S^\lambda, M^\lambda) =$ scalar maps.
   Take $\theta \in \text{Hom}_{\mathbb{C}S_n}(S^\lambda, M^\mu)$, $\theta \neq 0$.
   Extend θ to $\bar{\theta} \in \text{Hom}_{\mathbb{C}S_n}(M^\lambda, M^\mu)$, taking $\theta((S^\lambda)^\perp) \equiv 0$
   For some λ-tableau t, $0 \neq \theta(e_t) = \theta(K_t[t]) = K_t\,\theta([t]) = K_t \sum_i c_i [s_i] = \sum_i c_i K_t [s_i]$
   for some μ-tableaux s_i, and $c_i \in \mathbb{C}$
   By 3, $K_t[s_i] \neq 0 \Rightarrow \lambda$ dominates μ.
   If $\lambda = \mu$, then $K_t[s_i] = \pm e_t$ (by 3) $\Rightarrow \theta(e_t) = ce_t$ for some $c \in \mathbb{C}$
   S^λ is spanned by e_{gt}, $g \in S_n$, and $\theta(e_{gt}) = \theta(ge_t) = g\theta(e_t) = gce_t = ce_{gt}$
   $\therefore \theta$ is multiplication by c on all of S^λ.
   So, if $S^\lambda \cong S^\mu$, then $\exists$ maps $S^\lambda \to M^\mu$ and $S^\mu \to M^\lambda \Rightarrow \lambda$ dominates μ, μ dominates $\lambda \Rightarrow \lambda = \mu$.
   Hence S^λ are distinct.
   $\# S^\lambda$s $= \#$ partitions of $S_n = \#$ conjugacy classes $= \#$ irreducible representations $\Rightarrow$ there are no others.

It follows from 5 that the permutation modules decompose as

$$M^{\mu} = \bigoplus m_{\lambda\mu} S^{\lambda} \quad \text{over all } \lambda \text{ dominating } \mu.$$

and the multiplicity $m_{\mu\mu}$ of S^{μ} is 1

A tableau t is standard if the rows and columns of t are increasing sequences. In this case, the tabloid $[t]$ is also standard.
We shall show that $\{e_t : t \text{ a standard } \lambda\text{-tableau}\}$ is a basis for S^{λ}

Define a composition of n to be a sequence of positive integers summing to n.
(ie partitions but reordered)
Again, λ dominates μ if, $\forall i \geq 1$, $\lambda_1 + \lambda_2 + \dots \lambda_i \geq \mu_1 + \mu_2 + \dots \mu_i$
Given a λ-tabloid $[t]$, its composition sequence is λ^i where $\lambda^i_j = \#$ elts in $\{1, \dots i\}$ in row j
For λ-tabloids $[s], [t]$ with composition sequences λ^i, μ^i, $[s]$ dominates $[t]$ if λ^i dominates μ^i $\forall i$. (small numbers are towards the top)

Example: $\lambda = (2,2)$, $s = \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix}$ $\quad t = \begin{matrix} 2 & 3 \\ 1 & 4 \end{matrix}$ $\quad$ s is standard, t is not.

$\lambda^1 = (1,0)$, $\lambda^2 = (1,1)$ $\quad \mu^1 = (0,1)$, $\mu^2 = (1,1)$
$\lambda^3 = (2,1)$ $\quad \lambda^4 = (2,2)$ $\quad \mu^3 = (2,1)$, $\mu^4 = (2,2)$
$\therefore \lambda^1$ dominates μ^1, $\lambda^2 = \mu^2$, $\lambda^3 = \mu^3$, $\lambda^4 = \mu^4$ $\Rightarrow [s]$ dominates $[t]$.

1. If $k < l$ and k appears in a lower row of $[t]$ than l, then $(kl)[t]$ dominates $[t]$:
   let λ^i, μ^i be the composition series of $[t]$ and $(kl)[t]$ respectively.
   For $i < k$ and $i \geq l$, $\lambda^i = \mu^i$.
   For $k \leq i < l$, $\mu^i \quad \lambda^i + 1$ where $q =$ row of l
   $\mu^i_r = \lambda^i_r - 1$ where $r =$ row of k $\quad$ ($q < r$ - k further up)
   Since $q < r$, μ^i dominates $\lambda^i \Rightarrow (kl)[t]$ dominates $[t]$.

2. If t is standard and $[s]$ has non-zero coefficient in e_t, then $[t]$ dominates $[s]$:
   By definition of e_t, $[s]$ appears in $e_t \Leftrightarrow \exists g \in C_t$ with $[s] = [gt]$.
   Apply induction on the number of transpositions in $g = \#$ pairs $k < l$, k in lower row than l.
   Take one such pair k, l, and put $g = (kl)h$, $h \in C_t$. $\therefore [ht] = (kl)[gt]$
   $(kl) \in C_t$, so $[ht]$ appears in e_t also
   $\therefore$ by inductive hypothesis, $[t]$ dominates $[ht]$
   by 1, $[ht]$ dominates $[gt] = [s]$.

3. $\{e_t : t \text{ a standard } \lambda\text{-tableau}\}$ is linearly independent:
   Suppose $\exists c_i \in \mathbb{C}$ with $\sum_{i=1}^{k} c_i e_{t_i} = 0$, c_i nonzero, t_i distinct standard tableaux.
   Order the t_i so that $[t_1]$ is maximal (ie no $[t_i]$ dominates it — observe that $[t_i]$ are
?? distinct since t_i are distinct.
   By 2, $[t_1]$ cannot appear in e_{t_i} for any $i > 1$. Since $[t]$ is a basis for M^{λ}, $[t_1]$ must have coefficient 0 in $\sum_{i=1}^{k} c_i e_{t_i} \Rightarrow c_1 = 0$, a contradiction.

4. we say two tableaux are column equivalent if each of their columns contain the same numbers. Then we can consider the equivalence class of column tabloids, and define a dominance order using column composition series (measuring how far the small numbers are to the left). If $s,t \in$ same column tabloid, then $e_s = \pm e_t$.

If A,B are disjoint subsets of $\{1,2,\cdots n\}$, set $S_A \times S_B$ to be the subset of S_n permuting these two sets separately. let T be any set of coset representatives for $S_A \times S_B \subseteq S_{A\cup B}$.
Set $g_{A,B} = \sum_{g\in T} \mathrm{sgn}(g) g$ (which depends on T)

5. If $A \subseteq$ elts in j^{th} column, $B \subseteq$ elts in $j+1^{th}$ column with $|A\cup B| >$ # elts in column j.
Then $g_{A,B} e_t = 0$ (for any choice of T):
by pigeonhole principle, $\exists a\in A, b\in B$ in the same row of t.
$(ab) \in S_{A\cup B} \Rightarrow S_{A\cup B}^{-}[t] = 0$ (step 1 of previous theorem)
the same thing happens with every gt where $g \in C_t$.
$\therefore S_{A\cup B}^{-} e_t = \sum_{g\in C_t} \mathrm{sgn}(g) S_{A\cup B}^{-}[gt] = 0$
$S_{A\cup B}^{-} = \sum_{g\in A\cup B} \mathrm{sgn}(g) g = \sum_{x\in T} \sum_{h\in S_A\times S_B} \mathrm{sgn}(x)\mathrm{sgn}(h) xh = g_{A,B}(S_A\times S_B)^{-}$
$S_A\times S_B \subseteq C_t \Rightarrow (S_A\times S_B)^{-} e_t = \sum_{g\in S_A\times S_B} \mathrm{sgn}(g) g \sum_{x\in C_t} \mathrm{sgn}(x) x[t]$
$= \sum_{g\in S_A\times S_B} \sum_{y\in C_t} \mathrm{sgn}(y) y[t]$ $(y = gt)$
$= |S_A\times S_B| e_t$
$\therefore g_{A,B} e_t = \frac{1}{|S_A\times S_B|} g_{A,B}(S_A\times S_B)^{-} e_t = \frac{1}{|S_A\times S_B|} S_{A\cup B}^{-} e_t = 0$

6. $\{e_t : t \text{ a standard } \lambda\text{-tableau}\}$ spans S^λ
Set $V =$ span of $\{e_t : t \text{ a standard } \lambda\text{-tableau}\}$
For contradiction, take $e_t \in S^\lambda \setminus V$, $[t]$ maximal with respect to column dominance order
By 4, all tableau s column-equivalent to t also have $e_s \notin V$
$\therefore$ wlog assume the columns of t are increasing.
t is not standard $\Rightarrow$ some row i is not in increasing order
$\Rightarrow \exists j$ such that $a_i \in$ row i, column j $> b_i \in$ row i, column $j+1$.
Denote by $(a_i), (b_i)$ the elements in column j and column $j+1$.
Apply 5 to $A = \{a_i, a_{i+1}, \cdots\}$, $B = \{b_1, b_2, \cdots b_i\}$ — $g_{A,B} e_t = 0$
So $e_t = -\sum_{g\in T\setminus\{1\}} \mathrm{sgn}(g) g e_t = -\sum_{g\in T\setminus\{1\}} \mathrm{sgn}(g) e_{gt}$ (T defined as in 5)
Choose the transversal T to consist of elements transposing elements of A with elements of B (with no transpositions within each set)
Each $a\in A$ is larger than each $b\in B$ $\therefore$ by the column-analogue of step 1, every such transposition makes $[t]$ larger $\Rightarrow \forall g\in T\setminus\{1\}$, $[gt]$ dominates t.
By maximality of t, this means $e_{gt} \in V \Rightarrow -\sum_{g\in T\setminus\{1\}} \mathrm{sgn}(g) e_{gt} \in V$, a contradiction.

Example: Work in S^4, $\lambda = (2,2)$
The tableaux with increasing columns are:

13	12	12	21	21	31
24	34	43	34	43	42

(for any s, e_s is always $\pm e_t$ where t has increasing columns)

The first two of these are standard. The proof above gives an algorithm for expressing the other 4 tableaux in terms of these:

The second row of $\substack{12\\43}$ is out-of-order $\therefore A=\{4\}, B=\{2,3\}$; $T=\{\iota,(24),(34)\}$

$$\Rightarrow e_{\substack{12\\43}} = -(24)e_{\substack{12\\43}} - (34)e_{\substack{12\\43}} = -e_{\substack{14\\23}} - e_{\substack{12\\34}} = e_{\substack{13\\24}} - e_{\substack{12\\34}}$$

$\substack{21\\43}$ has both rows out-of-order, so we apply the algorithm twice, starting with either row. let's start with the first $\therefore A=\{2,4\}, B=\{1\}$; $T=\{\iota,(12),(14)\}$

$$\Rightarrow e_{\substack{21\\43}} = -(12)e_{\substack{21\\43}} - (14)e_{\substack{21\\43}} = -e_{\substack{12\\43}} - e_{\substack{24\\13}}$$
$$= -e_{\substack{13\\24}} + e_{\substack{12\\34}} - e_{\substack{12\\34}} = -e_{\substack{13\\24}}$$

Given any partition λ of n, the conjugate partition λ' is obtained by transposing the Young diagram (formally, we set $\lambda'_i = |\{j : \lambda_j \geq i\}| = \#$ rows of λ with at least length i)

Clearly $(\lambda')' = \lambda$.

The hook number h_{ij} of the i,j^{th} entry of a Young diagram is the # entries directly to the right and directly below this entry, including the entry itself.

(formally, $h_{ij} = (\lambda_i - j) + (\lambda'_j - i) + 1$)

The hook length formula says $\dim S^\lambda = \dfrac{n!}{\prod h_{ij}}$ producting over all entries i,j.

This is highly non-trivial — the most elegant proof is probabilistic.

let R_t be the row stabiliser of a tableau t (ie consisting of permutations within rows only)

Fix a λ-tableau t_0 and set $c_\lambda = C^-_{t_0} R^+_{t_0}$, a Young symmetriser.

Now we show $\mathbb{C}S_n c_\lambda \cong S^\lambda$ as representations ($\mathbb{C}S_n c_\lambda$ is left regular action)

We define $f: M^\lambda \longrightarrow (\mathbb{C}S_n)R^+_{t_0}$, $f([gt_0]) = gR^+_{t_0}$, and show this is a $\mathbb{C}S_n$-isomorphism

1. f is well-defined: if $[g_1 t_0] = [g_2 t_0]$, then $g_1^{-1}g_2 \in R_{t_0}$ (by definition of $[\ ]$)
   $\Rightarrow g_1^{-1}g_2 R^+_{t_0} = R^+_{t_0} \Rightarrow g_1 R^+_{t_0} = g_2 R^+_{t_0}$

2. f is $\mathbb{C}S_n$-homomorphism:
   $\forall h \in S_n$, $f(h[gt_0]) = f([hgt_0]) = hgR^+_{t_0} = h f([gt_0])$

3. $\{\sum_{h:[h]=[t]} h : [t]$ a λ-tabloid$\}$ is a basis for $(\mathbb{C}S_n)R^+_{t_0}$:
   as $g \in R_{t_0}$ sends t to gt with $[t]=[gt]$, and no non-trivial $g \in S_n$ fixes a tableau.
   S_n-action is transitive on tabloids $\therefore$ these vectors $\in (\mathbb{C}S_n)R^+_{t_0}$ for all tabloids $[t]$
   Each sum involves different basis vectors of $\mathbb{C}S_n$, so clearly linearly independent

4. f is bijective:
   $[t]$ is a basis of M^λ; and any basis element of $(\mathbb{C}S_n)R^+_{t_0}$ has the form $f([t])$ for any t appearing in the sum $\therefore$ surjective.
   if $f([t]) = f([s])$, then t appears in $\sum_{h:[h]=[t]} h = \sum_{h:[h]=[s]} h \Rightarrow [t]=[s]$ $\therefore$ injective

5. The image of S^λ under f is $\mathbb{C}S_n c_\lambda$:
   the image is spanned by $f(e_{gt_0}) = f(ge_{t_0}) = f(g C^-_{t_0}[t_0]) = g C^-_{t_0} R^+_{t_0} = gc_\lambda$

as g ranges over S_n, we get all of $\mathbb{C}S_n c_\lambda$.
So $S^\lambda \cong \mathbb{C}S_n c_\lambda$ (as the restriction of f remains injective, and 5 forces surjectivity)

Theorem: for any partition λ, $S^\lambda \otimes$ sign representation $\cong S^{\lambda'}$

1. Define linear maps: $\alpha: \mathbb{C}S_n \to \mathbb{C}S_n$, $\alpha(g) = \text{sgn}(g)g$
$\sigma: \mathbb{C}S_n \otimes \text{sgn} \to \mathbb{C}S_n$, $\sigma(g \otimes 1) = \text{sgn}(g)g$ (and extend linearly)
α is an algebra homomorphism, σ is a $\mathbb{C}S_n$ module homomorphism:
$\alpha(gh) = \text{sgn}(gh)gh = \text{sgn}(g)\text{sgn}(h)gh = \alpha(g)\alpha(h)$
$\sigma(h(g\otimes 1)) = \sigma(hg \otimes \text{sgn}(h)) = \sigma(\text{sgn}(h)hg \otimes 1) = \text{sgn}(hg)\,\text{sgn}(h)hg = h\text{sgn}(g)(g) = h\sigma(g\otimes 1)$

2. Let s_0 be the transpose of t_0 ie s_0 a λ'-tableau.
Then $\alpha(C_{t_0}^-) = R_{s_0}^+$, $\alpha(R_{t_0}^+) = C_{s_0}^-$:
for any subgroup H of S_n, $\alpha(H^+) = \sum_{h\in H}\alpha(h) = \sum_{h\in H}\text{sgn}(h)h = H^-$, and $\alpha(H^-) = H^+$.
also, since columns of t_0 are rows of s_0 (and vice versa), $C_{t_0} = R_{s_0}$, $R_{t_0} = C_{s_0}$

3. σ is an isomorphism: $\mathbb{C}S_n c_\lambda \otimes \text{sgn} \to \mathbb{C}S_n R_{s_0}^+ C_{s_0}^-$ (as $\mathbb{C}S_n$-modules):
all eigenvalues of σ are ± 1 $\therefore$ σ injective $\Rightarrow$ σ an isomorphism to its image.
$\sigma(gc_\lambda \otimes 1) = \sigma(g(c_\lambda \otimes \text{sgn}g)) = \sigma(g(\text{sgn}g\, c_\lambda \otimes 1)) = \text{sgn}(g)\, g\,\sigma(c_\lambda \otimes 1)$
$= \text{sgn}(g) g\, \alpha(c_\lambda)$
$= \text{sgn}(g)\, g R_{s_0}^+ C_{s_0}^-$ as α is alg hom, and by 2.
$\therefore$ as g ranges over S_n, we get all of $\mathbb{C}S_n R_{s_0}^+ C_{s_0}^-$.

4. Define $f: \mathbb{C}S_n \to \mathbb{C}S_n R_{s_0}^+$, $f(g) = gR_{s_0}^+$
f is a non-zero $\mathbb{C}S_n$-homomorphism: $\mathbb{C}S_n R_{s_0}^+ C_{s_0}^- \to \mathbb{C}S_n c_{\lambda'}$
f clearly a $\mathbb{C}S_n$-homomorphism, and $f(\mathbb{C}S_n R_{s_0}^+ C_{s_0}^-) \subseteq \mathbb{C}S_n R_{s_0}^+ C_{s_0}^- R_{s_0}^+ \subseteq \mathbb{C}S_n c_{\lambda'}$
$C_{s_0} \cap R_{s_0} = \{$permutations fixing each row and each column setwise$\} = \{1\}$
$\therefore$ if $x_1 y x_2 = 1$ with $x_i \in R_{s_0}$, $y \in C_{s_0}$, then $y = x_1^{-1}x_2 \Rightarrow x_1 = x_2$, $y = 1$.
so coefficient of 1 in $f(R_{s_0}^+ C_{s_0}^-) =$ coefficient of 1 in $R_{s_0}^+ C_{s_0}^- R_{s_0}^+ = |R_{s_0}^+| > 0$
$\therefore$ f is non-zero.
So $f: S^\lambda \otimes$ sign representation $\to S^{\lambda'}$. Since these are both simple modules, and f is non-zero, by Schur f must be an isomorphism.

Let χ^λ be the character afforded by S^λ. Now we describe all irreducible characters of A_n in terms of those for S_n

1. $(\chi^\lambda)_{A_n}$ is irreducible if $\lambda \neq \lambda'$
$(\chi^\lambda)_{A_n} = \theta_1^\lambda + \theta_2^\lambda$ distinct conjugates if $\lambda = \lambda'$ (self-transposing):
since $|S_n : A_n| = 2$ is prime, these are the only possible scenarios. Take θ a constituent of $(\chi^\lambda)_{A_n}$.
if $\lambda \neq \lambda'$, then $\chi^\lambda, \chi^{\lambda'}$ are both constituents of θ^{S_n} (since $(\chi^\lambda)_{A_n} = (\chi^{\lambda'})_{A_n}$)
so $2\theta(1) = \theta^{S_n}(1) \geq (\chi^\lambda + \chi^{\lambda'})(1)$. Also, $\theta(1) \leq \chi^\lambda(1) = \chi^{\lambda'}(1)$
$\therefore$ we must have equality ie $(\chi^\lambda)_{A_n} = \theta$ is irreducible.
Conversely, if $(\chi^\lambda)_{A_n}$ is irreducible, then $(\chi^\lambda)_{A_n}$ is extendible to an irreducible character

$\therefore$ by second Clifford correspondence, sign character $\neq 1_{S_n} \Rightarrow \chi^{\lambda'} \neq \chi^{\lambda}$ (by previous result)
$\therefore \lambda \neq \lambda'$.

2. The irreducible characters of A_n are precisely $\{(\chi^\lambda)_{A_n} : \lambda \neq \lambda'\} \cup \{\theta_1^\lambda, \theta_2^\lambda : \lambda = \lambda'\}$:
Since every irreducible character of A_n occurs as a constituent of some χ^λ, this list must exhaust all possibilities.
$(\theta_i^\lambda)^{S_n} = \chi^\lambda$ (since χ^λ lies over $\theta_i^\lambda \Rightarrow \chi^\lambda$ is constituent of $(\theta_i^\lambda)^{S_n}$, and by dimension)
and we saw above that $({\chi^\lambda}_{A_n})^{S_n} = \chi^\lambda + \chi^{\lambda'}$ if $\lambda \neq \lambda'$
$\therefore$ if we take one representative per pair $\lambda \neq \lambda'$, this list has no redundancy.

1. (i) Suppose $\mathcal{K}$ is a conjugacy class of S_n contained in A_n; then $\mathcal{K}$ is called *split* if $\mathcal{K}$ is a union of two conjugacy classes of A_n. Show that the number of split conjugacy classes contained in A_n is equal to the number of characters $\chi \in \mathrm{Irr}(S_n)$ such that χ_{A_n} is not irreducible. (*Hint.* Consider the vector space of class functions on A_n which are invariant under conjugation by the transposition (12).)

   (ii) Let $g \in A_n$ have a cyclic decomposition with cycle lengths
$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0.$$
Show that the conjugacy class of g in S_n is split if and only if the numbers μ_i are all distinct and odd. Deduce that the number of partitions λ of n such that $\lambda = \lambda'$ is equal to the number of partitions $(\mu_1, \ldots, \mu_k)$ of n with all parts μ_i distinct and odd.

   (iii)* Find an explicit combinatorial one-to-one correspondence between the set of self-conjugate partitions of n and the set of partitions of n with all parts distinct and odd.

i. Let $g \in A_n$, and let $\mathcal{K}$ denote the conjugacy class of g in S_n

$\bar{\mathcal{K}}$ denote the conjugacy class of g in A_n $\therefore \bar{\mathcal{K}} \subseteq \mathcal{K}$

If $C_{S_n}(g) \not\subseteq A_n$, then multiplication by any fixed $x \in C_{S_n}(g)$ is a bijection from the even elements of $C_{S_n}(g)$ to the odd elements

$\therefore |C_{A_n}(g)| = |C_{S_n}(g) \cap A_n| = \frac{1}{2}|C_{S_n}(g)|$

Then $|\bar{\mathcal{K}}| = |A_n| / |C_{A_n}(g)| = |S_n| / |C_{S_n}(g)| = |\mathcal{K}|$ $\therefore$ in this case $\mathcal{K}$ doesn't split.

otherwise, $C_{S_n}(g) \subseteq A_n \Rightarrow C_{S_n}(g) = C_{A_n}(g)$

$\Rightarrow |\bar{\mathcal{K}}| = |A_n| / |C_{A_n}(g)| = |S_n| / 2|C_{S_n}(g)| = \frac{1}{2}|\mathcal{K}|$ in this case $\mathcal{K}$ splits.

$\therefore$ Indicator class functions are S_n-invariant $\Leftrightarrow \mathcal{K}$ doesn't split

sum of indicator class functions for the 2 conjugacy classes split from $\mathcal{K}$ is S_n-invariant $\therefore$ difference of these is S_n-anti-invariant.

or view class fns in $\mathbb{C}G$

These three categories form a basis of eigenvectors of the action of conjugation-by-any-odd-element on the space of A_n class functions

ie # of -1 evalues = # of pairs of split conjugacy classes.

Similarly, S_n-invariant characters $\cup$ sum of pairs of non-S_n-invariant characters is a basis for the $+1$ eigenspace, difference of pairs of non-S_n-invariant characters is a basis for the -1 eigenspace (irreducibles)

$\therefore$ # of -1 evalues = # of pairs of non-S_n-invariant irreducible characters.

ii Let $g = g_1 g_2 \cdots g_k \in A_n$, each g_i a cycle.

Suppose first that g_i have odd distinct lengths, and take $x \in C_{S_n}(g)$, ie $x^{-1} g x = g$.

Since cycle decomposition is unique, and the cycles have distinct lengths, we must have $x^{-1} g_i x = g_i$ $\forall i$.

$\therefore$ if $g_i = (i_1 i_2 \cdots i_r)$, then $x^{-1} g_i x = (i_m i_{m+1} \cdots i_r i_1 \cdots i_{m-1})$ (cycle decomposition unique)

$r-1$ transpositions are needed to move the i_js down one place

$\therefore$ above transformation possible with $(m-1)(r-1)$ transpositions

(we assume x do not affect elements outside $i_1 i_2 \cdots i_r$, since we can write x as a product of permutations each affecting elements of one g_i)

This is an even number (as r odd) $\therefore$ total number of transpositions needed is even

$\therefore x \in A_n \Rightarrow$ split conjugacy class.

g_i itself is odd and commutes with g.

If $\exists g_i$ with even length, then by above an odd number of transpositions will shift this cycle by one place $\Rightarrow$ this is an odd permutation $\in C_{S_n}(g) \Rightarrow$ no split.

If all g_i have odd length, but g_i, g_j have same length, then x can swap g_i and g_j, which is an odd permutation (as g_i, g_j have odd length) $\Rightarrow$ no split

iii Given a self conjugate partition, μ_i = # of squares on row i, column $\geq i$ and on column i, row $\geq i$ (counting the i,i^{th} square once)

self conjugate $\Rightarrow \mu_i$ odd

partition $\Rightarrow \mu_i$ distinct