Representations of GL_n/U_n

Let $G = GL_n(\mathbb{C})$, $U \leq G$ be the unitary matrices, $T \leq U$, $T_c \leq G$ be diagonal
$N \leq G$ be the unipotent upper-triangular matrices
$N^T \leq G$ be the unipotent lower-triangular matrices
$B^T \leq G$ be the lower-triangular matrices

· We have equivalence of categories between: (all reps here are finite dimensional)
algebraic reps of G $\longleftrightarrow$ holomorphic reps of G $\longleftrightarrow$ continuous reps of U
where algebraic = matrix entries are polynomial in entries of g and $(\det g)^{-1}$
holomorphic = as $\mathbb{C}$-mfld maps = matrix entries are holomorphic in entries of g
Equivalence means "being a subrepresentation" is "preserved" between maps of the categories ie irreducibles correspond to irreducibles.
Also, direct sums are preserved $\Rightarrow$ algebraic reps of GL_n are completely reducible
(correspondence allows bootstrapping from compact group theory on U_n)

From now on, all reps considered are assumed to be in the appropriate category above.

Let V be an irrep of G (or U).
$\underline{k} = (k_1, k_2, \dots k_n) \in \mathbb{Z}^n$ is a weight of V if the corresponding weight space
$V_{\underline{k}} = \{ v \in V : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} v = t_1^{k_1} \cdots t_n^{k_n} v \} \neq 0$
The highest weight has maximal length $(\sqrt{\sum k_i^2})$ and $k_1 \geq k_2 \geq \dots \geq k_n$.
There is a unique such in every irrep, and the corresponding weight space is 1-dimensional. This highest weight determines the irrep uniquely (up to isomorphism).

In a G-irrep, the highest weight space can also be characterised as the line fixed pointwise by N.

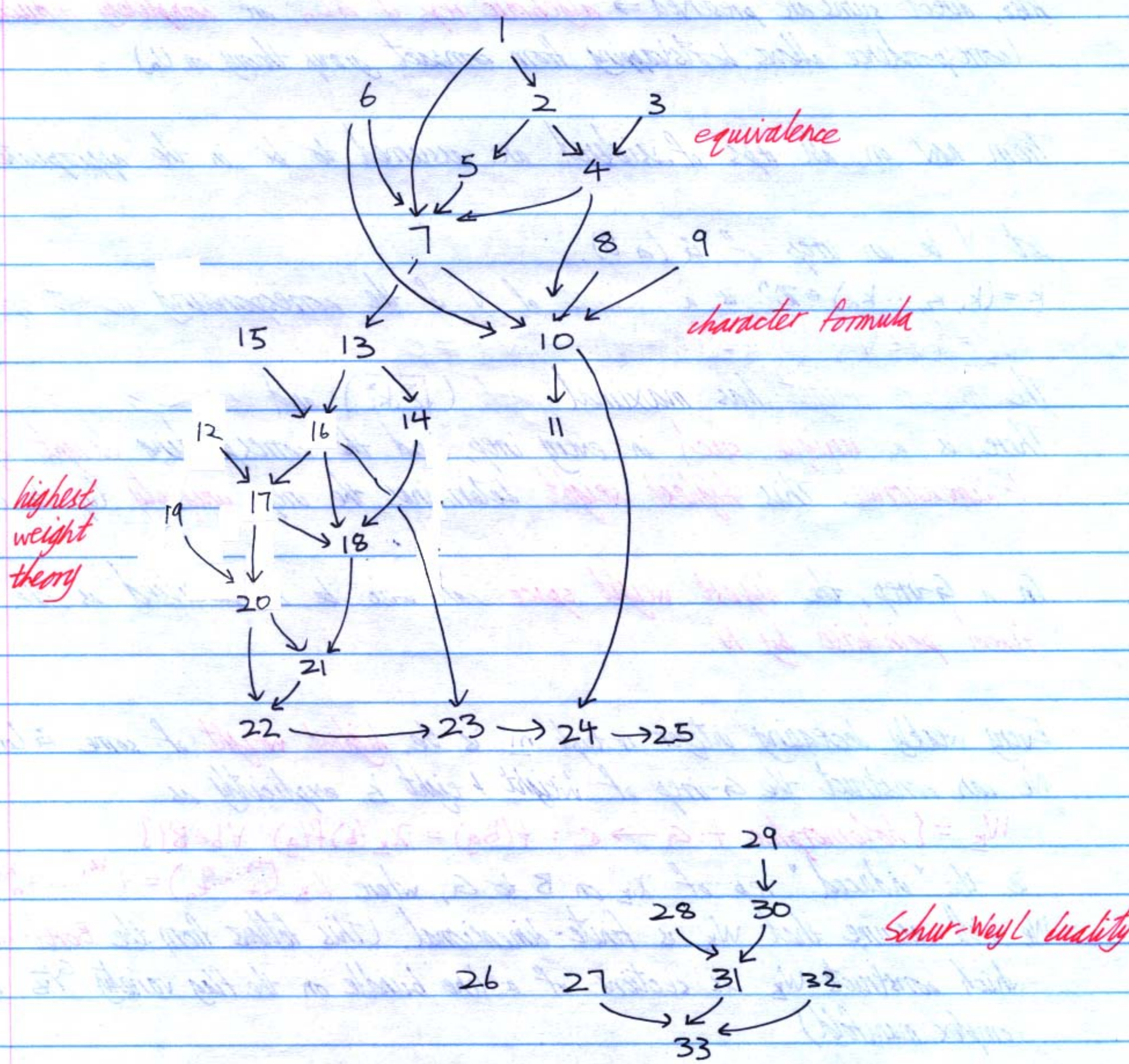
Every weakly decreasing integer n-tuple $\underline{k}$ is the highest weight of some G (or U)-irrep.
We can construct the G-irrep of highest weight $\underline{k}$ explicitly as
$W_{\underline{k}} = \{ \text{holomorphic } f : G \to \mathbb{C} : f(bg) = \chi_{\underline{k}}(b) f(g) \ \forall b \in B^T \}$
ie the "induced" rep of $\chi_{\underline{k}}$ on B to G, where $\chi_{\underline{k}} \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ * & & t_n \end{pmatrix} = t_1^{k_1} \cdots t_n^{k_n}$
We will assume that $W_{\underline{k}}$ is finite-dimensional. (This follows from the Borel-Weil theorem, which constructs $W_{\underline{k}}$ as sections of a line bundle on the flag variety G/B, a compact complex manifold)

The *character* of $W_{\underline{k}}$ on $\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}$ is given by the *Schur polynomial* $S_{\underline{k}+(n-1,n-2,\dots,1,0)}$. That is, $\chi_{W_{\underline{k}}}\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} = \dfrac{\det(t^{k_i+n-i})}{\det(t_j^{n-i})}$

The *dimension* of $W_{\underline{k}}$ is then $\displaystyle\prod_{i<j} \frac{k_i-k_j+j-i}{j-i}$

Schur Weyl duality: the *irrep* of S_l (the symmetric group) corresponding to the *partition* $\underline{k}$ is $\mathrm{Hom}_{GL_n(\mathbb{C})}(W_{\underline{k}}, (\mathbb{C}^n)^{\otimes l})$

Dependence chart of the following 33 sections:



1. The tangent space $T_e G$ is the complexification $T_e U \otimes \mathbb{C}$

$T_e G$ = all $n \times n$ matrices

$T_e U = \{X : (I + X\varepsilon)(I + \bar{X}\varepsilon)^T = I\} = \{X : X + \bar{X}^T = 0\}$

= real symmetric matrices $\oplus$ purely imaginary antisymmetric matrices

So $T_e U \otimes_{\mathbb{R}} \mathbb{C}$ = complex symmetric matrices $\oplus$ complex antisymmetric matrices $= T_e G$.

2. Holomorphic functions : $G \to ?$ that are constant on U are constant on G.

Using the exponential map and 1, we have a co-ordinate chart on a neighbourhood of $e \in G$ such that this neighbourhood $\cap U$ is the points with real co-ordinates.

Consider the given holomorphic function f as f: unit cube of $\mathbb{C}^n \to ?$, which is constant on all points with real co-ordinates (we can always scale our chart so it contains the unit cube) let $I_{\mathbb{C}}$ denote the unit cube of $\mathbb{C}$, I the interval $[-1,1]$.

Fix any real $z_1, z_2, \cdots z_{n-1}$, and consider $f(z_1, z_2, \cdots z_{n-1}, -) : I_{\mathbb{C}} \to ?$ This is constant $\forall z_n \in I$

$\therefore$ by 1D complex analysis, this is constant $\forall z_n \in I_{\mathbb{C}}$ (the same constant regardless of $z_1, \cdots z_{n-1}$)

Now fix $z_1, z_2, \cdots z_{n-2} \in I$, $z_n \in I_{\mathbb{C}}$, and look at $f(z_1, z_2, \cdots z_{n-2}, -, z_n) : I_{\mathbb{C}} \to ?$ By above, this is constant $\forall z_{n-1} \in I$ $\therefore$ constant $\forall z_{n-1} \in I_{\mathbb{C}}$

Repeat with $z_1, z_2, \cdots z_{n-3} \in I$, $z_{n-1}, z_n \in I_{\mathbb{C}}$, and so on. Finally, we conclude that $f(-, z_2, z_3, \cdots z_n)$ is constant for all fixed $z_2, z_3, \cdots z_n \in I_{\mathbb{C}} \Rightarrow f$ constant on $I_{\mathbb{C}}^n$.

So we have f constant on an open neighbourhood of G. The set where f agrees with this constant function is both open and closed $\therefore$ by connectedness of G, this set must be all of G (this is the standard argument regarding uniqueness of analytic continuations).

3 Every irreducible representation of U occurs as a constituent of $(\mathbb{C}^n)^{\otimes k} \otimes (\mathbb{C}^{n*})^{\otimes l}$

let $e_1, e_2, \cdots e_n$ be a basis of $\mathbb{C}^n$, $f_1, f_2, \ldots f_n$ the dual basis in $\mathbb{C}^{n*}$

$\therefore$ A basis of $(\mathbb{C}^n)^{\otimes k} \otimes (\mathbb{C}^{n*})^{\otimes l}$ is $e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes f_{j_1} \otimes \cdots \otimes f_{j_l}$ $(i_1, \cdots i_k, j_1, \cdots j_l \in \{1, \cdots n\})$

$\therefore$ the matrix coefficients of $g = (g_{ij})$ acting on $(\mathbb{C}^n)^{\otimes k} \otimes (\mathbb{C}^{n*})^{\otimes l}$ are

$$g_{i_1' i_1} g_{i_2' i_2} \cdots g_{i_k' i_k} (g^{-1})_{j_1' j_1} (g^{-1})_{j_2' j_2} \cdots (g^{-1})_{j_l' j_l} = g_{i_1' i_1} \cdots g_{i_k' i_k} \overline{g}_{j_1 j_1'} \cdots \overline{g}_{j_l j_l'}$$

for all choices of $i_1, \cdots i_k, i_1', \cdots i_k', j_1', \cdots j_l', j_1 \cdots j_l$ (equality above is because $g \in U \Rightarrow g^{-1} = \overline{g^T}$)

The $\mathbb{C}$-span of the matrix coefficients, across all k and l (ie polynomials in the entries of g and $\bar{g}$) are closed under addition, multiplication, complex conjugation and is point-separating, so, by Stone-Weierstrauss, it is dense in functions $U \to \mathbb{C}$ (since U is compact). This set is therefore also dense in $L^2(U)$.

By Schur orthogonality, matrix coefficients corresponding to distinct irreps are orthogonal $\therefore$ if an irrep is not contained in any $(\mathbb{C}^n)^{\otimes k} \otimes (\mathbb{C}^{n*})^{\otimes l}$, then its matrix coefficient is orthogonal to all matrix coefficients of $(\mathbb{C}^n)^{\otimes k} \otimes (\mathbb{C}^{n*})^{\otimes l}$ — but this cannot occur as they span a dense set.

4 Every U-irrep extends to an algebraic G-rep ($\therefore$ to a holomorphic G-rep also)
Since every U-rep is completely reducible, applying the process below to each constituent will extend any U-rep. So the restriction maps $\{G\text{-reps}\} \to \{U\text{-reps}\}$ are object preserving.
By 3, every U-irrep occurs in $(\mathbb{C}^n)^{\otimes k} \otimes (\mathbb{C}^{n*})^{\otimes l}$ - let the irrep be P.
so, $\forall g \in U, v \in P, w \in P^\perp$ (using any inner product, not necessarily interacting with U)
$\langle gv, w \rangle = 0$ ($\langle -, w \rangle = 0$ is how we detect "$\in P$")
By 2, this means $\langle gv, w \rangle = 0$ $\forall g \in G$ (since G-action is defined on $\mathbb{C}^{n \otimes k} \otimes \mathbb{C}^{n* \otimes l}$ and G-action on $\mathbb{C}^{n \otimes k} \otimes \mathbb{C}^{n* \otimes l}$ is algebraic $\therefore$ G-action on P is algebraic (as co-ordinate-changing is an algebraic map)

5 If V, V' are holomorphic G-reps and $f: V \to V'$ a U-rep map, then f is a G-rep map (because algebraic reps are holomorphic, we can replace holomorphic above by algebraic)
f is a U-rep homomorphism $\Rightarrow$ $g_{v'} f g_v^{-1} = f$ $\forall g \in U$.
$g_{v'} \cdot f \cdot g_v^{-1}$ is holomorphic in g (since multiplication of holomorphic functions is holomorphic) $\therefore$ by 1, $g_{v'} \cdot f \cdot g_v^{-1} = f$ $\forall g \in G$ so morphisms are also preserved.

6 All irreps of $(S^1)^m$ have the form $(z_1, z_2, \cdots z_m) \to z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m}$
we know that irreps of S^1 are $z \to z^k$
$\therefore$ since Haar measure on $(S^1)^m$ is the product measure, $(z_1, z_2, \cdots z_m) \to z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m}$ are indeed orthonormal.
To show they form a basis of class functions: suppose $\int f(\theta_1, \cdots \theta_m) e^{ik_1\theta_1} \cdots e^{ik_m\theta_m} \, d\theta_1 \cdots d\theta_m = 0$ $\forall k_1, \cdots k_m$. So, for any fixed $k_1, k_2, \cdots k_{m-1}$, $\int f(\theta_1, \cdots \theta_m) e^{ik_1\theta_1} \cdots e^{ik_{m-1}\theta_{m-1}} \, d\theta_1 \cdots d\theta_{m-1}$ has zero Fourier coefficients in θ_m $\Rightarrow$ it is the zero function. Continuing, we see $f = 0$.

7 All algebraic / holomorphic irreps of $(\mathbb{C}^\times)^m$ have the form $(z_1, z_2 \cdots z_m) \to z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m}$ because the analogue of 1-5 applies.
$T_e(\mathbb{C}^\times)^m = \mathbb{C}^m$, and $T_e(S^1)^m = \mathbb{R}^m$ (i-direction in each factor) - so we have 1.
Clearly $(z_1, z_2, \cdots z_m) \to z_1^{k_1} \cdots z_m^{k_m}$ is an algebraic extension of the corresponding $(S^1)^m$-rep
Now 5 implies that morphisms are preserved.
In particular, this means reps of $T_{\mathbb{C}}$ and reps of T correspond.

8. The conjugacy classes of U are labelled by unordered n-tuples with entries in S^1, and elements of each conjugacy class are matrices with those eigenvalues.
Every element of U has an orthonormal eigenbasis (spectral theorem), so conjugating by the matrix of these eigenvectors results in a diagonal matrix. By orthonormality of the eigenvectors, the matrix we conjugated with is in U. ie every $g \in U$ is conjugate in U to a diagonal matrix of its eigenvalues.
If two diagonal matrices are conjugate, they must have the same entries ordered differently (as conjugate matrices have the same eigenvalues). Conversely, two diagonal matrices with the same entries are conjugate via a permutation matrix (which is unitary, as it has a 1 in each row and column and 0s elsewhere).

9. Weyl integration formula: for class functions f, g on U:
$\langle f, g \rangle = \int_U \bar{f} g \, du = \frac{1}{n!} \int_T \bar{f} g \, |D|^2 \, dt$
where T has the product measure of S^1's, scaled so $\int_{S^1} 1 \, d\theta = 1$,
and $D = \prod_{i<j} |t_i - t_j|$
(I may or may not prove this later)

10 Calculation of all irreducible characters of G or U

Let χ be any irreducible U-character. By 8, it suffices to determine χ as a character on $(S^1)^m$ $\Rightarrow \chi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix}$ = a Laurent polynomial in $t_1, t_2, \cdots t_n$ (by 6)

$\therefore \chi D\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix}$ is also a Laurent polynomial in $t_1, t_2, \cdots t_n$

$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix}$ is conjugate to $\begin{pmatrix} t_{\sigma(1)} & & \\ & \ddots & \\ & & t_{\sigma(n)}\end{pmatrix}$ for all permutations σ (by 8) and $D\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix}$ is $\operatorname{sgn}\sigma\, D\begin{pmatrix} t_{\sigma(1)} & & \\ & \ddots & \\ & & t_{\sigma(n)}\end{pmatrix} \Rightarrow \chi D\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix} = \operatorname{sgn}\sigma\, \chi D\begin{pmatrix} t_{\sigma(1)} & & \\ & \ddots & \\ & & t_{\sigma(n)}\end{pmatrix}$. In other words, the coefficient of $t_1^{k_1}\cdots t_n^{k_n}$ is $\operatorname{sgn}\sigma \times$(coefficient of $t_{\sigma(1)}^{k_1}\cdots t_{\sigma(n)}^{k_n}$). In particular, if $k_i = k_j$ for $i \neq j$, then $t_1^{k_1}\cdots t_n^{k_n}$ does not appear in χD (coefficient is zero)

Weyl integration formula (9) gives $1 = \langle \chi, \chi \rangle = \frac{1}{n!}\int_T |\chi D|^2 \, dt$

By 6, the Laurent polynomials are orthogonal under $\int_T dt$, so $1 = \frac{1}{n!}\sum |\text{coefficient}|^2$

By the antisymmetry, each coefficient occurs $n!$ times with same or opposite sign

$\therefore \chi D$ contains only $n!$ terms which are all related by permutation, and their coefficients are 1 or -1, ie $\chi D\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix} = \pm \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, t_{\sigma(1)}^{\lambda_1} t_{\sigma(2)}^{\lambda_2} \cdots t_{\sigma(n)}^{\lambda_n}$, and we can choose $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.

ie up to sign, $\chi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix} = \dfrac{\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, t_{\sigma(1)}^{\lambda_1} t_{\sigma(2)}^{\lambda_2}\cdots t_{\sigma(n)}^{\lambda_n}}{\prod_{i<j}(t_i - t_j)} = \dfrac{\det(t_j^{\lambda_i})}{\det(t_j^{i-1})}$

To determine the sign, we calculate $\chi(1)$ and make it positive (see 11)

This shows that all characters must have the form $\pm\frac{\det(t_j^{\lambda_i})}{\det(t_j^{i-1})}$. All λ_i's must give valid characters: if not, then $\frac{\det(t_j^{\lambda_i})}{\det(t_j^{i-1})}$ is a class function orthogonal to all characters, which can't happen.

To find the character of a G-irrep V: we know T acts on V with character $\frac{\det(t_j^{\lambda_i})}{\det(t_j^{i-1})}$, and by 7, T_c-action on V has the same character. By continuity, this completely determines the character of the G-action. $\frac{\det(t_j^{\lambda_i})}{\det(t_j^{i-1})}$ is a character of some G-irrep for every $\underline{\lambda}$ by equivalence of G and U-irreps.

11 Weyl dimension formula

We apply L'hopital's rule to take the limit of $\frac{\det(t_j^{\lambda_i})}{\det(t_j^{i-1})}$ as $t_j \to 1\ \forall j$.

We apply $\dfrac{\partial^{n-1}}{\partial t_1^{n-1}}\dfrac{\partial^{n-2}}{\partial t_2^{n-2}}\cdots\dfrac{\partial}{\partial t_n}$ to the numerator and denominator (no fewer will do as the denominator has $(n-1)!$ factors which are all zero when $t_j = 1$)

View t_j's as fixed, and λ_i's as the variables in $\det(t_j^{\lambda_i})$.

Each time we take a partial derivative, we gain a linear factor of some λ_i

$\therefore \frac{\partial^{n-1}}{\partial t_1^{n-1}}\cdots\frac{\partial}{\partial t_n}\det(t_j^{\lambda_i})$ at $t_j = 1\ \forall j$ is a polynomial of degree $\frac{n(n-1)}{2}$ in the λ_i's.

$\det(t_j^{\lambda_i}) = 0$ whenever two of the λ_i's are equal, and in this case $\frac{\partial^{n-1}}{\partial t_1^{n-1}}\cdots\frac{\partial}{\partial t_n}\det(t_j^{\lambda_i})$ at $t_j = 1\ \forall j$ is zero too $\therefore \lambda_i - \lambda_j$ divides the polynomial in question, $\forall i \neq j$. This produces $\frac{n(n-1)}{2}$ linear factors $\Rightarrow$ product of these factors is,

up to a constant, the required polynomial.
To find the constant, explicitly calculate $\frac{\partial^{n-1}}{\partial t_1^{n-1}} \cdots \frac{\partial}{\partial t_n} \det(t_j^{\,i-1})$ at $t=1$, (ie substitute in $\lambda_i = i-1$, as in the denominator) and see that this equals $(n-1)!(n-2)!\cdots 2!1! = \prod_{i<j} ((j-1)-(i-1))$.
$\therefore \frac{\partial^{n-1}}{\partial t_1^{n-1}} \cdots \frac{\partial}{\partial t_n} \det(t_j^{\lambda_i}) = \prod_{i<j} (\lambda_j - \lambda_i)$ which has sign $(-1)^{\frac{n(n-1)}{2}}$ – ie this is the sign in the character formula, and the dimension is $\dfrac{\prod_{i<j}(\lambda_i-\lambda_j)}{\prod_{i<j}(j-i)}$.

12. The weights of any G-rep or U-rep is invariant under S_n action, so a highest weight always exists.
Let $(k_1, k_2, \cdots k_n)$ be a weight of V
$\Rightarrow$ there is a nonzero $v \in V$ such that $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} v = t_1^{k_1} \cdots t_n^{k_n} v$.
For any $\sigma \in S_n$, let g_σ be the associated permutation matrix
ie $g_\sigma^{-1} \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} g_\sigma = \begin{pmatrix} t_{\sigma(1)} & & \\ & \ddots & \\ & & t_{\sigma(n)} \end{pmatrix}$.
(right-multiplication by g_σ sends t_i to entry $i, \sigma^{-1}(i)$,
left-multiplication by g_σ^T sends entry $i, \sigma^{-1}(i)$ to entry $\sigma^{-1}(i), \sigma^{-1}(i)$.
g_σ is an orthogonal matrix, so $g_\sigma^{-1} = g_\sigma^T$)
Then $t_1^{\sigma^{-1}(k_1)} \cdots t_n^{\sigma^{-1}(k_n)} v = t_{\sigma(1)}^{k_1} \cdots t_{\sigma(n)}^{k_n} v = \begin{pmatrix} t_{\sigma(1)} & & \\ & \ddots & \\ & & t_{\sigma(n)} \end{pmatrix} v = g_\sigma^{-1} \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} g_\sigma v$
$\Rightarrow g_\sigma v \in$ the $(\sigma^{-1}k_1, \cdots \sigma^{-1}k_n)$ weight space ie $\sigma^{-1}(\underline{k})$ is also a weight.
So after finding a weight of maximal length, reorder the components so they are weakly decreasing. This reordered vector is also a weight, of the same length $\therefore$ a highest weight.

13. Any G or U representation is the direct sum of its weight spaces because T-representations are completely reducible (T is compact) and the weight spaces are precisely the (isotypical sum of) irreducible components (this is clear for U; for G, use 7, that restriction-to-T is an equivalence of $T_{\mathbb{C}}$ and T maps).

14. If $\underline{k}$ is a weight of V, $-\underline{k}$ is a weight of the dual V^*
Take a basis v_i of weight vectors of V (possible by 13) $v_i \in V_{\underline{k}(v_i)}$, and let v_i^* be the dual basis in V^*.
Then $\left[\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} v_i^*\right](v_j) = v_i^*\left[\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}^{-1} v_j\right] = v_i^*(t_1^{-k(v_j)_1} \cdots t_n^{-k(v_j)_n} v_j)$
$\Rightarrow \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} v_i^* = t_1^{-k(v_i)_1} \cdots t_n^{-k(v_i)_n} v_i^*$ ie $v_i^* \in V^*_{-k(v_i)}$ ie $-k(v_i)$ is a weight of V^*.

15. Fix any $a_i \in \mathbb{Z}$, with $a_1 > a_2 > \dots > a_n$. For $x \in \mathbb{C}^*$, set $t_x = \begin{pmatrix} x^{a_1} & & \\ & \ddots & \\ & & x^{a_n}\end{pmatrix} \in T_{\mathbb{C}}$

Then $t_x \cdot h \cdot t_x^{-1} \to e$ as $x \to 0$, $\forall h \in N$.

Write h_{ij} for the entries of h. ($h \in N$, so $h_{ij} = 0 \ \forall i > j$)

Right multiplication by t_x^{-1} scales column j by x^{-a_j}.

Left multiplication by t_x scales row i by x^{a_i}.

$\therefore$ entries of $t_x h t_x^{-1}$ are $h_{ij} x^{a_i - a_j}$ – 0s below the diagonal, unchanged on the diagonal.

As $a_i > a_j$ for $i < j$, $x^{a_i - a_j} \to 0$ as $x \to 0$

16. Action of N raises weights (V not necessarily irreducible)

Let $\underline{k}$ be any weight and choose $v \in V_{\underline{k}}$. By 15, $t_x h t_x^{-1} v \to v$ as $x \to 0$, $\forall h \in N$.

So $t_x h (x^{-a_1k_1 - a_2k_2 - \dots - a_nk_n}) v \to v$ as $x \to 0$

By 14, we can write hv as a sum of weight vectors $\sum v_i$, $v_i \in V_{\underline{k(i)}}$.

So $\sum_i x^{\langle a, k(i)\rangle - \langle a, k\rangle} v_i \to v$ as $x \to 0$ ($\langle a, k\rangle = a_1k_1 + \dots a_nk_n$)

So each coefficient $x^{\langle a, k(i)\rangle - \langle a, k\rangle}$ must $\to 0$ as $x \to 0$ whenever $k(i) \neq k$.

ie $\langle k(i) - k, a\rangle > 0$.

In other words, only $\underline{k}$, and $k(i)$ with $\langle k(i) - k, a\rangle > 0$ for all strictly decreasing $a \in \mathbb{Z}^n$, can appear in the weight-vector-expansion of $h(v)$, for any $h \in N$

17. Any highest weight space is N-fixed (V not necessarily irreducible)

Let $\underline{k}$ be a highest weight of V, $v \in V_{\underline{k}}$, $h \in N$ with $hv \notin V_{\underline{k}}$ (for contradiction)

$\therefore$ for each $a \in \mathbb{Z}^n$ with $a_1 > a_2 > \dots > a_n$, $\exists \underline{k}(a)$ a weight with $\langle \underline{k}(a) - \underline{k}, \underline{a}\rangle > 0$

This inequality is invariant under scaling of $\underline{a}$ $\therefore$ such $k(a)$s exist for all $a \in \mathbb{Q}^n$ with $a_1 > a_2 > \dots > a_n$.

Take a sequence of a's tending to k (we cannot necessarily set $k = a$ as k_i are only weakly decreasing). Then we have a sequence of $k(a)$'s with $\langle k(a) - k, a\rangle > 0$

V contains finitely many weights (because V is finite dimensional), so some weight k' satisfies $\langle k' - k, a\rangle > 0$ for a's arbitrarily close to $\underline{k}$ – ie $\langle k' - k, k\rangle \geq 0$.

So $\langle k', k'\rangle = \langle k, k\rangle + 2\langle k' - k, k\rangle + \langle k' - k, k' - k\rangle > \langle k, k\rangle$, a contradiction

So $hv \in V_{\underline{k}}$ $\forall v \in V_{\underline{k}}$, all $h \in N$. So $t_x h t_x^{-1} v = hv \Rightarrow$ (by) $hv = v$ $\forall h \in N$, all $v \in V_{\underline{k}}$.

18 Any lowest weight space is N^T-fixed.

A lowest weight has $k_1 \leq k_2 \leq \dots k_n$, and maximum length.

Repeat 15-17 with $a_1 < a_2 < \dots < a_n$ and $h^T \in N^T$. Now $a_i > a_j$ for $i > j$, so $t_x h^T t_x^{-1}$ sends entries below the diagonal towards 0 as $x \to 0$ (and entries above the diagonal are already 0, as $h^T \in N^T$)

19 There is an open set around $e \in G$ where all elements have the form bn for some $b \in B, n \in N$

The tangent space $T_e B = \{X : I+\varepsilon X \in B\}$ = lower triangular matrices

$T_e N = \{X : I+\varepsilon X \in N\}$ = strictly upper triangular matrices.

So $T_e B + T_e N$ = all matrices $= T_e G$

The exponential map then turns an expression $dg = db + dn$ into $g = bn$, for all g "near e". (This expression is unique as the sum $T_e B + T_e N$ is direct)

20 $W_{\underline{k}}$, if non-zero, is irreducible, and has a 1-dimensional N-fixed space

For all N-fixed $f \in W_{\underline{k}}$, $f(bn) = b_1^{k_1} \cdots b_n^{k_n} f(n) = b_1^{k_1} \cdots b_n^{k_n} f(e)$. ($N$-fixed f exists by 17)

Fix a ~~nonzero~~ $f \in W_{\underline{k}}$ which is N-fixed. We must have $f(e) \neq 0$, or f would be identically 0 on the open set specified by 19, forcing f to be 0 on all of G (as f is holomorphic)

Take any $f_2 \in W_{\underline{k}}$: $\frac{f_2(e)}{f(e)} f_2(bn) = f(bn)$ $\forall b \in B, n \in N$ $\therefore$ by uniqueness of analytic continuations, $\frac{f_2(e)}{f(e)} f_2 \equiv f$

So all N-fixed elements of $W_{\underline{k}}$ are multiples of f ie the N-fixed space is 1-dimensional.

By 17, $W_{\underline{k}}$ has an N-fixed space within each of its irreducible components.

$\therefore$ a 1-dimensional N-fixed space means that $W_{\underline{k}}$ is irreducible.

21 If V is an irreducible G-rep with $\underline{k}$ as a highest weight, then $V \simeq W_{\underline{k}}$.

Fix $v^* \in V^*_{-\underline{k}}$, which exists by 14.

Define $\phi : V \to \{$holomorphic functions $G \to \mathbb{C}\}$, $\phi(v)$ is the map $g \to v^*(gv)$.

This is a G-map: $\phi(gv) = v^*(-gv) = g\phi(v)$, since G-action on $\{$holomorphic functions $: G \to \mathbb{C}\}$ is precomposition with right-multiplication.

The image lies entirely in $W_{\underline{k}}$: $\phi(v)(bg) = v^*(bgv) = (b^{-1}v^*)(gv)$

b^{-1} has the form $\begin{pmatrix} b_1^{-1} & \\ * & b_n^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}\begin{pmatrix} b_1^{-1} & \\ & b_n^{-1} \end{pmatrix}$

so $b^{-1}v^* = \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}(b_1^{-1})^{-k_1} \cdots (b_n^{-1})^{-k_n} v^* = b_1^{k_1} \cdots b_n^{k_n} v^*$, as v^* is N^T fixed, since $-\underline{k}$ is a lowest weight (by 14, 18).

So ϕ is a map between irreducible representations $\therefore$ by Schur, ϕ is an isomorphism.

22. # irreducible constituents of a G-rep V = dimension of N-fixed space.
By 21, V decomposes as a direct sum of $W_{\underline{k}}$'s, who each, by 20, has a 1-dimensional N-fixed space.
In particular, an irreducible rep has a single N-fixed line $\therefore$ unique highest weight, by 17, and the highest weight space is precisely the N-fixed space.

23. Let ρ be the vector $(n-1, n-2, \dots 0)$. Then, if $\underline{k}$ is a weight of an irrep with $\underline{k} \cdot \rho$ maximal, then $\underline{k}$ is the highest weight.
Take $\underline{k}$ with $\underline{k} \cdot \rho$ maximal and suppose there exists $v \in V_{\underline{k}}$ which is not N-fixed.
As in 16, the weight vector expansion of $h(v)$ (for some $h \in N$) contains some $v' \in V_{\underline{k}'}$ with $\langle k'-k, \rho \rangle > 0$ (as ρ has strictly decreasing entries)
So $\langle k', \rho \rangle > \langle k, \rho \rangle$, contradicting maximality of $\underline{k} \cdot \rho$.
Hence $V_{\underline{k}}$ is N-fixed. Since V is irreducible, 22 implies $\underline{k}$ is the unique highest weight.

24. The representation with character $\frac{\det(t_j^{\lambda_i})}{\det(t_j^{n-i})}$ has highest weight $(\lambda_1-(n-1), \lambda_2-(n-2), \cdots \lambda_n)$
We have $\chi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \det(t_j^{n-i}) = \det(t_j^{\lambda_i})$
By 23, the highest weight $\underline{k}$ is the only exponent in $\chi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$ with $\langle -, \rho \rangle$ maximal.
All exponents in $\det(t_j^{n-i})$ have the same length $\therefore$ by Cauchy-Schwartz, $(n-1, n-2, \dots 0)$ is the unique exponent with $\langle -, \rho \rangle$ maximal.
So the sole exponent on the left hand side with $\langle -, \rho \rangle$ maximal is $(k_1+(n-1), k_2+(n-2), \dots k_{n-1}+1, k_n)$. (ie this exponent has non-zero coefficient)
$\underline{k}$ is a highest weight $\Rightarrow k_1 \geq k_2 \geq \cdots \geq k_n \Rightarrow$ the exponent above is strictly decreasing.
On the right hand side, there is only one strictly decreasing exponent: $(\lambda_1, \lambda_2, \cdots \lambda_n)$.
So we must have $k_i + (n-i) = \lambda_i$ ie the highest weight $\underline{k}$ is $(\lambda_1-(n-1), \lambda_2-(n-2), \cdots \lambda_n)$

25. All weakly decreasing n-tuples occur as the highest weight of some irrep
Because any weakly decreasing n-tuple can be expressed as $(\lambda_1-(n-1), \lambda_2-(n-2), \dots \lambda_n)$ for a strictly decreasing λ, and $\frac{\det(t_j^{\lambda_i})}{\det(t_j^{n-i})}$ is a valid character for all strictly decreasing λ.

26. Suppose $\sigma: G_1 \times G_2 \to \mathrm{End}(W)$ and $W = \rho_1 \otimes \tau_1 \oplus \rho_2 \otimes \tau_2 \oplus \cdots \oplus \rho_n \otimes \tau_n$.
with G_1-irreps ρ_i and G_2-irreps τ_i all distinct. Then $\mathbb{C}G_1 \simeq \mathrm{Hom}_{G_2}(W,W)$
ie $\sigma(G_1 \times \{e\}) \subseteq \mathrm{End}(W)$ spans the centraliser of all G_2-action.
Since τ_i are distinct, $\mathrm{Hom}_{G_2}(W,W) \simeq \bigoplus \mathrm{Hom}_{G_2}(\rho_i \otimes \tau_i, \rho_i \otimes \tau_i)$ (as v.s.)
By density theorem, (since ρ_i are all distinct) every map of the form
$\bigoplus f_i \otimes \mathrm{id}$, for $f_i \in \mathrm{End}(\rho_i)$, lies in $\sigma(\mathbb{C}G_1 \times \{e\})$.
And $\mathrm{Hom}_{G_2}(\rho_i \otimes \tau_i, \rho_i \otimes \tau_i)$ is precisely $\mathrm{End}(\rho_i) \otimes \mathrm{id}$ (since, by Schur, the "second component" must stay fixed. Easiest to see with a basis)

27. The converse holds: if $\mathbb{C}G_1 \simeq \mathrm{Hom}_{G_2}(W,W)$, then the decomposition of W into $G_1 \times G_2$-irreps has all G_1-irreps distinct and all G_2-irreps distinct.
We show the contrapositive.
Suppose first that the G_2-irreps τ_i are distinct, but the G_1-irreps ρ_i are not.
Then $\mathrm{Hom}_{G_2}(W,W) \simeq \bigoplus_i \mathrm{End}(\rho_i) \otimes \mathrm{id}$ where some summands are repeated, and the image of $\mathbb{C}G_1$ must have identical components in the repeated summands.
If the τ_i's repeat, then group these together and write $W = \sigma_1 \otimes \tau_1 \oplus \cdots \sigma_m \otimes \tau_m$
with τ_i's distinct. Now $\mathrm{Hom}_{G_2}(W,W) = \bigoplus_i \mathrm{End}(\sigma_i) \otimes \mathrm{id}$
Some σ_i is not irreducible, so $\mathbb{C}G_1$ does not surject to $\mathrm{End}(\sigma_i)$
(we only get "diagonal" elements if σ_i contains repeated summands; if σ_i contains two distinct irreps, then $\mathbb{C}G_1$-action preserves these, so the image of $\mathbb{C}G_1$ in $\mathrm{End}(\sigma_i)$ only contains block matrices, not all of $\mathrm{End}(\sigma_i)$).

28 For any finite dimensional W, $\{w^{\otimes L} : w \in W\}$ spans the S_L-invariants of $W^{\otimes L}$.

$\{\sum_{\sigma \in S_n} e_{\sigma(i_1)} \otimes e_{\sigma(i_2)} \otimes \cdots e_{\sigma(i_n)} : 1 \le i_1 \le i_2 \le \cdots \le i_n \le \dim W\}$ is a basis of weight vectors for $GL(W)$ action on the S_L-invariants of $W^{\otimes L}$. The only N-fixed basis vector is $n!\, e_1 \otimes e_1 \otimes \cdots \otimes e_1$ $\therefore$ there is only one N-fixed weight space, which is one-dimensional. So S_L-invariants of $W^{\otimes L}$ is a $GL(W)$-irrep.

The span of $\{w^{\otimes L} : w \in W\}$ is $GL(W)$-invariant and a subspace of the S_L-invariants $\therefore$ $\{w^{\otimes L} : w \in W\}$ span the S_L-invariants.

29 The map $(\operatorname{End} V)^{\otimes L} \longrightarrow \operatorname{Hom}(V^{\otimes L}, V^{\otimes L})$, ($V$ finite-dimensional)

$X_1 \otimes X_2 \cdots \otimes X_L \longrightarrow (v_1 \otimes \cdots \otimes v_L \longrightarrow X_1 v_1 \otimes \cdots \otimes X_L v_L)$ is an isomorphism

let $e_1, e_2, \dots e_n$ be a basis of $V \Rightarrow e_{i_1} \otimes \cdots \otimes e_{i_L}$, $i_1, i_2 \cdots i_L \in \{1, 2, \cdots n\}$ is a basis of $V^{\otimes L}$ $\therefore$ we have a basis of $\operatorname{Hom}(V^{\otimes L}, V^{\otimes L})$ indexed by $i_1, i_2, \cdots i_L, j_1, \cdots j_L \in \{1, 2, \cdots n\}$: $e_{i_1} \otimes \cdots \otimes e_{i_L}$ goes to $e_{j_1} \otimes \cdots \otimes e_{j_L}$, and all other basis vectors of $V^{\otimes L}$ go to 0.

End V has a basis given by elementary matrices E_{ij}, which sends e_i to e_j and all other basis vectors to 0.

So a basis for $(\operatorname{End} V)^{\otimes L}$ is $E_{i_1 j_1} \otimes \cdots \otimes E_{i_L j_L}$. The map sends $E_{i_1 j_1} \otimes \cdots \otimes E_{i_L j_L}$ to the basis vector of $\operatorname{Hom}(V^{\otimes L}, V^{\otimes L})$ labelled by the same indices i.e. the bases are mapped to each other bijectively $\therefore$ map is an isomorphism.

30 The S_L-invariants of $(\operatorname{End} V)^{\otimes L}$ is isomorphic to $\operatorname{Hom}_{S_L}(V^{\otimes L}, V^{\otimes L})$ via the above map because the map respects S_L-action: $\sigma(X_1 \otimes X_2 \cdots \otimes X_L)$ is sent to $(v_1 \otimes \cdots \otimes v_L \longrightarrow X_{\sigma(1)} v_1 \otimes \cdots \otimes X_{\sigma(L)} v_L = \sigma(X_1 v_{\sigma(1)} \otimes X_2 v_{\sigma(2)} \otimes \cdots \otimes X_L v_{\sigma(L)}))$

31. $\{g^{\otimes L} : g \in GL_n\}$ spans $\operatorname{Hom}_{S_L}(\mathbb{C}^{n \otimes L}, \mathbb{C}^{n \otimes L})$

By 30 and 28, it suffices to show that $\{g^{\otimes L} : g \in GL_n\}$ spans $\{m^{\otimes L} : m \in \operatorname{End} \mathbb{C}^n\}$.

Suppose this is false: then there is a linear functional on $\{m^{\otimes L} : m \in \operatorname{End} \mathbb{C}^n\}$ that vanishes on $\{g^{\otimes L}\}$ (but is non-trivial)

But GL_n is dense in $\operatorname{End} \mathbb{C}^n \Rightarrow \{g^{\otimes L}\}$ dense in $\{m^{\otimes L}\}$ (as $-^{\otimes L}$ is a continuous map), so a continuous function vanishing on $\{g^{\otimes L}\}$ vanishes on $\{m^{\otimes L}\}$ also, by continuity. This gives the desired contradiction.

32 Every irrep of S_L occurs in $(\mathbb{C}^n)^{\otimes L}$ if $n \ge L$.

Send $1 \in \mathbb{C}S_L$ to $e_1 \otimes e_2 \otimes \cdots \otimes e_L$, where e_i are basis vectors of $\mathbb{C}^n$, and extend to an S_L-map. This sends a basis of $\mathbb{C}S_L$ to distinct basis vectors in $(\mathbb{C}^n)^{\otimes L}$ $\therefore$ is injective. So $(\mathbb{C}^n)^{\otimes L}$ contains the regular representation, and hence all ireps of S_l.

33 Schur-Weyl duality

By 31, we have the hypothesis of 27, so $(\mathbb{C}^n)^{\otimes l} = \rho_1 \otimes \tau_1 \oplus \cdots \oplus \rho_m \otimes \tau_m$ as $GL_n \times S_l$ reps, with ρ_i distinct GL_n-ireps and τ_j distinct S_l-ireps.

So we have a bijection $\rho_i \leftrightarrow \tau_i$.

By 32, all S_l-ireps occur as τ_i's $\therefore$ the number of ρ_i's occurring is the number of partitions of l.

If $e_1, e_2, \dots e_n$ is a basis for $\mathbb{C}^n$, then $e_{i_1} \otimes \cdots \otimes e_{i_l}$ is a weight basis for GL_n-action on $(\mathbb{C}^n)^{\otimes l}$, where the j^{th} component of the corresponding weight is the number of times j occurs in $\{i_1, \dots i_l\}$ $\therefore \sum k_j = l$ and each $k_j \in \mathbb{N}$.

The number of weakly-decreasing n-tuples satisfying these conditions is the number of partitions of l. As the ρ_i's have distinct highest weights, these must all occur as highest weights of the ρ_i (by counting)

Examples of reps of $GL_n(\mathbb{C})$:

- $\mathrm{Sym}^k(\mathbb{C}^n)$: basis $= \{e_1^{a_1} e_2^{a_2} \cdots e_n^{a_n} : a_1 + a_2 + \cdots + a_n = k,\ a_i \in \mathbb{N}\}$

  and $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} (e_1^{a_1} \cdots e_n^{a_n}) = t_1^{a_1} \cdots t_n^{a_n} (e_1^{a_1} \cdots e_n^{a_n})$

  $\therefore$ weights $= \{(a_1, a_2, \cdots a_n) : a_i \in \mathbb{N}, \sum_i a_i = k\}$

  weights of maximal length $= \{(k,0,0,\cdots 0), (0,k,0,0,\cdots 0), \cdots (0,0,\cdots k)\}$

  $\therefore$ highest weight $= (k,0,0,\cdots 0)$

  corresponding weight space = span of e_1^k, which is indeed fixed by N.

- $\Lambda^k(\mathbb{C}^n)$: basis $= \{e_{i_1} \wedge e_{i_2} \wedge \cdots e_{i_k} : i_1 < i_2 < \cdots < i_k \leq n\}$

  and $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = t_{i_1} t_{i_2} \cdots t_{i_k} (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k})$

  $\therefore$ weights $= \{$ n-tuples consisting of 1 in k entries, 0 in $n-k$ entries $\}$

  all weights have same length

  highest weight $= (1,1,1,\cdots,1,0,0\cdots 0)$ (first k are 1s)

  corresponding weight space $= e_1 \wedge \cdots \wedge e_k$ is fixed by N as all other terms in the expansion $N(e_1 \wedge \cdots \wedge e_k)$ have the first i vectors in span$\{e_1, \cdots e_{i-1}\}$, so the wedge is 0 (some e_j is repeated for $j<i$).

- the 1-dim rep: $g \to \det g \in \mathbb{C}^\times$: only weight is $(1,1,\cdots\cdots 1)$
- for $m \in \mathbb{Z}$, $g \to (\det g)^m \in \mathbb{C}^\times$ has only weight $(m, m, \cdots m)$

In general, if V is any irrep and W is 1-dimensional rep, then

$\therefore$ highest weight of $V \otimes W$ = highest weight of V + highest weight of W, using the N-invariant characterisation

- action by conjugation on sl_n: $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$ scales column i by t_i^{-1}, row i by t_i

  $\therefore$ weights have 1 in one entry, -1 in one entry, 0s elsewhere

  or are all 0 (multiplicity $n-1$)

  $\therefore$ highest weight $= (1,0,0,\cdots 0,-1)$,

  corresponding weight space $= \begin{pmatrix} & 1 \\ & \end{pmatrix}$, which is preserved pointwise by left and right multiplication by N.

Example: let V be the representation of GL_3 with highest weight $\underline{k}=(2,1,1)$
$\Rightarrow \underline{\lambda}=(4,2,1)$

So the character $\chi_V\begin{pmatrix}t_1 & & \\ & t_2 & \\ & & t_3\end{pmatrix} = \dfrac{t_1^4t_2^2t_3+t_2^4t_3^2t_1+t_3^4t_1^2t_2-t_1^4t_3^2t_2-t_2^4t_1^2t_3-t_3^4t_2^2t_1}{(t_1-t_2)(t_2-t_3)(t_1-t_3)}$

$$= \frac{t_1^4t_2t_3+t_2^2t_3^2t_1(t_2+t_3)-t_1^2t_2t_3(t_2^2+t_2t_3+t_3^2)}{(t_1-t_2)(t_1-t_3)}$$

$$= \frac{t_1^2t_2t_3(t_1+t_2)-t_3^2t_2^2t_1-t_3^3t_1t_2}{(t_1-t_3)}$$

$$= t_1t_2t_3(t_1+t_3)+t_2^2t_1t_3$$

$$= t_1^2t_2t_3+t_1t_2^2t_3+t_1t_2t_3^2$$

ie weights are $(2,1,1)$, $(1,2,1)$, $(1,1,2)$ each with multiplicity 1.
and the dimension is indeed $\dfrac{(4-2)(4-1)(2-1)}{(2-0)(1-0)(2-1)}=3$

Recall highest weight of $\mathbb{C}^3\otimes\det$
$=$ highest weight of $\mathbb{C}^3$ + highest weight of $\det$
$=(1,0,0)+(1,1,1)=(2,1,1)$
$\therefore V\subseteq\mathbb{C}^3\otimes\det$. As both sides are 3-dimensional, in fact $V=\mathbb{C}^3\otimes\det$.

To see N raising weights:
let h_{ij} denote the entries of $h\in N$: $h(e_1)=e_1$, $h(e_2)=e_2+h_{12}e_1$
$h(e_3)=e_3+h_{23}e_2+h_{13}e_1$ ... etc.
So, take $e_3^k\in\mathrm{Sym}^k\mathbb{C}^n$. (ie the weight space $(0,0,k,0,\dots)$)
Then $h(e_3^k)=(e_3+h_{23}e_2+h_{13}e_1)^k$, which expands out to a linear combination of $e_1^ie_2^je_3^l$ with $i+j+l=k$.
$e_1^ie_2^je_3^l$ lies in higher weight spaces: if $a_1>a_2>a_3$, then
$a_1i+a_2j+a_3l > a_3i+a_3j+a_3l=a_3k$