Character Table of $GL_2(\mathbb{F}_p)$

Let $G = GL_2(\mathbb{F}_p)$
$|G| = (p^2-1)(p^2-p) = p(p-1)^2(p+1)$

Recall that all $GL_2\mathbb{C}$-irreps have the form $\mathrm{Sym}^{a-b}(\mathbb{C}^2) \otimes \det^b$ for $a \geq b \in \mathbb{Z}$
(since these are irreducible with the correct highest weights)

Fact: Over $\overline{\mathbb{F}}_p$, G-irreps are $\{\mathrm{Sym}^a(\overline{\mathbb{F}}_p^{\,2}) \otimes \det^b : 0 \leq a \leq p-1,\ 0 \leq b \leq p-2\}$
($\mathrm{Sym}^p V$ is never irreducible over a field of characteristic p, as the p^{th} powers form a subrep)
($\det^{p-1}$ is trivial since it takes values in $\mathbb{F}_p^\times$)
To prove this, one would check that all these are irreducible and non-isomorphic. There are $p(p-1)$ possibilities for a, b. We show below that this is exactly the number of conjugacy classes whose elements have order coprime to p — so we know these are all the irreducible representations.

Over $\mathbb{C}$, the character table of G admits an interesting symmetry:

	$C(x,y)$	$C(\alpha)$	$Z'(x)$	$Z(x)$
$I(\chi_1,\chi_2)$	$\chi_1(x)\chi_2(y) + \chi_2(x)\chi_1(y)$	0	$(\chi_1\chi_2)(x)$	$(p+1)\chi_1\chi_2(x)$
$I(\psi)$	0	$-\psi(\alpha)-\psi(\alpha^p)$	$-\overline{\psi}(x)$	$-(p-1)\overline{\psi}(x)$
$st(\chi)$	$\chi(xy)$	$-\chi(\alpha^{p+1})$	0	$p\,\chi(x)^2$
$\chi\cdot\det$	$\chi(xy)$	$\chi(\alpha^{p+1})$	$\chi(x^2)$	$\chi(x^2)$

- The conjugacy classes

These are labelled by invariant factors ie all degree ≤ 2 elements of $\mathbb{F}_p[T]$. The possibilities are:

$T-x$	(ccl of type $Z(x)$)
$(T-x)^2$	(ccl of type $Z'(x)$)
$(T-x)(T-y)$ with $x \neq y$	(ccl of type $C(x,y)$)

a quadratic irreducible over $\mathbb{F}_p$. (ccl of type $C(\alpha)$)

As $\mathbb{F}_{p^2}$ is the splitting field of all quadratics over $\mathbb{F}_p$, irreducible quadratics over $\mathbb{F}_p$ split as $(T-\alpha)(T-\beta)$ over $\mathbb{F}_{p^2}$, where β is the Galois conjugate of α. Since the Galois group is generated by the Frobenius map, in fact $\beta = \alpha^p$.

$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in Z(x)$ and is central $\therefore Z(x) = \left\{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right\}$

there are $p-1$ ccls of this form (x can take any value in $\mathbb{F}_p^\times$)

$Z'(x) = \{$non-scalar matrices of trace $2x$ and determinant $x^2\}$

$= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+d = 2x,\ ad-bc = x^2,\ b, c \text{ not both } 0\right\}$

$= \left\{\begin{pmatrix} a & b \\ c & 2x-a \end{pmatrix} : bc = -(x-a)^2,\ b, c \text{ not both } 0\right\}$

if $a \neq x$, then b can be anything in $\mathbb{F}_p^\times$, and a, b, x determines c.

if $a = x$, then $b = 0$, $c \in \mathbb{F}_p^\times$ or $c = 0$, $b \in \mathbb{F}_p^\times$

$\Rightarrow (p-1)^2 + p-1 + p-1 = (p-1)(p+1)$ elements.

and there are $p-1$ ccls of this form (x can take any value in $\mathbb{F}_p^\times$)

$C(x,y) = \{$matrices of trace $x+y$ and determinant $xy\}$

$= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+d = x+y,\ ad-bc = xy\right\}$

$= \left\{\begin{pmatrix} a & b \\ c & x+y-a \end{pmatrix} : bc = -a^2 + a(x+y) - xy\right\}$

there are 2 distinct values of a for which RHS $= 0$: then $b = 0$, $c \in \mathbb{F}_p^\times$
or $c = 0$, $b \in \mathbb{F}_p$.

away from these 2 values, $b \in \mathbb{F}_p^\times$ is arbitrary, then c is determined

$\Rightarrow 2(2p-1) + (p-2)(p-1) = p^2 + p$ elements; $\binom{p-1}{2} = \frac{(p-1)(p-2)}{2}$ ccls of this form.

$C(\alpha) = \{$matrices of trace $\alpha + \alpha^p$ and determinant $\alpha^{p+1}\}$

$= \left\{\begin{pmatrix} a & b \\ c & \alpha+\alpha^p - a \end{pmatrix} : bc = -a^2 + a(\alpha+\alpha^p) - \alpha^{p+1}\right\}$

and RHS $\neq 0$ for all $a \in \mathbb{F}_p$ $\therefore a \in \mathbb{F}_p$, $b \in \mathbb{F}_p^\times$ arbitrary, c determined

$\Rightarrow p(p-1)$ elements.

$\alpha \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ and $C(\alpha) = C(\alpha^p)$ $\therefore$ there are $\frac{p^2-p}{2}$ ccls of this form.

Consider the subgroup H of lower triangular matrices, of which $N^T = \{\begin{pmatrix}1&0\\ *&1\end{pmatrix} : * \in \mathbb{F}_p\}$ is a normal subgroup. So every irreducible character of H/N^T = the diagonal matrices T (ie H is a semidirect product $N^T \rtimes T$) lifts to an irreducible character of H.

$T \cong \mathbb{F}_p^\times \times \mathbb{F}_p^\times$ $\therefore$ irreducible characters of T have the form $\begin{pmatrix}t_1&\\&t_2\end{pmatrix} \to \chi_1(t_1)\chi_2(t_2)$ where χ_1, χ_2 are irreducible characters of $\mathbb{F}_p^\times \cong$ cyclic group of order $p-1$

So $\begin{pmatrix}t_1&0\\ *&t_2\end{pmatrix} \to \chi_1(t_1)\chi_2(t_2)$ is an irreducible character of T.

Induce this up to G and call the result $I(\chi_1, \chi_2)$

As χ_1, χ_2 are 1-dimensional, $I(\chi_1, \chi_2)$ has dimension $|G:H| = \frac{p(p-1)^2(p+1)}{p(p-1)^2} = p+1$.

Take $g \in C(x,y)$. If $g \in H$ also, then the diagonal entries of g must be x and y (as $C(x,y)$ is characterised as the matrices with trace $x+y$ and determinant xy)

$\therefore C(x,y) \cap H = \{\begin{pmatrix}x&0\\ *&y\end{pmatrix} : * \in \mathbb{F}_p\} \sqcup \{\begin{pmatrix}y&0\\ *&x\end{pmatrix} : * \in \mathbb{F}_p\}$

So $I(\chi_1,\chi_2)(g) = \frac{|C_G(g)|}{|H|} \sum_{\begin{pmatrix}t_1&*\\0&t_2\end{pmatrix} \in C(x,y)\cap H} \chi_1(t_1)\chi_2(t_2) = \frac{|G|}{|C(x,y)||H|} p\left(\chi_1(x)\chi_2(y) + \chi_1(y)\chi_2(x)\right)$

$= \frac{p(p-1)^2(p+1)}{p(p+1)p(p-1)^2} p\left(\chi_1(x)\chi_2(y) + \chi_1(y)\chi_2(x)\right)$

$= \chi_1(x)\chi_2(y) + \chi_1(y)\chi_2(x)$

$C(\alpha) \cap H = \{\begin{pmatrix}t_1&0\\ *&t_2\end{pmatrix} : t_1 + t_2 = \alpha + \alpha^p,\ t_1 t_2 = \alpha^{p+1},\ t_1, t_2 \in \mathbb{F}_p\} = \emptyset$

(the solutions to these equations are over $\mathbb{F}_{p^2}$).

So $I(\chi_1,\chi_2)(g) = 0$ if $g \in C(\alpha)$.

$Z'(x) \cap H = \{\begin{pmatrix}x&0\\ *&x\end{pmatrix} : * \in \mathbb{F}_p^\times\}$

So, for $g \in Z'(x)$, $I(\chi_1,\chi_2)(g) = \frac{|C_G(g)|}{|H|} \sum_{\begin{pmatrix}x&0\\ *&x\end{pmatrix} \in Z'(x)\cap H} \chi_1(x)\chi_2(x)$

$= \frac{|G|}{|Z'(x)||H|} (p-1)\chi_1(x)\chi_2(x)$

$= \frac{p(p-1)^2(p+1)}{(p-1)(p+1)p(p-1)^2} (p-1)\chi_1(x)\chi_2(x) = (\chi_1\chi_2)(x)$.

$Z(x) \leq H$, so, for $g \in Z(x)$, $I(\chi_1,\chi_2)(g) = |G:H|\chi_1(x)\chi_2(x) = (p+1)(\chi_1\chi_2)(x)$.

Observe that these values are symmetric in χ_1 and χ_2, so $I(\chi_1,\chi_2) \cong I(\chi_2,\chi_1)$ (as reps), though there is no obvious map.

$$\langle I(\chi_1,\chi_2), I(\chi_1,\chi_2)\rangle = \frac{1}{|G|}\sum_{x\neq y\in\mathbb{F}_p^\times} |\chi_1(x)\chi_2(y)+\chi_2(x)\chi_1(y)|^2 \frac{(p^2+p)}{2}$$
$$+\frac{1}{|G|}\sum_{x\in\mathbb{F}_p^\times} |\chi_1\chi_2(x)|^2 (p-1)(p+1) + \frac{1}{|G|}\sum_{x\in\mathbb{F}_p} |(p+1)\chi_1\chi_2(x)|^2$$

(we divide by 2 as we're counting both (x,y) and (y,x))

$$= \frac{1}{2|G|}(p^2+p)\sum_{x\neq y\in\mathbb{F}_p^\times}\left(|\chi_1(x)\chi_2(y)|^2 + \overline{\chi_1(x)}\,\overline{\chi_2(y)}\,\chi_2(x)\chi_1(y) + |\chi_2(x)\chi_1(y)|^2 + \chi_1(x)\chi_2(y)\overline{\chi_2(x)}\,\overline{\chi_1(y)}\right)$$
$$+\frac{1}{|G|}2(p^2+p)\sum_{x\in\mathbb{F}_p^\times}|\chi_1\chi_2(x)|^2$$
$$= \frac{1}{2(p-1)^2}\sum_{x,y\in\mathbb{F}_p^\times}\left(|\chi_1(x)\chi_2(y)|^2 + \overline{\chi_1(x)}\,\overline{\chi_2(y)}\,\chi_2(x)\chi_1(y) + |\chi_2(x)\chi_1(y)|^2 + \chi_1(x)\chi_2(y)\overline{\chi_2(x)}\,\overline{\chi_1(y)}\right)$$
$$= \langle\chi_1,\chi_1\rangle\langle\chi_2,\chi_2\rangle + \langle\chi_1,\chi_2\rangle\langle\chi_2,\chi_1\rangle$$

(inner producting over $\mathbb{F}_p^\times$)

As χ_1, χ_2 are irreducible, $I(\chi_1,\chi_2)$ is irreducible if $\chi_1\neq\chi_2$

otherwise, $I(\chi_1,\chi_2)$ is the sum of two irreducible characters

Observe that
$$I(\chi_1\chi_3,\chi_2\chi_3)\begin{pmatrix}x&0\\0&x\end{pmatrix} = (p+1)(\chi_1\chi_3\chi_2\chi_3)(x) = [I(\chi_1,\chi_2)\otimes\chi_3\cdot\det]\begin{pmatrix}x&0\\0&x\end{pmatrix}$$
$$I(\chi_1\chi_3,\chi_2\chi_3)\begin{pmatrix}x&1\\0&x\end{pmatrix} = (\chi_1\chi_3\chi_2\chi_3)(x) = [I(\chi_1,\chi_2)\otimes\chi_3\cdot\det]\begin{pmatrix}x&1\\0&x\end{pmatrix}$$
$$I(\chi_1\chi_3,\chi_2\chi_3)\begin{pmatrix}x&0\\0&y\end{pmatrix} = (\chi_1\chi_3)(x)(\chi_2\chi_3)(y) + (\chi_2\chi_3)(x)(\chi_1\chi_3)(y)$$
$$= \chi_3(xy)\left[I(\chi_1,\chi_2)\begin{pmatrix}x&0\\0&y\end{pmatrix}\right] = [I(\chi_1,\chi_2)\otimes\chi_3\cdot\det]\begin{pmatrix}x&0\\0&y\end{pmatrix}$$

$\therefore I(\chi_1\chi_3,\chi_2\chi_3) = I(\chi_1,\chi_2)\otimes\chi_3\cdot\det$

$\therefore I(\chi_1,\chi_2)$ can all be expressed as $I(\chi_1,1)\otimes\chi_3\cdot\det$.

To decompose $I(\chi,\chi)$, we only need to decompose $I(1,1)$, which is the induction of the trivial character on H to G, ie the permutation character of G on the cosets of H. As H is precisely the stabiliser of the line $\binom{0}{*}$, this is the same as the permutation action of G on lines $=\mathbb{P}^1(\mathbb{F}_p)$, as this action is transitive.

$\therefore$ the usual decomposition of a transitive permutation representation shows $I(1,1) = 1 + st(1)$.

$st(1)(g) = 0-1 = -1$ for $g\in C(\alpha)$

$st(1)\begin{pmatrix}x&1\\0&x\end{pmatrix} = 1-1 = 0$ — $\begin{pmatrix}x&1\\0&x\end{pmatrix}$ fixes $\binom{*}{0}$

$st(1)\begin{pmatrix}x&0\\0&x\end{pmatrix} = p+1-1 = p$ — $\begin{pmatrix}x&0\\0&x\end{pmatrix}$ fixes all $p+1$ lines

$st(1)\begin{pmatrix}x&0\\0&y\end{pmatrix} = 2-1 = 1$ — $\begin{pmatrix}x&0\\0&y\end{pmatrix}$ fixes $\binom{*}{0}, \binom{0}{*}$.

Define $st(\chi)$ to be $st(1)\otimes\chi\cdot\det$ (χ a character of $\mathbb{F}_p^\times$).

so $st(\chi)$ has values as shown in the table. (and is p-dimensional)

- The characters $I(\psi)$

Take $\zeta \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ with $\zeta^2 \in \mathbb{F}_p$ (such ζ always exists for a quadratic extension)

View $\mathbb{F}_p^{\,2}$ as $\mathbb{F}_{p^2}$ by identifying $\binom{a}{b}$ with $a+b\zeta$

Multiplication by $a+b\zeta$ is an $\mathbb{F}_p$-linear on $\mathbb{F}_{p^2}$; under above identification, this has matrix $\begin{pmatrix} a & b\zeta^2 \\ b & a \end{pmatrix}$

Claim $\begin{pmatrix} a & b\zeta^2 \\ b & a \end{pmatrix} \in C(a+b\zeta)$: $\operatorname{tr}\begin{pmatrix} a & b\zeta^2 \\ b & a \end{pmatrix} = 2a = a+b\zeta+a+b\zeta^p = (a+b\zeta)+(a+b\zeta)^p$

$$\det\begin{pmatrix} a & b\zeta^2 \\ b & a \end{pmatrix} = a^2-b^2\zeta^2 = (a+b\zeta)(a-b\zeta) = (a+b\zeta)(a+b\zeta^p) = (a+b\zeta)^{p+1}$$

(here we have used $\zeta+\zeta^p=0$; this is because $\zeta^2=(\zeta^2)^p=(\zeta^p)^2$) (assuming $b \neq 0$)

As $C(a+b\zeta) = C((a+b\zeta)^p) = C(a-b\zeta)$, each such conjugacy class contains two such matrices.

Given an irreducible character ψ of $\mathbb{F}_{p^2}^\times$, we have a character of the subgroup $\left\{\begin{pmatrix} a & b\zeta^2 \\ b & a \end{pmatrix} : a, b \text{ not both } 0,\ a,b \in \mathbb{F}_p\right\}$. Induce it to $\tilde{\psi}$, a character of $GL_2\mathbb{F}_p$.

Since this subgroup lies in the $C(\alpha)$'s and $Z(x)$'s, $\tilde{\psi}=0$ on $C(x,y)$ and $Z'(x)$.

On $C(\alpha)$, $\tilde{\psi}$ has value $\dfrac{|G|}{|C(\alpha)|(p^2-1)}\left(\psi(\alpha)+\psi(\alpha^p)\right) = \psi(\alpha)+\psi(\alpha^p)$

On $Z(x)$, $\tilde{\psi}$ has value $\frac{|G|}{p^2-1}\psi(x) = p(p-1)\psi(x)$ So $\tilde{\psi}=\tilde{\psi}^p$.

Consider the virtual character $st(1)I(1,\psi) - I(1,\psi) - \tilde{\psi}$, which has value:

0	on $C(x,y)$	
$-\psi(\alpha)-\psi(\alpha^p)$	on $C(\alpha)$	
$\psi(x)$	on $Z'(x)$	(In $I(1,\psi)$, we are restricting ψ to
$(p-1)\psi(x)$	on $Z(x)$	$\mathbb{F}_p^\times \leq \mathbb{F}_{p^2}^\times$ ie as a character of $\mathbb{F}_p^\times$)

So its inner product with itself is

$$\frac{1}{|G|}\left(\frac{1}{2}\sum_{\alpha\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p}|\psi(\alpha)+\psi(\alpha^p)|^2\,p(p-1) + \sum_{x\in\mathbb{F}_p^\times}|\psi(x)|^2(p-1)(p+1) + \sum_{x\in\mathbb{F}_p^\times}|(p-1)\psi(x)|^2\right)$$

$$= \frac{1}{|G|}\left(\frac{p^2-p}{2}\sum_{\alpha\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p}|\psi(\alpha)|^2+|\psi(\alpha^p)|^2+\psi(\alpha)\overline{\psi(\alpha^p)}+\overline{\psi(\alpha)}\psi(\alpha^p) + (2p^2-2p)\sum_{x\in\mathbb{F}_p^\times}|\psi(x)|^2\right)$$

$$= \frac{1}{|G|}\left(\frac{p^2-p}{2}\right)\sum_{\alpha\in\mathbb{F}_p^\times}|\psi(\alpha)|^2+|\psi(\alpha^p)|^2+\psi(\alpha)\overline{\psi(\alpha^p)}+\overline{\psi(\alpha)}\psi(\alpha^p)$$

$$= \frac{1}{2}\left(\langle\psi,\psi\rangle+\langle\psi^p,\psi^p\rangle+\langle\psi,\psi^p\rangle+\langle\psi^p,\psi\rangle\right) = 1 \qquad \text{if } \psi\neq\psi^p.$$

and its value at $1 \in GL_2\mathbb{F}_p$ is $p-1$ $\therefore$ this is an irreducible character if $\psi\neq\psi^p$.

(this condition is equivalent to sending the generator of $\mathbb{F}_{p^2}^\times$ to a p^2-1 th root of unity that isn't a $p-1$th root of unity — ie p^2-p choices)

Call this character $I(\psi)$. I claim that, unless $\psi_1=\psi_2^{\,p}$, $I(\psi_1)\neq I(\psi_2)$. Hence we have $\frac{p(p-1)}{2}$ distinct $I(\psi)$s, which must complete the character table.

$I(\psi_1)=I(\psi_2) \Rightarrow \psi_1(\alpha)+\psi_1(\alpha)^p = \psi_2(\alpha)+\psi_2(\alpha)^p \quad \forall\alpha\in\mathbb{F}_{p^2}^\times\setminus\mathbb{F}_p^\times$

$\Rightarrow \psi_1(g)^s+\psi_1(g)^{sp} = \psi_2(g)^s+\psi_2(g)^{sp} \quad \forall s\in(0,p^2)$ which are coprime to p, where g denotes the multiplicative generator of $\mathbb{F}_p$

$\Rightarrow \zeta_1^s+\zeta_1^{sp} = \zeta_2^s+\zeta_2^{sp} \quad \forall s\in(0,p^2)$ coprime to p, where $\zeta_i=\psi_i(g)$.

Write ψ_1 as ζ^{n_1} where ζ is a primitive p^2-1-th root of unity. Let $\psi_2 = \zeta^{n_2}$.
Then ζ^s, $\forall s \in (0, p^2)$ coprime to p, is a root of $x^{n_1} + x^{n_1 p} - x^{n_2} - x^{n_2 p}$.
By multiplying by some power of x, we can make this a polynomial of degree at most $\frac{3}{4}(p^2-1)$, by pigeonhole principle (replacing any x^{p^2-1} by 1, as we are only interested in solutions within the p^2-1-th roots of unity)
But, for all odd primes, $(p-3)(p-1) > 0$

$$\Rightarrow \quad 4p^2 - 4p > 3p^2 - 3$$
$$\Rightarrow \quad p^2 - p > \tfrac{3}{4}(p^2-1)$$

ie this polynomial has too many roots $\therefore$ it must be the zero polynomial.
As $\psi_i^p \neq \psi_i$, $n_1 \neq n_1 p \bmod p^2-1$, so we must have $n_1 = n_2$ or $n_1 = n_2 p$.

$$\Rightarrow \psi_1 = \psi_2 \text{ or } \psi_1 = \psi_2^p.$$

How to actually construct the representations $I(\psi)$ (Deligne, Lusztig):
If G acts on a discrete set X, the permutation character π_X sends g to the number of points in X fixed by g.
A natural extension of this to action on topological spaces X is the Lefschetz character:
$L_X(g) = \sum_{i=0}^{\dim X} (-1)^i \operatorname{tr}(g : H^i(X) \longrightarrow H^i(X))$
This is an alternating sum of representations (of G acting on the cohomology of X), so strictly speaking it is a virtual character.

Groups like $GL_2(\mathbb{F}_p)$ (ie finite groups of Lie type) act naturally on many algebraic varieties over $\mathbb{F}_p$, and the corresponding Lefschetz characters (using étale cohomology) are a source of representations.
e.g. $GL_2(\mathbb{F}_p) \times \mathbb{F}_p$ act on $\{(x,y) \in \mathbb{F}_{p^2}^2 : (xy^p - x^p y)^{p-1} = 1\}$, with $GL_2(\mathbb{F}_p) \subseteq GL_2(\mathbb{F}_{p^2})$ action on $\mathbb{F}_{p^2}^2$, and $\mathbb{F}_{p^2}^\times$ action is scalar multiplication.
Since irreps of a product group are products of the irreps, we know the associated Lefschetz character is $\sum_i a_{ij} \rho_i \otimes \psi_j$ where ρ_i are irreducible characters of $GL_2(\mathbb{F}_p)$, ψ_j are irreducible characters of $\mathbb{F}_{p^2}^\times$ and $a_i \in \mathbb{Z}$.

We know all the irreducible characters ψ_j of $\mathbb{F}_p$, so we can find $\sum_i a_{ij} \rho_i$ for fixed j by taking the inner product of the Lefschetz character (viewed as a character of $\mathbb{F}_p$, with $g \in GL_2(\mathbb{F}_p)$ fixed) with ψ_j.
We then find $\sum_i a_{ij} \rho_i = I(\psi_j)$.

Now consider only $GL_2(\mathbb{F}_p)$-action on the above variety, X. ψ_j are all 1-dimensional, so the Lefschetz representation decomposes as $\oplus I(\psi_j)$.
For each $g \in GL_2(\mathbb{F}_p)$, we have

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\ & X/GL_2(\mathbb{F}_p) & \end{tikzcd} \quad , \quad \text{so} \quad \begin{tikzcd} H^*(X) & & H^*(X) \arrow[ll, "g^*"'] \\ & H^*(X/GL_2(\mathbb{F}_p)) \arrow[ul, "\pi^*"] \arrow[ur, "\pi^*"'] & \end{tikzcd}$$

so $\pi^*(H^*(X/GL_2(\mathbb{F}_p)))$ is fixed by g^* $\forall g \in GL_2(\mathbb{F}_p)$
ie $\pi^*(H^*(X/GL_2(\mathbb{F}_p))) \subseteq$ trivial representation $\subseteq GL_2(\mathbb{F}_p)$-action on $H^*(X)$.
In fact, this is an equality. To find the other irreps, we still take cohomology of this orbit space, but with coefficients in a local system.