Lie Algebras and their Representations

We will "linearise" Lie groups to form Lie algebras, then study their representations with linear algebra.

A Lie group G is a differentiable manifold which is also a group, where the multiplication and inverse maps are smooth. (as manifold maps $\therefore$ defined for collar around G)
e.g. SL_n
The associated Lie algebra $\mathfrak{g} = T_1 G$, the tangent space of G at the identity. (ie this is a vector space)

G is a linear algebraic group if: $\cdot$ G is a subgroup of GL_n
G is defined as the solution of polynomials in the matrix coefficients
e.g. $SL_n = \{g \in GL_n : \det(g) = 1\}$
$O_n = \{A \in GL_n : AA^T = I\}$ (set of quadratic equations) (this has two components)
Then G is an affine algebraic group, ie an affine algebraic variety where multiplication and inverse are polynomial maps (in terms of varieties). In fact, the converse is true – ie every affine algebraic group has a faithful representation.

For these groups, there is a naïve way of working out the Lie algebra:
Let E be the dual numbers $= \mathbb{C}[\varepsilon]/\varepsilon^2 = \{\alpha + \beta\varepsilon : \alpha, \beta \in \mathbb{C}, \varepsilon^2 = 0\}$.
Then $\mathfrak{g} = \{X \in M_n : I + X\varepsilon \in G(E)\}$
e.g. $\mathfrak{sl}_n = \{X \in M_n : tr(X) = 0\}$ since $\det(I + X\varepsilon) = 1 + trace(X)\varepsilon$.
$\mathfrak{o}_n = \{X \in M_n : X + X^T = 0\}$ since $(I + X\varepsilon)(I + X\varepsilon)^T = I + (X + X^T)\varepsilon$
$\mathfrak{gl}_n = \{X \in M_n\}$ since $(A + BC)^{-1} = (A^{-1} - A^{-1}BA^{-1}C) \Rightarrow (I + X\varepsilon)^{-1} = (I - X\varepsilon)$ (true $\forall$ groups)
Observe that $\mathfrak{so}_n = \{X \in M_n : X + X^T = 0 \text{ and } tr(X) = 0\} = \{X \in M_n : X + X^T = 0\} = \mathfrak{o}_n$, so we usually write $\mathfrak{so}_n$.
This happens because SO_n is the connected component of O_n containing the identity, and the tangent space at the identity only sees that component.

This works because G is characterised by $f_i(g) = 0$. For any $X \in \mathfrak{g}$, we can take a curve $\alpha(t) \in G$ with $\alpha(0) = I$, $\alpha'(0) = X$. Then $f_i(\alpha(t)) = 0 \ \forall t \Rightarrow 0 = df_i[\alpha'(0)] = \lim_{t \to 0} \frac{1}{t}[f_i(I + \alpha'(0)t) - f_i(I)] = $ first order term of $f_i(I + Xt)$

The fact that $\mathfrak{g}$ came from a group gives it extra structure. Consider the commutator map: $G \times G \to G$:
$PQP^{-1}Q^{-1} = (I + A\varepsilon)(I + B\delta)(I - A\varepsilon)(I - B\delta) = I + (AB - BA)\varepsilon\delta$.
so we have a map: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $[A, B] = AB - BA$ (ie take 2^{nd} mixed derivative of commutator at $I \times I$)
Since $I + [B, A]\varepsilon\delta = QPQ^{-1}P^{-1} = (PQP^{-1}Q^{-1})^{-1} = (I + [A, B]\varepsilon\delta)^{-1} = I - [A, B]\varepsilon\delta$, we see $[B, A] = -[A, B]$.

$\therefore$ we can define a Lie algebra as a vector space over any field equipped with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is skew-symmetric and satisfies the Jacobi identity.
(we have linearity in above case as all derivatives are linear)

Observe that upper triangular matrices and strictly upper triangular matrices are closed under $[X,Y] = XY - YX$ $\therefore$ these are Lie algebras.
These and so_n, sl_n are vector subspaces of gl_n closed under $[\,,\,]$ ie they are Lie subalgebras.

For any vector space V, $[X,Y]=0$ defines a Lie algebra. This is an abelian Lie algebra, obtained from an abelian Lie group.

A representation of a Lie algebra g on a vector space V is a homomorphism of Lie algebras $\phi: g \to gl_V$. Then g is said to act on V.
ie we have a linear map $\phi: g \to \operatorname{End} V$, with $\phi[X,Y] = [\phi(X), \phi(Y)] = \phi(X)\phi(Y) - \phi(Y)\phi(X)$

If $g \subseteq gl_V$, trivially g acts on V (by inclusion) $\therefore$ so_n, sl_n etc. act on $\mathbb{C}^n$.

Lie-algebra representations arise from differentiating Lie group representations:
Suppose G is an affine algebraic group and $\rho: G \to GL_V$ is an algebraic representation (defined by polynomials). Then $\rho(I + A\varepsilon) = I + \varepsilon d\rho(A)$
and $I + \varepsilon[d\rho(A), d\rho(B)] = \rho(P)\rho(Q)\rho(P)^{-1}\rho(Q)^{-1} = \rho(PQP^{-1}Q^{-1}) = I + \varepsilon d\rho[A,B]$ (if $P = I + A\varepsilon$, $Q = I + B\varepsilon$)
$I + \varepsilon(d\rho A + d\rho B) = \rho(P)\rho(Q) = \rho(PQ) = I + \varepsilon d\rho(A+B)$
$\therefore d\rho$ is a Lie-algebra representation.

Observe that, if $W \subseteq V$ is invariant under ρ, so is $d\rho$
$\therefore$ we have a map: algebraic representations of G $\to$ representations of g (derivative map)
and irreducible representations $\to$ irreducible representations

In general, this is not a bijection: consider $\mathbb{C}^\times$, with tangent space $\mathbb{C}$.
The irreducible representations of $\mathbb{C}^\times$ are $z \mapsto z^n$, $n \in \mathbb{Z}$, and representations of $\mathbb{C}^\times$ are completely reducible.
Any representation of $\mathbb{C}$ is completely determined by $\rho(1)$, which can be any matrix.
All linear maps: $\mathbb{C}^n \to \mathbb{C}^n$ have a linear invariant subspace $\therefore$ irreducible representations of $\mathbb{C}$ are uniquely determined by $\rho(1): \mathbb{C} \to \mathbb{C}$. Identifying these maps with multiplication by $n \in \mathbb{C}$, we have $\rho(z) = nz$.
This is expected as $d(z \to z^n)$ at $z=1$ is $z \to nz$. But only $z \to nz$, $n \in \mathbb{Z}$ are derivatives, not all the irreducible representations.
Also, ρ may not be completely reducible: not every matrix can be diagonalised.

Lie's theorem states that, if G is simple, connected and simply connected, these problems do not occur. (Any interesting topology of G will have restrictions on its algebraic representations, which the Lie algebra does not detect.

Any group G acts on G itself by conjugation, which produces a representation on $\mathbb{C}$-linear combinations of the group elements.
The derivative of this is the *adjoint representation*: $\operatorname{ad}(x): g \to g$, $\operatorname{ad} x(y) = [x,y]$
$[\,,\,]$ is bilinear $\Rightarrow$ ad is linear. Skew-symmetry and Jacobi identity $\Rightarrow$ $\operatorname{ad}[x,y] = [\operatorname{ad} x, \operatorname{ad} y]$.

The center of $\mathfrak{g}$ is $\{x : [x,y]=0\ \forall y\in\mathfrak{g}\} = \ker \mathrm{ad}$. These are derivatives of elements of $Z(G)$.
$\therefore$ $\mathfrak{g}$ has trivial center $\Leftrightarrow$ $\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$ is an embedding, ie a faithful representation.

Lie algebras with non-trivial centers may of course have other representations that are faithful – in fact, Ado's theorem says any finite dimensional Lie algebra is a subalgebra of $\mathfrak{gl}_n$ for some n.

Representation of $\mathfrak{sl}_2$

$\mathfrak{sl}_2$ = matrices of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$
$\mathfrak{sl}_2$ has basis $e=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$
$\therefore$ a representation of $\mathfrak{sl}_2$ is precisely a choice of linear operators E, F, H
with $[H,E]=2E$, $[H,F]=-2F$, $[E,F]=H$

Let $L(n)$ be the vector space of homogeneous polynomials in x, y
ie $L(n)$ has basis $X^n, X^{n-1}Y, X^{n-2}Y^2, \dots XY^{n-1}, Y^n$, and hence dimension $n+1$
Let ρ_n be an action of GL_2 on $L(n)$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(X^iY^j) = (aX+cY)^i(bX+dY)^j$ and extend linearly ($i+j=n$)
ie if we think of $f\in L(n)$ as a function on $\mathbb{R}^2$, $Af(\underline{x}) = f A(\underline{x})$.
$\therefore$ ρ_0 is the trivial representation
ρ_1 is the standard representation.

The restriction of ρ_n to SL_2 gives a representation of SL_2 on $L(n)$
$\therefore$ we can differentiate to obtain a representation of $\mathfrak{sl}_2$ on $L(n)$:
$(1+\varepsilon e)(X^iY^j) = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}(X^iY^j) = X^i(\varepsilon X+Y)^j = X^iY^j + j\varepsilon X^{i+1}Y^{j-1}$ $\quad\therefore E = X\frac{\partial}{\partial Y}$
$(1+\varepsilon f)(X^iY^j) = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}(X^iY^j) = (X+\varepsilon Y)^iY^j = X^iY^j + i\varepsilon X^{i-1}Y^{j+1}$ $\quad F = Y\frac{\partial}{\partial X}$
$(1+\varepsilon h)(X^iY^j) = \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix}(X^iY^j) = (X+\varepsilon X)^i(Y-\varepsilon Y)^j = (1+i\varepsilon-j\varepsilon)X^iY^j$ $\quad H = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$
we can check that these satisfy the correct $[\,,\,]$ relations.
The infinite sum $\oplus\rho_n$ gives a representation of $\mathfrak{sl}_2$ on $\mathbb{C}[x,y]$.

Identifying e with X^2, f with $-Y^2$ and h with $-2XY$, we see that ρ_2 is the adjoint representation.

Observe that X^iY^j are the distinct eigenspaces of H – in this basis, H has matrix $\begin{pmatrix} n & & & \\ & n-2 & & \\ & & \ddots & \\ & & & 2-n \\ & & & & -n \end{pmatrix}$.
E and F permute these eigenspaces, and no subset of them stay invariant under both E and F $\therefore$ these representations are irreducible.

In fact, these are all the irreducible representations:
Theorem: For every $n\geq 0$, there is a unique irreducible representation of $\mathfrak{sl}_2$ of dimension $n+1$, and every finite dimensional representation of $\mathfrak{sl}_2$ is a direct sum of irreducibles ie they are completely reducible.

For $\lambda\in\mathbb{C}$, let $V_\lambda = \{v\in V : Hv=\lambda v\}$, the λ-weight space of eigenvectors of H with eigenvalue λ, e.g. $L(n)_{i-j} = \mathrm{span}\{X^iY^j\}$ (these always exist if we work over an algebraically closed field)

Now $H(Ev) - E(Hv) = [H,E](v) = 2Ev$

$H(Ev) = E(\lambda v) + 2Ev = (\lambda+2)Ev$ for $v \in V_\lambda$.

and similarly $H(Fv) = (\lambda-2)Fv$

$\therefore$ E, F send V_λ to $V_{\lambda+2}$ and $V_{\lambda-2}$ respectively

Since this is a finite-dimensional representation, some V_λ must be sent to 0 by E.
If $Hv = \lambda v$ and $ev = 0$, then v is a highest weight vector with weight λ. $(v \neq 0)$

Let $W = \langle v, Fv, F^2v, \dots \rangle$ $\therefore v \in V_\lambda$, $F^k v \in V_{\lambda-2k}$
By construction, $F(W) \subseteq W$.
$H(F^k v) = (\lambda-2k)(F^k v) \in W$ $\therefore H(W) \subseteq W$.
By induction, we can show $E(F^k v) = k(\lambda-k+1)F^{k-1}v$ (as $EF = FE + [E,F] = FE + H$)

V is finite dimensional, so $F^k v$ must be zero for some k
(otherwise, since they live in different H-eigenspaces, $F^i v$ are linearly independent)
Take minimal such k.
Then $0 = E(F^k v) = k(\lambda-k+1)F^{k-1}v$. Since $F^{k-1}v \neq 0$, we have $k(\lambda-k+1) = 0$
Since $k \neq 0$ $(v \neq 0)$, $\lambda = k-1 \in \mathbb{Z}$ ie all weights are integral.

We showed before that $W = \langle v, Fv, F^2v, \cdots \rangle$ is an invariant subspace under sl_2.
$\therefore$ if V is an irreducible representation it must be $\langle v, Fv, \cdots F^k v \rangle$ for some k.
Since $\lambda = k-1$, k completely determines the action of E, F, H on V
$\therefore$ there is precisely one irreducible representation for each k, which proves the first statement of the theorem.

Now define the Casimir of sl_2: $\Omega = EF + FE + \frac{1}{2}H^2 = \frac{1}{2}H^2 + H + 2FE$
By direct computation $E\Omega = \Omega E$, $F\Omega = \Omega F$, $H\Omega = \Omega H$
$\therefore$ if V is an irreducible representation, by Schur's lemma, then Ω acts of V by scalar multiplication. To find this scalar, it is sufficient to apply Ω to any (non-zero) vector in V. Applying Ω to the highest weight vector, we find that Ω acts on $L(n)$ as multiplication by $\frac{1}{2}n^2 + n$, which are distinct for distinct n.

Consider the Jordan decomposition of Ω, which splits V into generalised λ-eigenspaces
$V^\lambda = \{v \in V : (\Omega - \lambda)^{\dim V} v = 0\}$, $V = \oplus V^\lambda$
For each $g \in sl_2$, we have $(\Omega-\lambda)^{\dim V}(gv) = g(\Omega-\lambda)^{\dim V} v = 0$ if $v \in V^\lambda$
$\therefore g$ preserves each V^λ ie the V^λs are subrepresentations of V, for each fixed λ.
Since there is a highest weight vector in each of these subrepresentations, this shows λ can only take values $\frac{1}{2}n^2 + n$.
To show these subrepresentations are completely reducible, we need slightly more mechanism:

Let g be a Lie-algebra, and W a g-module.
A composition series for W is a sequence of g-submodules $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_r = W$ with each subquotient W_i/W_{i-1} an irreducible representation.

Lemma: if W is finite dimensional, a composition series exists

Proof: We apply induction on $\dim W$. If $\dim W = 1$, W is irreducible $\therefore 0 \subseteq W$ is a composition series.
If $\dim W > 1$, take any irreducible submodule $W_1 \subseteq W$.
W/W_1 is a $\mathfrak{g}$-module of lower dimension $\therefore \exists$ composition series $0 = W_1/W_1 \subseteq W_2/W_1 \subseteq \cdots \subseteq W_r/W_1 = W/W_1$.
Then $0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_r = W$ is a composition series. (3rd isomorphism theorem)

Take a composition series for V^λ. Work in a basis so the first $\dim W_1 + \dim W_2 + \cdots \dim W_i$ vectors span W_i.
Each W_i is invariant under sl_2, so all of sl_2 act on V^λ by a matrix of the form
$$\begin{pmatrix} B_1 & * & * \\ & B_2 & * \\ & & \ddots \\ & & & B_n \end{pmatrix}$$
where B_i describes an irreducible representation $L(n_i)$, and $*$ are unknown entries.
We will prove that $*$ is in fact all 0.

Each W_i/W_{i-1} is irreducible, so Ω acts on them by scalars. Hence, in this basis, Ω has the form shown. Since $(\Omega - \lambda)^{\dim V}$ is the zero matrix, we have $n_i = n$ $\forall i$, with $\lambda = \frac{1}{2}n^2 + n$ (as $\frac{1}{2}n^2 + n$ are distinct for distinct n)
$$\left(\begin{array}{cc|cc} \frac{1}{2}n_1^2+n_1 & & & \\ & \frac{1}{2}n_1^2+n_1 \ \ddots & & * \\ \hline & & \frac{1}{2}n_2^2+n_2 & \\ & & & \frac{1}{2}n_2^2+n_2 \ \ddots \end{array}\right)$$

1. H acts on V^λ with eigenvalues in the set $\{-n, -n+2, \cdots, n-2, n\}$
   suppose v is an eigenvector of H. Take i minimal with $W_i \ni v$. Then $v + W_{i-1} \neq 0$ in W_i/W_{i-1}, and $H(v + W_{i-1}) = \text{eigenvalue}(v + W_{i-1})$, so this eigenvalue must be an eigenvalue of the H action on W_i/W_{i-1}.
   $\therefore$ all eigenvalues of H on V^λ are eigenvalues of H on W_i/W_{i-1} for some i = eigenvalues of H on $L(n)$.

2. H acts on $\operatorname{Ker}(E : V^\lambda \to V^\lambda)$ with eigenvalue n. ie $(H-n)^{\dim V^\lambda} x = 0$ $\forall x \in \operatorname{Ker} E$.
   by the commutation relation, we see H maps $\operatorname{Ker} E$ to itself.
   Now apply the above argument to $\operatorname{Ker} E \cap W_i$ in place of W_i. The eigenvalues of H on $\operatorname{Ker} E \cap W_i / \operatorname{Ker} E \cap W_{i-1}$ is n only.

3. $\operatorname{Ker} E$ generates all of V^λ as an sl_2-module ie $\sum_i F^i(\operatorname{Ker} E) = V^\lambda$. (as this is sl_2-invariant)
   we apply induction on the W_i.
   $0 \to W_{i-1} \to W_i \to W_i/W_{i-1} = L(n) \to 0$, so W_i is generated by $\{$generators of $W_{i-1}\} \cup \{$pullback of generators of $W_i/W_{i-1}\}$.
   We know $L(n)$ is generated by any highest weight vector $\bar{x}$ with $E(\bar{x}) = 0$, $H(\bar{x}) = n\bar{x}$.
   In the basis where the B_i's for H are $\begin{pmatrix} n & & \\ & n-2 & \\ & & \ddots \ -n \end{pmatrix}$, the vector x with 1 in position $\dim W_1 + \cdots + \dim W_i + 1$ and 0s elsewhere is a lift of $\bar{x}$, and we have $(H-n)^{\dim V^\lambda} x = 0$
   Since $[H, E] = 2E$, $(H-n-2)^{\dim V^\lambda} Ex = 0$ but H has no eigenvalue of $n+2$, so $Ex = 0$ ie $x \in \operatorname{Ker} E$.

4. $(H+2k)F^k = F^k H$, $EF^{n+1} = F^{n+1}E + (n+1)F^n(H-n)$
   by direct computation, from the commutation relations.

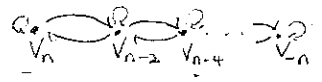
5. For $x \in \operatorname{Ker} E$, $(H-n+2k)^{\dim V^\lambda} F^k x = 0$ from 2 and first part of 4.
   $\therefore F^k x \in$ generalised $n-2k$-eigenspace of H. This shows that $\sum_{i,k} \lambda_{ik} F^k x_i = 0 \Rightarrow \sum_i \lambda_i F^k x_i = 0$ $\forall k$.

6. For non-zero $y \in \operatorname{Ker} E$, $F^n y \neq 0$
   take minimal i with $y \in W_i$. let $\bar{y}$ = the projection of y in W_i/W_{i-1}. $\bar{y}$ is a highest weight vector so $F^n \bar{y} \neq 0$ in W_i/W_{i-1} $\Rightarrow F^n y \neq 0$.
   In particular, $\sum_i \lambda_i F^k x_i = 0 \Rightarrow \sum_i \lambda_i x_i = 0$ $\therefore$ if x_i is a basis of $\operatorname{Ker} E$, $F^k x_i$ is a basis of V^λ. (using 3).

7. $Hx = nx \quad \forall x \in \operatorname{Ker} E$.

From 5, we see that $F^{n+1}x \in$ generalised-$(-n-2)$-eigenspace of H, which doesn't exist
$\therefore F^{n+1}x = 0 \quad \therefore$ by second part of 4, $0 = EF^{n+1}x = (n+1)F^n(H-n)x + 0$.
since $E(H-n)x = 0$, F^n kills $(H-n)x \in \operatorname{Ker} E$. By 6, $(H-n)x = 0 \Rightarrow Hx = nx$.

By 4, this means H preserves $\{x, Fx, F^2x \cdots F^nx\}$, and, by commutation relations, so does E.
So, for each basis vector x_i of $\operatorname{Ker} E$, $\{x_i, Fx_i, \cdots F^nx_i\}$ is a basis for an irreducible representation, and their direct sum is V^λ.

It is useful to think of an sl_2 representation as a series of strings, each corresponding to one irreducible component  E F H

Let V, W be representations of any g.
If G acts on V, W, then $g(v \otimes w) = gv \otimes gw$ is an action of G on $V \otimes W$.
Differentiating this gives the action $X(v \otimes w) = Xv \otimes w + v \otimes Xw$, on $V \otimes W$
This is clearly linear, and $[X,Y] = XY - YX$ follows from this fact applied to V and W.
$\therefore$ this is a valid Lie-algebra representation.

So, given $L(m) \otimes L(n)$, what are its irreducible components?
One method to solve this is to find all the highest weight vectors – such a formula does exist, but it's very complicated.
e.g. let v_n, v_m be highest weight vectors of $L(n)$, $L(m)$ respectively
Then $E(v_n \otimes v_m) = 0 \otimes v_m + v_n \otimes 0 = 0$
$H(v_n \otimes v_m) = n v_n \otimes v_m + v_n \otimes m v_m = (n+m) v_n \otimes v_m \quad \therefore v_n \otimes v_m$ is a h.w. vector for $L(n+m)$.

Instead, given any finite-dimensional representation of sl_2, set its character to be
$\chi(V) = \sum_{n \in \mathbb{Z}} \dim V_n z^n$. (a laurent polynomial)

1. $\chi(V)|_{z=1} = \dim V$ since H is diagonalisable, and all its eigenvalues are integral

2. $\chi(L(n)) = z^n + z^{n-2} + \cdots + z^{2-n} + z^{-n} = \dfrac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}$

3. $\chi(V) = \chi(W) \Leftrightarrow V, W$ are isomorphic representations
By 'peeling off' the leading coefficient of $\chi(V)$, we can find the number of copies of $L(n)$ (for each n) in V. ie $V = \oplus\, a_n L(n)$ and a_n are determined uniquely by $\chi(V)$.

4. $\chi(V \otimes W) = \chi(V)\chi(W)$
since, $\forall v_i \in V_i, w_j \in W_j$, $H(v_i \otimes v_j) = i v_i \otimes v_j + v_i \otimes v_j = (i+j) v_i \otimes v_j$ and $E(v_i \otimes v_j) = 0$
$\therefore V_i \otimes W_j \subseteq (V \otimes W)_{i+j}$. $\sum_{ij} V_i \otimes W_j = V \otimes W \Rightarrow (V \otimes W)_r = \sum_{i+j=r} V_i \otimes W_j$
$\dim (V \otimes W)_r = \sum_{i+j=r} \dim V_i \dim W_j$ which is the way coefficients transform under multiplication of polynomials.

So $\chi(L(1)\otimes L(3)) = (z+z^{-1})\left(\frac{z^4-z^{-4}}{z-z^{-1}}\right) = \frac{z^5+z^3-z^{-3}-z^{-5}}{z-z^{-1}} = \chi(L(2))+\chi(L(4))$ $\therefore L(1)\otimes L(3) = L(2)+L(4)$

Applying this algorithm to $L(n)\otimes L(m)$ for general $m\leq n$ gives the **Clebsch-Gordon rule**:
$L(n)\otimes L(m) = L(n+m)\oplus L(n+m-2)\oplus\cdots\oplus L(n-m)$

Example: $L(1)\otimes L(n) = L(n+1)\oplus L(n-1)$

H has eigenvalues $1,-1$ on $L(1)$ $\therefore$ let v_1, v_{-1} be eigenvectors with these eigenvalues respectively. similarly, let w_i be eigenvectors of $L(n)$ with eigenvalue i.

$E(v_1\otimes w_n) = 0\otimes w_n + v_1\otimes 0 = 0$ $\quad H(v_1\otimes w_n) = v_1\otimes w_n + v_1\otimes n w_n = (n+1)(v_1\otimes w_n)$

$E(v_1\otimes w_{n-2} - v_{-1}\otimes w_n) = 0\otimes w_{n-2} + v_1\otimes w_n - v_1\otimes w_n - v_{-1}\otimes 0 = 0$

$H(v_1\otimes w_{n-2} - v_{-1}\otimes w_n) = v_1\otimes w_{n-2} + v_1\otimes(n-2)w_{n-2} - (-1)v_{-1}\otimes w_n - v_{-1}\otimes n w_n = (n-1)(v\otimes w_{n-2} - v_{-1}\otimes w_n)$

$\therefore$ highest weight vectors are $v_1\otimes w_n$ and $v\otimes w_{n-2} - v_{-1}\otimes w_n$

Similarly, highest weight vectors of $L(2)\otimes L(n) = L(n+2)\oplus L(n)\oplus L(n-2)$ are:
$v_2\otimes w_n$, $v_2\otimes w_{n-2} - v_0\otimes w_n$, $v_2\otimes w_{n-4} - v_0\otimes w_{n-2} + v_{-2}\otimes w_n$

We now proceed to do the above to general semi-simple lie algebras.

Aside / Example: Heisenburg lie algebras

let L be any vector space, and L^* its dual.
$W = L\oplus L^*$ is sympletic with respect to the inner product $\langle (l_1,f_1),(l_2,f_2)\rangle = f_2(l_1) - f_1(l_2)$

Consider $W\oplus \mathbb{C}c$ where W is any sympletic vector space.
Define $[c,w]=0$, $[w,w'] = \langle w,w'\rangle c$ and extend linearly. (check this satisfies axioms)

This is nilpotent: $g^1 = \mathbb{C}c$, $g^2 = 0$

Take $L=\mathbb{C}$, and let the generator of L and its corresponding dual be p, q respectively
$\therefore [p,c]=[q,c]=0$, $[p,q]=c$ $\quad *$
This has a representation on $\mathbb{C}[x]$: $p\to\frac{d}{dx}$
$q\to$ multiplication by x
$c\to$ identity
(valid representation as RHS satisfies $*$)
This representation is irreducible: take any $f\in\mathbb{C}[x]$. After many applications of p, f becomes a constant $\therefore 1\in$ irreducible component of f. $q^n(1)=x^n$ also belongs to this component, and these span all of $\mathbb{C}[x]$.

But all Heisenburg algebras have a finite-dimensional representation.
all sympletic spaces have a basis e_i, f_i such that $\langle e_i,e_j\rangle = \langle f_i,f_j\rangle = 0$, $\langle e_i,f_j\rangle = 1$.
$\therefore$ we require $[e_i,e_j]=[f_i,f_j]=[c,e_i]=[c,f_j]=0$, $[e_i,f_j]=\delta_{ij}c$. $\quad (1\leq i,j\leq n)$
Set $e_i\to A_{1\,i+1}$ $f_j\to A_{(j+1)(n+2)}$, $c = A_{1\,(n+2)}$
(A_{ij} denotes matrix with 1 in entry i,j and 0 elsewhere, $(n+2)\times(n+2)$)

Heisenberg algebra over $\mathbb{F}_p$ shows that Lie's theorem does not hold in characteristic p:
we obtain an irreducible representation on the p-dimensional space $\mathbb{F}_p[x]/(x^p)$, by setting

$p \to \frac{d}{dx}$

$q \to$ multiplication by x

$c \to$ identity

(show irreducibility by same argument)

'Structure of semi-simple lie algebras

A lie algebra is simple if: its only ideals are $0, g$, and g is non-abelian (to exclude dim 1 case)
" " " is semisimple if it is a direct sum of simple subalgebras.
" " " is reductive if it is semisimple subalgebra $\oplus$ abelian subalgebra.
(an ideal being a subalgebra h where $[X,Y] \in h$ $\forall X \in g, Y \in h$. Then quotients make sense)

Define $[g,g] = \{[X,Y] : X,Y \in g\}$ which is clearly an ideal, and $g/[g,g]$ is abelian.
By induction the subalgebras $g^n, g^{(n)}$ below are ideals (using the Jacobi identity)

The central series of g is : $g^0 = g,\ g^n = [g^{n-1}, g]$ $\quad \therefore g^0 \supseteq g^1 \supseteq \cdots$ $\quad g^{n-1}$ in center of g/g^n
derived series of g is : $g^{(0)} = g,\ g^{(n)} = [g^{(n-1)}, g^{(n-1)}]$ $\quad g^{(0)} \supseteq g^{(1)} \supseteq \cdots$ $\quad g^{(n)}/g^{(n+1)}$ abelian

g is nilpotent if $g^n = 0$ for some n
solvable if $g^{(n)} = 0$ for some n $\quad$ (ie built out of abelian subalgebras)

We can show inductively that $g^{(n)} \subseteq g^n$ $\therefore$ nilpotent groups are solvable.

Example: Let n = strictly upper triangular matrices = linear maps A with $Ae_i \in \langle e_1, \cdots e_{i-1}\rangle$.
$\therefore [A,B](e_i) \in \langle e_1, \cdots e_{i-2}\rangle \Rightarrow n^0(e_i) \in \langle e_1, \cdots, e_{i-2}\rangle$
Inductively we see $n^{(n)}(e_i) \in \langle e_1, \cdots, e_{i-2-n}\rangle$ $\therefore n^{(\dim - 2)}(e_i) = 0$ $\forall i$ $\therefore n^{(\dim-2)} = 0$
$\therefore n$ is nilpotent. (These are called nilpotent endomorphisms)

Let b = upper triangular matrices = linear maps A with $Ae_i \in \langle e_1, \cdots e_i\rangle$
$\therefore [A,B](e_i) \in \langle e_1, \cdots e_{i-1}\rangle$ (since i^{th}-component of $AB(e_i)$ = i^{th} component of $BA(e_i)$)

$$\left[\begin{pmatrix} & & \\ a_i & \cdots & a_r \\ & & \end{pmatrix}, \begin{pmatrix} & & \\ 1 & \cdots & 1 \\ & & \end{pmatrix}\right] = \begin{pmatrix} & & & \\ a_i - a_i & a_i - a_{i+1} & \cdots & a_i - a_r \\ & & & \end{pmatrix}$$

where the entries shown are in row i, for arbitrary i, and all other entries are 0.

$\therefore [b,b] = n$.
$[b,n] = n$ (set $a_i = 0$ in example above) $\quad \therefore b$ is not nilpotent.
But $b^{(i)} = n^{(i-1)} \subseteq n^{i-1}$ which terminates $\quad \therefore b$ is solvable

Let h be a subalgebra of g, k an ideal in g.
Then $[h,h] \subseteq g^1 = g^{(1)}$, and we can show inductively that $h^n \subseteq g^n$, $h^{(n)} \subseteq g^{(n)}$ $\therefore$
and $[g/k, g/k] = \frac{g^1+k}{k} = \frac{g^{(1)}+k}{k}$ " " " " that $(g/k)^n = \frac{g^n+k}{k}$, $(g/k)^{(n)} = \frac{g^{(n)}+k}{k}$
$\therefore$ subalgebras and quotients of nilpotent/solvable g are nilpotent/solvable.

The partial converses are:
k, g/k solvable $\Rightarrow g$ solvable
Proof: $\exists n$ with $(g/k)^{(n)} = 0$ ie $g^{(n)} \subseteq k$. Then $g^{(n+i)} \subseteq k^{(i)}$ and we have $k^{(m)} = 0$ for some m.
center of $g \neq 0$, $g/$center of g nilpotent $\Leftrightarrow g$ nilpotent
Proof: $\Leftarrow$: Let n be minimal with $g^n = 0$. Then $0 = [g^{n-1}, g]$ and $g^{n-1} \neq 0$ $\therefore g^{n-1} \subseteq$ center of $g \neq 0$.
$\Rightarrow$: $\exists n$ with $g^n \subseteq$ center of g. Then $g^{n+1} = 0$.

Since $g/\text{center of } g$ is precisely the adjoint representation, so the last fact says:
g is nilpotent $\Leftrightarrow \mathrm{ad}(g) \subseteq gl(g)$ is nilpotent.

Lie's theorem: let $g \subseteq gl_V$ be a solvable Lie algebra over an algebraically closed k of characteristic 0. Then $\exists$ a basis $v_1, v_2, \dots, v_n$ of V such that, with respect to which all elements of g are upper triangular. ie $g \subseteq b$.
Equivalently, $\exists$ a common eigenvector for g (= a 1-dimensional subrepresentation, or a linear map $\lambda : g \to k$ and $x \in V$ such that $A(x) = \lambda(A)(x)$ $\forall A \in g$.)
We will not prove this.

Corollary: g is a solvable finite-dimensional Lie algebra $\Rightarrow [g,g]$ is nilpotent
Proof: Apply Lie to the adjoint representation $g \to gl_g$: $[\mathrm{ad}\,g, \mathrm{ad}\,g] \subseteq [b,b] = n$
which is nilpotent. $\Rightarrow [\mathrm{ad}\,g, \mathrm{ad}\,g]$ nilpotent.
ad is a representation $\therefore \mathrm{ad}[g,g] = [\mathrm{ad}\,g, \mathrm{ad}\,g]$
and we have shown $\mathrm{ad}[g,g]$ nilpotent $\Rightarrow [g,g]$ nilpotent.

Engel's theorem: let $\pi : g \to gl_V$ be a finite dimensional representation such that, $\forall x \in g$, $\pi(x)$ is nilpotent. Then $\exists$ a common eigenvector $v \in V$ ie $\pi(x)v = 0$ $\forall x \in g$, or a 1-dimensional trivial representation.
By considering $V/\langle v \rangle$, then the quotient of this by the 1-dimensional trivial subspace, we see that $\pi(g)$ is represented by nilpotent endomorphisms $\therefore g$ is nilpotent $\Leftrightarrow \mathrm{ad}(g)$ are nilpotent endomorphisms

A symmetric bilinear form: $g \times g \to k$ is invariant if $([x,y],z) = (x,[y,z])$
(this is the derivative of a G-invariant form $G \times G \to k$)
For any ideal h, $h^\perp = \{x : (x,y) = 0 \ \forall y \in h\}$ is also an ideal. In particular, $g^\perp$ is an ideal.
The form is degenerate if $g^\perp \neq 0$

Define the trace form: given $\rho : g \to gl_V$ a representation, set $(x,y)_V = \mathrm{tr}(\rho(x)\rho(y))$
Linearity follows from linearity of the trace function.
This is invariant: $([x,y],z) = \mathrm{tr}(XYZ - YXZ) = \mathrm{tr}(XYZ - XZY) = (x,[y,z])$

e.g. Killing form $= (x,y)_{\mathrm{ad}} = \mathrm{tr}(\mathrm{ad}\,x\,\mathrm{ad}\,y)$

Not every invariant form is a trace form:
let g have basis c,p,q,d, with $[c,x] = 0$ $\forall x \in g$, $[p,q] = c$, $[d,p] = p$, $[d,q] = -q$.
$g^{(1)} = \mathrm{span}\ c,p,q$, $g^{(2)} = \mathrm{span}\ c$, $g^{(3)} = 0 \Rightarrow g$ is solvable.
Invariant forms on g must satisfy: $(c,c) = (c,p) = (c,q) = 0$, $(p,p) = (q,q) = (d,p) = (d,q) = 0$, $(c,d) = (p,q)$, and these are the only restrictions.
$\therefore$ Set $(c,d) = (p,q) = (d,d) = 1$. This is non-degenerate $\therefore$ not a trace form (by Cartan)

(This is the extended Heisenberg algebra. Given any representation of the usual Heisenberg algebra, we obtain a representation of g above by setting $\rho(d) = -\rho(q)\rho(p)$.)

Theorem: Cartan's criteria. Let $g \subseteq gl_V$ over a field k of characteristic 0.
Then g solvable $\Leftrightarrow (x,y)_V = 0 \quad \forall x \in g,\ y \in [g,g]$ ie $[g,g] \subseteq g^\perp$.
$\Leftrightarrow$ all trace forms are degenerate (since $g/_{ker}$ = solvable)

In fact, it suffices to check that the Killing form is degenerate: $\Rightarrow$ is in fact a consequence of Lie's theorem, as (upper triangular matrix, strictly upper triangular matrix) = 0. Conversely, if the Killing form is degenerate, ad(g) is solvable by Cartan. Since center(g) is also solvable, g is solvable.

The sum of solvable ideals is solvable (each term in its derived series is the sum of the corresponding terms for the derived series of its summands) so we can define $R(g)$ = sum of solvable ideals = maximal solvable ideal.

Observe $R(g/R(g)) = 0$: if $h/R(g) \subseteq g/R(g)$ is solvable, then, as $R(g)$ is solvable, so is $h \Rightarrow h \subseteq R(g)$
$\therefore h/R(g) = 0$.

Theorem: The following are equivalent: (ie ii, iii, iv can be taken as definitions of semi-simple)

i g is semi-simple

ii $R(g) = 0$

iii g has no non-zero abelian ideals

iv the Killing form is non-degenerate — this is Killing's criteria

Moreover, if this holds, every derivation $D: g \to g$ is inner, and the simple components are unique.

Proof: ii $\Rightarrow$ iii immediate, as an abelian ideal is certainly solvable.

iii $\Rightarrow$ ii suppose h is a solvable ideal. $\therefore \exists n$ with $h^{(n)} = 0 \Rightarrow [h^{(n-1)}, h^{(n-1)}] = 0$
$\Rightarrow h^{(n-1)}$ is an abelian ideal of g, $h^{(n-1)} \neq 0$.

iv $\Rightarrow$ iii suppose h is a non-zero abelian ideal. Let $g = h + W$
$\forall x \in g$, $[x,h] \subseteq g$ $\therefore$ ad x is represented by a matrix of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$
(first column corresponding to h, second column to W)
$\forall y \in h$, $[y,h] = 0$, $[y,g] \subseteq h$ $\therefore$ ad y is represented by $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$
$\therefore$ ad$x$ad$y$ have the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, which has zero trace $\therefore h \subseteq g^\perp$ under Killing form.

ii $\Rightarrow$ iv suppose $g^\perp \neq 0$. By Cartan, this means g is solvable $\Rightarrow R(g) = g \neq 0$.

iii, iv $\Rightarrow$ i let $h \subseteq g$ be a minimal non-zero ideal.
$h^\perp \cap h \subseteq h$ $\therefore$ by minimality, $h^\perp \cap h = 0$ or h
if $h \cap h^\perp = h$, then (,) is the zero form when restricted to h. By Cartan, this means h is solvable, contradicting iii.
$\therefore h \cap h^\perp$ must be $0 \Rightarrow$ (,) is non-degenerate on h.
$h + h^\perp = g$ (for any symmetric bilinear form). As $h \cap h^\perp = 0$, we have $g = h \oplus h^\perp$.
Then $x \in h^\perp$, $[x, h^\perp] = 0$ would imply $[x, g] = 0$ $\therefore$ (,) is non-degenerate on $h^\perp$ also.
Applying this argument repeatedly to $h, h^\perp$, we obtain $g = \oplus_i g_i$, with g_i minimal ideals. By iii, g_i are not abelian $\therefore g_i$ are simple $\Rightarrow g$ is semisimple.

$i \Rightarrow iii$: let I be an ideal of $g = \oplus_i g_i$, g_i simple. $\therefore I \cap g_i = 0$ or g_i
let π_i be the projection to g_i. $\pi_i(I)$ is an ideal of g_i. $\therefore \pi_i(I) = 0$ or g_i.
g_i, I are ideals $\therefore [I, g_i] \subseteq I \cap g_i$.
Then $I \cap g_i \supseteq \pi_i[I, g_i] = [\pi_i I, \pi_i g_i] = \pi_i I$
$\therefore$ either $\pi_i(I) = 0$, or $\pi_i(I) = g_i \subseteq I \cap g_i \Rightarrow I \supseteq g_i$.
$\therefore I =$ sum of some of the g_is. Assume I is non-zero.
Since, $\forall x, y \in I$, the components of $[x,y]$ belong in $[g_i, g_i] \neq 0$, I is non-abelian.

Observe that $0 \to R(g) \to g \to g/R(g) \to 0$ is always exact.
$\therefore$ we can always view g as being composed of a solvable part and a semisimple part.
Moreover, in characteristic 0, Levi's theorem asserts that this sequence splits
ie $g = s + R(g)$, or that $\exists$ a complement to $R(g)$ that is a subalgebra.

It fails in characteristic p: let $g = sl_p(F_p)$.
$F_p \subseteq$ center of g $\therefore F_p$ is solvable (scalar matrices)
By considering matrices with only 1 non-zero entry, we see g/F_p has no non-zero abelian ideals
$\therefore$ it is semisimple $\Rightarrow F_p = R(sl_p(F_p))$.
Suppose s is a complement subalgebra. Then, $\forall x, y \in g$, $x = a + b$, $y = c + d$ where $a, c \in s$, $b, d \in F_p$.
and $s \ni [a,c] = [x,y]$ as $b, d \in$ center of g
$\therefore s = [g,g]$. $[A_{ij}, A_{ji}] = A_{ii} - A_{jj}$ $\therefore \sum_{i=1}^{p-1} (A_{ii} - A_{pp}) = \sum_{i=1}^{p} A_{ii} = I \in s$, a contradiction
as $s \cap F_p$ should be zero (A_{ij} = matrix with only non-zero entry being 1 in row i, column j)

A derivation is a map which satisfies the analogue of the Jacobi identity:
$D[a,b] = [Da, b] + [a, Db]$.
These are inner if $Da = [x, a]$ for some x (ie D is the map $\operatorname{ad} x$).

Proof (ctd): let $D : g \to g$ be a derivation.
consider a linear function $l(x) = tr(D \cdot \operatorname{ad} x) \in g^*$ (dual space)
$(\,,\,)_{ad}$ is non-degenerate $\therefore g^*$ is isomorphic to g via $(\,,\,)_{ad}$ ie $l(x) = (x, y)_{ad}$ for some y
let $E = D - \operatorname{ad} y$, which is another derivation.
$\operatorname{ad}(E(x))(a) = [Ex, a] = E[x,a] - [x, Ea] = E \operatorname{ad} x(a) - \operatorname{ad} x \cdot E(a)$
$\therefore \operatorname{ad}(Ex)$ is the map $[E, \operatorname{ad} x]$ ($[\,,\,]$ in gl_g)
$\therefore (Ex, z)_{ad} = tr(\operatorname{ad}(Ex) \operatorname{ad} z)$
$= tr([E, \operatorname{ad} x] \operatorname{ad} z)$
$= tr(E [\operatorname{ad} x, \operatorname{ad} z])$ traceforms are invariant (considering tr as a traceform on gl_g, standard rep)
$= tr(D \operatorname{ad}[x,z] - \operatorname{ad} y \operatorname{ad}[x,z])$
$= tr(D \operatorname{ad}[x,z]) - tr(D \operatorname{ad}[x,z]) = 0$ by definition of y.
This holds $\forall z$ $\therefore Ex = 0 \Rightarrow Dx = \operatorname{ad} y\, x$ (as $(\,,\,)_{ad}$ is non-degenerate)
And this holds $\forall x$ $\therefore D = \operatorname{ad} y$ as maps.

Let $\oplus g_i$ be a decomposition of g into simple ideals.
Then any ideal of g is a sum of the g_i's $\therefore$ we can define g_i as the minimal non-zero ideals of g.

Proposition: a nilpotent Lie algebra always has non-inner derivations.
Proof: Take a basis of $\mathfrak{g}'$ and extend by $v_1, v_2, \dots, v_m, w$ to a basis of $\mathfrak{g}$.
$\mathfrak{g}' \oplus \operatorname{span} v_i$ is an ideal (since all $[.,.] \in \mathfrak{g}'$). Denote this by $\mathfrak{h}$.
Consider $\{x \in \mathfrak{g} : [x,\mathfrak{h}] = 0\} \supseteq$ center of $\mathfrak{g} \neq 0$.
$\therefore \exists n$ such that $\{x \in \mathfrak{g} : [x,\mathfrak{h}]=0\} \subseteq \mathfrak{g}^{n-1} \setminus \mathfrak{g}^n$.
$\therefore$ we can choose $z \in \{x \in \mathfrak{g} : [x,\mathfrak{h}]=0\} \setminus \mathfrak{g}^n$.
Define $D(w) = z$, $D(\mathfrak{h}) = 0$ and extend linearly.
This is not inner: if $\exists y \in \mathfrak{g}$ with $[y,\mathfrak{h}]=0$, then $y \in \{x \in \mathfrak{g} : [x,\mathfrak{h}]=0\} \subseteq \mathfrak{g}^{n-1}$.
but $[y,w]$ would then be in $\mathfrak{g}^n$.

The converse does not hold: let a, b be a basis of $\mathfrak{g}$, $[a,b]=b$.
$\therefore$ all derivations must satisfy $D(b) = [Da,b] + [a,Db] \in \mathfrak{g}' = \operatorname{span} b$
$\therefore [a,Db] = Db \Rightarrow [Da,b]=0 \Rightarrow D(a) \in \operatorname{span} b$ also.
Let $Da = \lambda b$, $Db = \mu b$. Then $D = \operatorname{ad}(-\lambda b + \mu a)$
$\therefore$ every derivation is inner, but $\mathfrak{g}$ is not nilpotent ($\mathfrak{g}^n = \operatorname{span} b \ \forall n$)

Theorem: Let $\mathfrak{g}$ be a simple Lie algebra with $(\,,\,)$, $\langle\,,\,\rangle$ both non-degenerate invariant bilinear forms. Then $\exists \lambda \in k^*$ with $(\,,\,) = \lambda \langle\,,\,\rangle$. In particular, every non-degenerate invariant bilinear form is some multiple of the Killing form

Proof: $(\,,\,)$, $\langle\,,\,\rangle$ are non-degenerate $\therefore$ given any linear $f : \mathfrak{g} \to k$, $f(z) = (x,z) = \langle y, z\rangle$ for some $x, y \in \mathfrak{g}$, and these correspondences: $\mathfrak{g}^* \to \mathfrak{g}$ are bijections.
$\therefore$ we can define $\phi(x) = y$ above ie $(x,z) = \langle \phi(x), z\rangle \ \forall z \in \mathfrak{g}$. (note ϕ is linear)
Observe that, for any $x,y,z \in \mathfrak{g}$: $\langle \phi[x,y], z\rangle$
$\langle \phi[x,y], z\rangle = ([x,y],z) = (x,[y,z]) = \langle \phi(x), [y,z]\rangle = \langle [\phi(x),y], z\rangle$
$\langle\,,\,\rangle$ is non-degenerate $\therefore \phi[x,y] = [\phi(x), y]$, or equivalently $\phi \cdot \operatorname{ad} x = \operatorname{ad} \cdot \phi x$.
$\mathfrak{g}$ is simple so the adjoint representation is faithful and irreducible.
Since ϕ commutes with $\mathfrak{g}$-action (given by ad), by Schur, it must be scalar multiplication
$\therefore (x,z) = \langle \lambda x, z\rangle = \lambda \langle x, z\rangle \ \forall x, z$.

We will show that $\mathfrak{sl}_n$ is simple. $(A,B) = \operatorname{tr}(AB)$ is a traceform $\therefore$ invariant.
Let A_{ij} denote the matrix with 1 in entry i,j and 0 elsewhere ($i \neq j$)
B_i " " " " 1 in entry i,i, -1 in entry n,n and 0 elsewhere ($i \neq n$), $n \geq 2$.
Let $X = A_{21}$, $Y = A_{12}$ $\therefore \operatorname{tr}(X,Y) = \operatorname{tr} A_{22} = 1$. $\{A_{ij}, B_i\}$ form a basis of $\mathfrak{sl}_n$.

Left multiplication by X sends 1st row to 2nd row. (all other entries are removed)
Right " " " " 2nd column to 1st column
$\therefore \operatorname{ad} X$ sends: $A_{1j} \to A_{2j}$, $A_{i2} \to -A_{i1}$ ($i,j \neq 1,2$), $A_{12} \to B_2 - B_1$, $B_1 \to A_{21}$, $B_2 \to -A_{21}$.
Left multiplication by Y sends 2nd row to 1st row
Right " " " " 1st column to 2nd column
$\therefore \operatorname{ad} Y$ sends: $A_{2j} \to A_{1j}$, $A_{i1} \to -A_{i2}$ ($i,j \neq 1,2$), $A_{21} \to B_1 - B_2$, $B_1 \to -A_{12}$, $B_2 \to A_{12}$
all other basis elements sent to 0 (same for ad X)
$\therefore \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y) = 2(n-2) + 2 = 2n-2$ $\therefore \lambda$ in the theorem above is $2n-2$.

$\mathfrak{t} \subseteq \mathfrak{g}$ is a torus if it is an abelian lie subalgebra such that, $\forall x \in \mathfrak{t}$, $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is diagonalisable.
A torus is maximal if it is not contained in any other torus.

Lemma: let $t_1, t_2, \cdots t_r : V \to V$ be pairwise commuting diagonalisable linear maps.
let λ denote a vector in k^r.
Define $V_\lambda = \{v \in V : t_i v = \lambda_i v\}$, a simultaneous eigenspace for $t_1, t_2, \cdots t_r$.
Then $V = \bigoplus_{\lambda \in k^r} V_\lambda$.

Proof: Apply induction on r. For $r=1$, this is just the statement that t_1 is diagonalisable.
If $r>1$, by inductive hypothesis $V = \bigoplus V_{(\lambda_1, \lambda_2, \cdots \lambda_{r-1})}$.
For $v \in V_{(\lambda_1, \lambda_2, \cdots \lambda_{r-1})}$, $t_i(t_r v) = t_r(t_i v) = \lambda_i (t_r v)$ $\forall i$ $\therefore t_r v \in V_{(\lambda_1, \cdots \lambda_{r-1})}$
$\therefore V_{(\lambda_1, \cdots \lambda_{r-1})}$ is invariant under t_r, so we can decompose it into eigenspaces for t_r.

Now suppose t_i are a basis for a lie algebra $\mathfrak{t}$, and t_i are diagonalisable w.r.t. the action of $\mathfrak{t}$ on some V. Then we have a decomposition $V = \bigoplus_{\lambda \in k^r} V_\lambda$.
Let $x \in \mathfrak{t}$ be $\sum x_i t_i$. Then, for $v \in V_\lambda$, $x(v) = \sum x_i (t_i v) = (\sum x_i \lambda_i) v$
$\therefore V_\lambda$ are eigenspaces of x also $\Rightarrow$ all of $\mathfrak{t}$ is diagonalisable.
If we view λ as a linear function on $\mathfrak{t}$, $\lambda(t_i) = \lambda_i$, then $V_\lambda = \{v \in V : xv = \lambda(x) v \ \forall x \in \mathfrak{t}\}$.
Since $\mathfrak{t}$ acts as a direct sum of 1-dimensional representations on each V_λ, we see that V is completely reducible, and the irreducible representations are one-dimensional, indexed by $\mathfrak{t}^*$.

If we apply this to the adjoint representation restricted to $\mathfrak{t} : \mathfrak{g} \to \mathfrak{g}$, we get a decomposition $\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\lambda \in \mathfrak{t}^*} \mathfrak{g}_\lambda$ (where $\mathfrak{g}_0$ is $\mathfrak{t}$, $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} : [t, x] = \lambda(t) x \ \forall t \in \mathfrak{t}\}$)

Example: $\mathfrak{g} = \mathfrak{sl}_n$. let $\mathfrak{t} = \operatorname{span} B_i$ = all diagonal matrices in $\mathfrak{sl}_n$. (with previous notation)
$\mathfrak{t} \subseteq$ diagonal matrices, which is an abelian subalgebra (viewed in $\mathfrak{gl}_n$)
right multiplication by B_i removes all but column i, and column n with reversed sign.
left multiplication by B_i removes all but row i, and row n with reversed sign
$\therefore$ if all B_i's commute with some $A \in \mathfrak{g}$, then A can only have diagonal entries
$\therefore \mathfrak{t}$ is a maximal torus. (we show below that $\operatorname{ad} B_i$ are diagonalisable)

For $i, j \neq k$ or n, $[B_k, A_{ij}] = 0$.
$i \neq k$ or n, $[B_k, A_{ik}] = -A_{ik}$, $[B_k, A_{in}] = A_{in}$ $\forall k \neq i$, $[B_i, A_{in}] = 2A_{in}$
$j \neq k$ or n, $[B_k, A_{kj}] = A_{kj}$, $[B_k, A_{nj}] = -A_{nj}$ $\forall k \neq j$, $[B_j, A_{nj}] = -2A_{nj}$.
$\therefore A_{ij} \in \mathfrak{g}_{\lambda_{ij}}$ where $\lambda_{ij}(\underline{x}) = x_i - x_j$ $\forall i, j \neq n$, $i \neq j$.
$A_{in} \in \mathfrak{g}_{\alpha_i}$, $A_{ni} \in \mathfrak{g}_{\beta_j}$ where $\alpha_i(\underline{x}) = \sum x_j + x_i$, $\beta_j(\underline{x}) = x_j - \sum x_i$
and B_i, A_{ij} span $\mathfrak{sl}_n$.

Now let $\varepsilon_i \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} = a_i$, the dual of B_i for $i < n$, and $\varepsilon_n = -\sum x_i$.
Then $\mathfrak{sl}_n = \mathfrak{g}_0 \bigoplus_{i \neq j,\ 1 \leq i, j \leq n} \mathfrak{g}_{\varepsilon_i - \varepsilon_j}$

Theorem: let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{t}$ a maximal torus.
Then $\mathfrak{t} \neq 0$, and $\mathfrak{g}_0 = \mathfrak{t}$ ie if $[x,t]=0$ $\forall t \in \mathfrak{t}$, then $x \in \mathfrak{t}$.

The roots of $\mathfrak{g}$ are $R_{\mathfrak{g}} = \{\lambda \in \mathfrak{t}^* : \mathfrak{g}_\lambda \neq 0, \lambda \neq 0\}$ ie the non-zero weights of $\mathfrak{g}$.
e.g. the roots of $\mathfrak{sl}_n$ are $\{\varepsilon_i - \varepsilon_j : i \neq j\}$. Observe that these span $\mathfrak{t}^*$

Proposition: $\mathfrak{sl}_n$ is a simple Lie algebra
Proof: suppose $\mathfrak{h} \subseteq \mathfrak{sl}_n$ is a non-zero ideal $\forall x \in \mathfrak{h}$, $x = x_0 + x_{\alpha_1} + x_{\alpha_2} + \cdots x_{\alpha_n}$ where $\alpha_i \in R$ are distinct.
pick $x \in \mathfrak{h}$ with a minimal number of terms in the expansion above, $x \neq 0$.
if $x_0 \neq 0$, then take $h_0 \in \mathfrak{t}$ with distinct diagonal terms. $\therefore \alpha_i(h_0) \neq 0$ for any i.
$\mathfrak{h} \ni [x, h_0] = [x_0, h_0] + [x_{\alpha_1}, h_0] + \cdots [x_{\alpha_n}, h_0] = \alpha_1(h_0) x_{\alpha_1} + \cdots + \alpha_n(h_0) x_{\alpha_n}$.
If x_{α_i} are not all zero, the above shows $[x, h_0] \neq 0$ and has fewer terms, a contradiction
$\therefore$ we must have $x = x_0$.
$x_0 \neq 0$ $\therefore \exists \alpha_r$ with $\alpha_r(x_0) \neq 0$. let α_r be the root $\varepsilon_i - \varepsilon_j$.
$\therefore \mathfrak{h} \ni [x_0, A_{ij}] = \alpha_r(x_0) A_{ij}$.
$\mathfrak{h} \ni [A_{ij}, A_{jk}] = A_{ik}$ for any $k \neq i$.
$\mathfrak{h} \ni [A_{li}, A_{ik}] = A_{lk}$ for any $l \neq k$. $\therefore \forall l \neq k$, $A_{lk} \in \mathfrak{h}$.
$[A_{ln}, A_{nl}] = B_l$ $\therefore$ all B_is $\in \mathfrak{h}$ $\Rightarrow \mathfrak{h}$ is all of $\mathfrak{sl}_n$.
if $x_0 = 0$, then $x = x_{\alpha_1} + x_{\alpha_2} + \cdots x_{\alpha_n}$. If there is more than 1 term, then we can find $h \in \mathfrak{t}$ with $\alpha_1(h) \neq \alpha_2(h)$. Then $[h, x] - \alpha_1(h) x$ has no α_1 component $\therefore$ is a shorter expression, a contradiction. $\therefore$ we must have $x = x_{\alpha_1} = A_{ij}$ for some $i \neq j$.
by same argument as above, this forces $\mathfrak{h} \ni$ all A_{ij} s, all B_i s $\therefore \mathfrak{h} = \mathfrak{sl}_n$.

We can carry out the same proof for $\mathfrak{so}_n$ and $\mathfrak{sp}_n$. (exercise)

Structure Theorem: let $\mathfrak{g}$ be a semi-simple Lie algebra with maximal torus $\mathfrak{t}$ and a root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus_{\alpha \in R} \mathfrak{g}_\alpha$. Then:

- (1) the elements of R span $\mathfrak{t}^*$
- (7) $\mathfrak{g}_\alpha$ are one-dimensional
- (2,12) if $\alpha, \beta \in R$ with $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$
- (3) " " " $\alpha + \beta \notin R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$
- (5,6) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional, and $\mathfrak{g}_\alpha + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] + \mathfrak{g}_{-\alpha}$ is isomorphic to $\mathfrak{sl}_2$
- (9) this copy of $\mathfrak{sl}_2$ acts on $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$, and this is an irreducible representation
- (4,8,13,15) there is an inner product on R such that, $\forall \alpha, \beta \in R$, $\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$, $\langle \alpha, \beta \rangle \in \mathbb{Q}$
- (3,11) if $\alpha \in R$, then $-\alpha \in R$, and no other multiple of $\alpha \in R$.
- (10) R is invariant under reflection: define $s_\alpha : \mathfrak{t}^* \to \mathfrak{t}^*$: $s_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \alpha$
  Then, for $\alpha, \beta \in R$, $s_\alpha \beta \in R$
- (3) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)_{ad} = 0$ $\forall \alpha \neq -\beta$

All these will follow from the discussion below.

1. If R does not span $\mathfrak{t}^*$, then take $t \in \mathfrak{t}$ whose dual $\in \mathfrak{t}^* \setminus R$
ie $\alpha(t) = 0$ $\forall \alpha \in R$. $\Rightarrow \forall x \in \mathfrak{g}_\alpha$, $[t, x] = \alpha(t) x = 0$ $\therefore [t, \mathfrak{g}] = 0 \Rightarrow t$ belongs to an abelian ideal.
This is a contradiction as $\mathfrak{g}$ is semi-simple.

2. Take $x \in g_\alpha$, $y \in g_\beta$, $t \in t$. $[t[x,y]] = [[t,x],y] + [x,[t,y]]$
$$= \alpha(t)[x,y] + \beta(t)[x,y] = (\alpha+\beta)t[x,y]$$
$$\therefore [g_\alpha, g_\beta] \subseteq g_{\alpha+\beta}$$

3. Take $z \in g_\gamma$. $(\mathrm{ad}\,x\,\mathrm{ad}\,y)\cdot z = [x[y,z]] \in g_{\gamma+\alpha+\beta}$ by (2)
$\therefore (\mathrm{ad}\,x\,\mathrm{ad}\,y)^N \cdot z \in g_{\gamma+N(\alpha+\beta)}$
if $\alpha+\beta \neq 0$, $\exists N$ with $(\mathrm{ad}\,x\,\mathrm{ad}\,y)^N$ a zero map (as g finite-dimensional)
$\therefore \mathrm{ad}\,x\,\mathrm{ad}\,y$ is a nilpotent map $\Rightarrow$ it has zero trace $\therefore (g_\alpha, g_\beta)_{ad} = 0$ if $\alpha \neq -\beta$.
Since $(\,,\,)_{ad}$ is non-degenerate, it is non-degenerate when restricted to $g_\alpha \oplus g_{-\alpha}$.
$-\alpha$ is actually a root: $(g_\alpha, g_\alpha)_{ad} = 0$ so the non-degenerate condition means $g_{-\alpha} \neq \{0\}$.

4. In particular, $(\,,\,)_{ad}$ is non-degenerate when restricted to t.
$\therefore$ for every linear function $\alpha \in t^*$, $\exists t_\alpha \in t$ with $\alpha(t) = (t_\alpha, t)_{ad}\ \forall t$.
then we can define a bilinear form on t^* (and therefore on R): $\langle\alpha,\beta\rangle = (t_\alpha, t_\beta)_{ad}$.

5. Take $x \in g_\alpha$, $y \in g_{-\alpha}$. $[x,y] \in t$ by (2).
$(t, [x,y])_{ad} = ([t,x],y)_{ad} = \alpha(t)(x,y)_{ad} = (t_\alpha, t)_{ad}(x,y)_{ad} \quad \forall t$
Since $(\,,\,)_{ad}$ is non-degenerate on t, this shows $[x,y] = (x,y)_{ad}t_\alpha$.

6. Now take any non-zero $e_\alpha \in g_\alpha$, and choose $e_{-\alpha} \in g_{-\alpha}$ with $(e_\alpha, e_{-\alpha})_{ad} \neq 0$.
$\Rightarrow [e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})_{ad} t_\alpha$, $[t_\alpha, e_{\pm\alpha}] = \pm\alpha(t_\alpha)e_{\pm\alpha} = \pm\langle\alpha,\alpha\rangle e_{\pm\alpha}$
$\therefore e_\alpha, e_{-\alpha}, t_\alpha$ span a sub-algebra - let this be m_α.
If $\langle\alpha,\alpha\rangle = 0$, $[m_\alpha, m_\alpha] = \mathrm{span}\, t_\alpha$, so m_α is solvable.
By Lie's theorem, $\mathrm{ad}\,m_\alpha$ is a collection of upper triangular matrices
$\Rightarrow \mathrm{ad}[m_\alpha, m_\alpha] = [\mathrm{ad}\,m_\alpha, \mathrm{ad}\,m_\alpha]$ = nilpotent endomorphisms
$\Rightarrow t_\alpha$ is a nilpotent endomorphism, and is diagonalisable (as $t_\alpha \in t$) $\Rightarrow t_\alpha = 0$.
Hence we must have $\langle\alpha,\alpha\rangle \neq 0$.
Set $h_\alpha = \frac{2t_\alpha}{\langle\alpha,\alpha\rangle}$ and rescale e_α so $(e_\alpha, e_{-\alpha})_{ad} = 2/\langle\alpha,\alpha\rangle$.
$\Rightarrow [e_\alpha, e_{-\alpha}] = h_\alpha$, $[h_\alpha, e_{\pm\alpha}] = \pm 2e_{\pm\alpha}$ and this is isomorphic to sl_2.

7. Suppose $\dim g_{-\alpha} > 1$.
$\mathrm{ad}\,e_\alpha$ sends $g_{-\alpha}$ to $\{\mathrm{span}\, t_\alpha\}$, so it must have a kernel. by (5).
ie $\exists v \in g_\alpha$ with $[e_\alpha, v] = 0$, $[h_\alpha, v] = -\alpha(h_\alpha)v = -2v$
$\Rightarrow v$ is a highest weight vector for sl_2 with weight -2.
g is finite dimensional so, by the representation theory of sl_2, this is impossible
$\therefore \dim g_{-\alpha} = 1$ $\therefore$ all root spaces are 1-dimensional.

8. let $q = \max\{k \in \mathbb{Z} : \beta + k\alpha \in R\}$. Take $v \in g_{\beta+q\alpha}$
$e_\alpha(v) = [e_\alpha, v] \in g_{\beta+(q+1)\alpha}$ $\therefore e_\alpha(v) = 0$ by construction of q
$h_\alpha(v) = (\beta+q\alpha)(h_\alpha)v = \frac{2\langle\alpha,\beta+q\alpha\rangle}{\langle\alpha,\alpha\rangle}v$ $\therefore v$ is a highest weight vector of weight $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} + 2q$
highest weights are integral $\therefore$ as $q \in \mathbb{Z}$, $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$.

9. From the representation theory of sl_2, precisely $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} + 2q + 1$ iterates of $\mathrm{ad}\,e_{-\alpha}$ is required to kill v.
$\therefore \{\beta + q\alpha - k\alpha : 0 \le k \le 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} + 2q\} \setminus \{0\}$ are all roots. We want to show there are no more roots of this form

let $p = \max\{k \in \mathbb{Z} : \beta - k\alpha \in R\}$, and take $w \in \mathfrak{g}_{\beta - p\alpha}$

$e_{-\alpha}(w) = 0$, $h_\alpha(w) = \left(\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} - 2p\right) w$

By representation theory of $\mathfrak{sl}_2$, $\beta - p\alpha$, $\beta - p\alpha + \alpha$, $\cdots$ $\beta - p\alpha + \left(2p - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\right)$ are all roots.

By definition, $q \geq p - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$, $p \geq q + \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ $\therefore p = q + \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ ie we have a single irreducible rep.

10. s_α just flips this string around e.g. sends $\beta + q\alpha$ to $\beta - p\alpha$ (since $s_\alpha(\alpha) = -\alpha$)
String runs from $\beta + q\alpha = \beta + \left(p + \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\right)\alpha$ (top) to $\beta - p\alpha = \beta - \left(q + \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\right)\alpha$ (bottom)
p, q are positive integers $\therefore -\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} - q \leq -\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \leq -\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} + p$ $\therefore \beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha$ lies on string.

11. Suppose $\alpha, k\alpha \in R$.
by 8, $\frac{2\langle\alpha,k\alpha\rangle}{\langle\alpha,\alpha\rangle}, \frac{2\langle\alpha,k\alpha\rangle}{\langle k\alpha,k\alpha\rangle} \in \mathbb{Z} \Rightarrow 2k, 2/k \in \mathbb{Z} \Rightarrow k = \pm 1, 2$.
by 3, $\mathfrak{g}_{2\alpha}$, $\mathfrak{g}_{-2\alpha}$ are both zero or both non-zero.
in the latter case, take $v \in \mathfrak{g}_{-2\alpha} \Rightarrow e_\alpha(v) \in \mathfrak{g}_{-\alpha} \Rightarrow (e_{-\alpha}, [e_\alpha, v])_{ad} = 0$
$(e_{-\alpha}, [e_\alpha, v])_{ad} = ([e_{-\alpha}, e_\alpha], v)_{ad} = 0$.
but the killing form is non-degenerate on $\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$ (3), so this means $e_\alpha(v) = 0$
$h_\alpha(v) = (-2\alpha)(h_\alpha)v = -4v$
$\therefore v$ is a highest weight vector with highest weight -4, which is impossible $\therefore \mathfrak{g}_{2\alpha} = \mathfrak{g}_{-2\alpha} = 0$.

12. Suppose α, β are roots with $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0 \Rightarrow e_\alpha(v) = 0$ $\forall v \in \mathfrak{g}_\beta$; $h_\alpha(v) = 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} v$
Take $w \in \mathfrak{g}_{\alpha+\beta} \Rightarrow h_\alpha(w) = \left(2 + 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\right)w$. But v is a highest weight vector, and $\oplus_k \mathfrak{g}_{\beta + k\alpha}$ consists of a single irreducible representation of $\mathfrak{m}_\alpha$ $\therefore w = 0$
$\therefore$ if $\mathfrak{g}_{\alpha+\beta} \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$. Since root spaces are 1-dimensional, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

13. Work in the basis $\{e_\alpha, \alpha \in R\}$. For any $t \in \mathfrak{t}$, the diagonal matrix $\mathrm{ad}\, t$ has entries $\alpha(t)$ for every $\alpha \in R$, and $\dim \mathfrak{t}$ 0s.

$\therefore \forall \alpha, \beta \in R$, $\langle\alpha,\beta\rangle = (t_\alpha, t_\beta) = \mathrm{tr}(\mathrm{ad}\, t_\alpha \cdot \mathrm{ad}\, t_\beta) = \sum_{\lambda \in R} \lambda(t_\alpha)\lambda(t_\beta)$
$= \sum_{\lambda \in R} \langle\lambda,\alpha\rangle\langle\lambda,\beta\rangle$
$\therefore \langle\beta,\beta\rangle = \sum_{\lambda \in R} \langle\lambda,\beta\rangle^2 = \langle\beta,\beta\rangle^2 \sum_{\lambda \in R}\left(\frac{\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle}\right)^2$

$\therefore 4/\langle\beta,\beta\rangle = \sum_{\lambda \in R}\left(\frac{2\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle}\right)^2 \in \mathbb{Z} \Rightarrow \langle\beta,\beta\rangle \in \mathbb{Q} \Rightarrow \langle\alpha,\beta\rangle = \langle\beta,\beta\rangle\left(\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\right)\frac{1}{2} \in \mathbb{Q}$.

14. Choose a basis $\beta_1, \beta_2, \cdots \beta_l$ for $\mathfrak{t}^*$. wlog β_i are roots.
Let B be the matrix $\langle\beta_i, \beta_j\rangle$ $\therefore B$ has rational entries.
$B\underline{x} = 0 \Rightarrow \sum_j \langle\beta_i,\beta_j\rangle x_j = 0 \Rightarrow \sum_j \beta_j x_j \perp \mathfrak{t}^* \Rightarrow \sum_j \beta_j x_j = 0$ as $\langle\, ,\rangle$ non-degenerate on $\mathfrak{t}^*$.
as β_i form a basis, this forces $x_i = 0$ $\forall i$ $\therefore \ker B = \{0\} \Rightarrow B$ invertible
Now take any $\beta = \sum c_i\beta_i \in R$. $\langle\beta, \beta_j\rangle = \sum c_i \langle\beta_i,\beta_j\rangle = (B\underline{c})_j$ $\Rightarrow \underline{c} = B^{-1}(\langle\beta,\beta_j\rangle)$ has entries in $\mathbb{Q}$.

15. Take $\lambda \in$ rational span of roots $\Rightarrow \lambda = \sum c_i\beta_i$ with $c_i \in \mathbb{Q}$ by above.
$(\forall \beta \in R)$ $\Rightarrow \langle\lambda,\beta\rangle = \sum c_i \langle\beta_i,\beta\rangle \in \mathbb{Q}$ since $\langle\, ,\rangle \in \mathbb{Q}$ when restricted to roots
$\therefore \langle\lambda,\lambda\rangle = \sum_{\beta \in R} \langle\lambda,\beta\rangle^2 \geq 0$ (sum of rational squares)
with equality if and only if $\langle\lambda,\beta\rangle = 0$ $\forall \beta \in R \Rightarrow \lambda = 0$ as $\langle\, ,\rangle$ non-degenerate on $\mathbb{R}R$
$\therefore \langle\, ,\rangle$ does define an inner product.

Root Systems

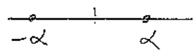
Let V be a vector space with an inner product $\langle\ ,\ \rangle$ (over $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$)
Given any $\alpha \in V$, define $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ $\quad \therefore \langle\alpha,\alpha^\vee\rangle = 2$
and $s_\alpha: V \to V$ $\quad s_\alpha(v) = v - \langle v,\alpha^\vee\rangle\alpha$ (clearly this is a linear map)
s_α is a reflection in the hyperplane $\perp \alpha$: $\forall v \in$ this hyperplane, $\langle v,\alpha^\vee\rangle = 0$
$\therefore s_\alpha$ preserves v. The complement of this hyperplane is spanned by α, with $s_\alpha(\alpha) = -\alpha$

A root system $R \subseteq V$ is a finite set such that, $\forall \alpha,\beta \in R$:
 R spans V
 $\langle\alpha,\beta^\vee\rangle \in \mathbb{Z}$
?? $s_\alpha(R) = R$, or, equivalently, $\{\beta + k\alpha \in R : k \in \mathbb{Z}\} = \{\beta - p\alpha, \beta-(p-1)\alpha, \cdots \beta+q\alpha\} \setminus \{0\}$ where $p - q = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$
it is reduced if $\alpha, k\alpha \in R \Rightarrow k = \pm 1$ (the converse always holds, since $s_\alpha(\alpha) = -\alpha$)

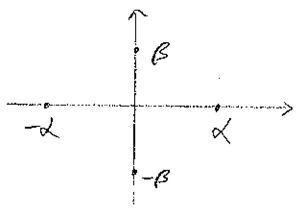
The associated Weyl group W is group generated by $\{s_\alpha : \alpha \in R\}$
By definition, W permutes R and any elements fixing R must fix V $\therefore$ this action is faithful $\therefore$
 $\Rightarrow W \leq \mathrm{Sym}_{|R|} \Rightarrow W$ is finite.
Rank $R = \dim V = \#$ of linearly independent elements in R.

If (R,V) and (R',V') are root systems, their direct sum $(R \amalg R', V \amalg V')$ is a root system
A root system which is not a non-trivial direct sum is irreducible. Otherwise it is decomposable into orthogonal R, R', and V, V' are invariant under W.
An isomorphism of root systems is a bijective linear map (not necessarily isometric, so rescaling allowed) of the corresponding vector spaces, mapping the root systems to each other.

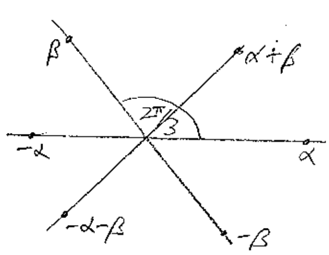
Examples: In rank 1, there is only one reduced root system, given by $R = \{\alpha, -\alpha\}$ $(\alpha \neq 0)$
This is A_1, and its associated Weyl group is $\mathbb{Z}/2\mathbb{Z}$.



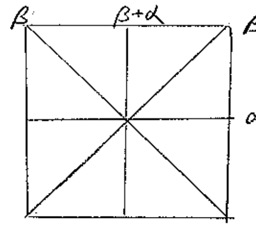
In rank 2, $A_1 \times A_1$ is a decomposable root system, with associated Weyl group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



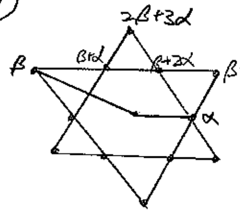
A_2 consists of the roots shown, scaled such that
$\langle\alpha,\alpha\rangle = \langle\beta,\beta\rangle = 2$, $\langle\alpha,\beta\rangle = -1$
The associated Weyl group is S_3
This is irreducible as the Weyl group does not leave any line invariant.



B_2 consists of the eight roots shown, scaled so that
$\langle\alpha,\alpha\rangle=1$, $\langle\beta,\beta\rangle=2 \Rightarrow \alpha^\vee=2\alpha$, $\beta^\vee=\beta$
$\langle\alpha,\beta^\vee\rangle=-1$, $\langle\beta,\alpha^\vee\rangle=2\langle\beta,\alpha\rangle=-2$
Again, this is irreducible. Associated Weyl group is D_8



G_2 consists of the twelve roots shown (including 6 unlabelled)
$\langle\alpha,\alpha\rangle=2$, $\langle\beta,\beta\rangle=6 \Rightarrow \alpha^\vee=\alpha$, $\beta^\vee=\frac{1}{3}\beta$.
$\langle\alpha^\vee,\beta\rangle=\sqrt{2}\sqrt{6}\frac{\sqrt{3}}{2}=-3$, $\langle\alpha,\beta^\vee\rangle=-1$
This is irreducible.
Associated Weyl group is D_{12}



Observe that $\langle\beta-\langle\beta,\alpha^\vee\rangle\alpha, \beta-\langle\beta,\alpha^\vee\rangle\alpha\rangle = \langle\beta,\beta\rangle+\langle\beta,\alpha^\vee\rangle^2\langle\alpha,\alpha\rangle-2\langle\beta,\alpha\rangle\langle\beta,\alpha^\vee\rangle=\langle\beta,\beta\rangle$
and $(\lambda\alpha)^\vee=\frac{2\lambda\alpha}{\langle\lambda\alpha,\lambda\alpha\rangle}=\frac{1}{\lambda}\alpha^\vee \Rightarrow s_{\lambda\alpha}(v)=v-\langle v,(\lambda\alpha)^\vee\rangle(\lambda\alpha)=s_\alpha(v)$
$\therefore s_{\alpha^\vee}(\beta^\vee)=\beta^\vee-\langle\beta^\vee,\alpha^\vee\rangle\alpha=\frac{2}{\langle\beta,\beta\rangle}(\beta-\langle\beta,\alpha^\vee\rangle\alpha)=(\beta-\langle\beta,\alpha^\vee\rangle\alpha)^\vee$
$\therefore R^\vee$ is also a root system.
By linearity of s_α, rescaling R creates an isomorphic root system.

lemma: The above shows all the rank 2 reduced root systems.
Proof: Take $\alpha,\beta\in$ rank 2 root system. Rescale so $\langle\alpha,\alpha\rangle=2$ ie $\alpha^\vee=\alpha$. let $\theta=$ angle between α,β.
$\frac{4\langle\alpha,\alpha\rangle}{\langle\beta,\beta\rangle}>\left(\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\right)^2\geq 1 \Rightarrow \langle\beta,\beta\rangle<8$, and $\langle\alpha,\beta\rangle=\langle\alpha^\vee,\beta\rangle\in\mathbb{Z}$
$\langle\alpha,\beta\rangle^2\leq\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle<16 \quad \therefore \langle\alpha,\beta\rangle=0,\pm1,\pm2,\pm3.$
• $\langle\alpha,\beta\rangle=0 \Rightarrow A_1\times A_1$ (we can rescale α,β separately.
$\Longrightarrow$ $s_\beta\cdot s_\alpha=$ rotation through 2θ. If α,β are simple, $s_\beta\cdot s_\alpha(\alpha)$, $s_\beta\cdot s_\alpha(\beta)=\mathbb{Z}$-combinations of α,β.
matrix representing $s_\beta\cdot s_\alpha$ has same trace in standard basis and α,β basis $\Rightarrow 2\cos 2\theta\in\mathbb{Z}$
$\Rightarrow \theta=\frac{3\pi}{2},\frac{5\pi}{6},\frac{2\pi}{3}$ (as $\theta>\pi$)

lemma: let g be a semisimple Lie algebra.
$g=g'\oplus g'' \Leftrightarrow R=R'\perp\!\!\!\perp R''$
In particular, g is simple $\Leftrightarrow$ the corresponding root system is irreducible
Proof: $\Rightarrow$: set $R'=\{\alpha: g_\alpha\subseteq g'\}$, $R''=\{\beta: g_\beta\subseteq g''\}$
$g_{\alpha+\beta}=[g_\alpha,g_\beta]\subseteq g'\cap g''=\{0\}$. $\therefore \alpha+\beta$ is not a root. As $-\alpha\in R'$ also, $\beta-\alpha$ is not a root.
Since roots of the form $\beta+k\alpha$ form a continuous string, the only possible value of k is 0.
$s_\alpha(\beta)=\beta+k\alpha$ for some $k \Rightarrow s_\alpha$ fixes $\beta \Rightarrow s_\alpha$ permutes R', fixes R'' pointwise. $R'\perp R''$.
$\Leftarrow$: set $g'=\bigoplus_{\alpha\in R'} g_\alpha\oplus\operatorname{span} t_\alpha$, $g''=\bigoplus_{\beta\in R''} g_\beta\oplus\operatorname{span} t_\beta$.
g', g'' are clearly subalgebras. To show they are ideals, we must show $[g_\alpha,g_\beta]=0$
$\langle\alpha+\beta,\alpha\rangle=\langle\alpha,\alpha\rangle\neq0$, $\langle\alpha+\beta,\beta\rangle=\langle\beta,\beta\rangle\neq0 \Rightarrow \alpha+\beta\notin R''$ or $R' \Rightarrow \alpha+\beta$ not a root

R is simply laced if all roots have the same length.
e.g. A_1, A_2, $A_1\times A_1$ are simply laced; B_2, G_2 are not.
Any simply laced root system is isomorphic (via rescaling) to one where $\langle\alpha,\alpha\rangle=2$ $\forall\alpha\in R$.

We look for root systems by examining lattices:

A lattice L is a finitely generated free abelian group with bilinear form: $\langle\,,\,\rangle : L\times L\to\mathbb{Z}$ which is a positive definite inner product when extended to the $\mathbb{R}$-span of L.
L is even if $\langle l,l\rangle\in 2\mathbb{Z}$ $\forall l\in L$.

The roots of L are $R_L=\{l\in L : \langle l,l\rangle=2\}=\{l\in L : l=l^\vee\}$
R_L is a finite set since it is the intersection of the compact set $\{l\in\mathbb{R}L : \langle l,l\rangle=2\}$ with the discrete set L.
For α,β roots, $\langle s_\alpha\beta, s_\alpha\beta\rangle=\langle\beta,\beta\rangle-2\langle\beta,\alpha\rangle\langle\beta,\alpha\rangle+\langle\beta,\alpha\rangle^2\langle\alpha,\alpha\rangle=2$
$\therefore s_\alpha$ sends R_L to itself $\Rightarrow R_L$ is a root system in $\mathbb{R}R_L$, and is simply laced.

L is generated by roots if $\mathbb{Z}R_L=L$.
$\therefore \forall l\in L$, $l=\sum a_i\alpha_i$ where $\alpha_i\in R_L$, $a_i\in\mathbb{Z}$ $\Rightarrow \langle l,l\rangle=\langle\alpha_i,\alpha_i\rangle+\sum_{i\neq j}2a_ia_j\langle\alpha_i,\alpha_j\rangle\in 2\mathbb{Z}$
$\therefore L$ is even.

Example: Consider $\mathbb{Z}^{n+1}=\mathbb{Z}$-span of $e_1,e_2,\cdots e_{n+1}$, $\langle e_i,e_j\rangle=\delta_{ij}$ (a square lattice)
Put $L=\{l\in\mathbb{Z}^{n+1} : \langle l, e_1+\cdots+e_{n+1}\rangle=0\}=\{\sum a_ie_i : a_i\in\mathbb{Z}, \sum a_i=0\}\cong\mathbb{Z}^n$ (subgp of free gp)
$R_L=\{e_i-e_j : i\neq j\}$ and this generates the lattice. This is A_n. $(n\geq 1)$
Number of roots $=n(n+1)$
$s_{e_i-e_j}(\sum x_ie_i)=\sum x_ie_i-(x_i-x_j)(e_i-e_j)$
$=x_1e_1+\cdots+x_je_i+\cdots+x_ie_j+\cdots x_ne_n$ ie swap x_i and x_j.
$\therefore$ associated Weyl group $=$ generated by transpositions $=S_{n+1}$
This acts transitively on R_L $\therefore A_n$ are irreducible $\forall n$.

Consider $\mathbb{Z}^n=\mathbb{Z}$-span of $e_1,e_2,\cdots e_n$. $\langle e_i,e_j\rangle=\delta_{ij}$ (a square lattice)
$R_L=\{\pm e_i\pm e_j : i\neq j\}$ $\mathbb{Z}R_L=\{l=\sum a_ie_i : a_i\in\mathbb{Z}, \sum a_i\in 2\mathbb{Z}\}$. This is D_n. $(n>1)$.
Number of roots $=2n(n-1)$
$s_{e_i+e_j}(\sum x_ie_i)=\sum x_ie_i-(x_i+x_j)(e_i+e_j)$
$=x_1e_1+\cdots-x_je_i+\cdots-x_ie_j+\cdots x_ne_n$ ie transposition + sign change
$\therefore$ associated Weyl group $=$ generated by transpositions and an even number of sign changes
$=(\mathbb{Z}/2\mathbb{Z})^{n-1}\rtimes S_n$ ($(\mathbb{Z}/2\mathbb{Z})^{n-1}$ normal, intersect S_n trivially)
D_2 has roots $\{e_1+e_2, e_1-e_2, -e_1-e_2, -e_1+e_2\}\cong$ roots of $A_1\times A_1$: $f_1=e_1+e_2$, $f_2=e_1-e_2$.
D_3 and A_3 have the same roots after rewriting e_4 (in A_3) as $e_1+e_2+e_3$.
(can check inner product preserved ie (e_i-e_4, e_j-e_k) and (e_i-e_4, e_i-e_k))
$\therefore$ we usually consider D_n, $n\geq 4$. Then the Weyl group acts transitively on roots $\therefore$ irreducible

Let $T_n=\{(k_1,k_2,\cdots k_n) : \sum k_i\in 2\mathbb{Z}\,;$ all $k_i\in\mathbb{Z}$ or all $k_i\in\mathbb{Z}+\tfrac12\}\subseteq\{(k_1,\cdots k_n) : k_i\in\mathbb{Z}[\tfrac12]\}$
$R_{T_{8n}}=R_{D_{8n}}$ for $n>1$: $\forall\alpha\in[T_{8n}\cap(k_i\in\mathbb{Z}+\tfrac12)]$, $\langle\alpha,\alpha\rangle\geq 8n(\tfrac14)=2n>2$.
$R_{T_8}=\{\pm e_i\pm e_j : i\neq j\}\cup\{\tfrac12(\pm e_1\pm\cdots\pm e_8) :$ even number of minus signs$\}=E_8$
$\therefore$ number of roots $=2(8)(7)+2^7=240$.
(Observe T_{8n} is indeed a lattice: $(\tfrac12,\tfrac12,\cdots\tfrac12)\cdot(k_1,k_2\cdots k_n)=\tfrac12\sum k_n\in\mathbb{Z}$, so, by linearity, all inner products are indeed integers)

Observe that, for $\alpha \in R$ a root system, $\alpha^\perp \cap R$ is a root system (in the subspace of $\alpha^\perp$; not necessarily all of $\alpha^\perp$):
$\forall \beta,\gamma \in \alpha^\perp \cap R$, $s_\beta(\gamma) \in R \cap \mathrm{span}\,\beta,\gamma \subseteq \alpha^\perp \cap R$.

∴ Define $\alpha = \frac{1}{2}(e_1+e_2+\cdots+e_8)$, $\beta = e_7+e_8$, $\alpha,\beta \in E_8$.
E_7 has roots $\alpha^\perp \cap R_{E_8} = \{e_i - e_j : i \neq j\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm \cdots \pm e_8) : \text{four minus signs}\}$
E_6 has roots $\alpha^\perp \cap \beta^\perp \cap R_{E_8} = \{e_i - e_j : i \neq j,\ i,j \neq 7,8\} \cup \{e_7 - e_8, e_8 - e_7\}$
$\cup \{\frac{1}{2}(\pm e_1 \pm e_2 \cdots + e_7 - e_8) : \text{four minus signs in total}\}$
$\cup \{\frac{1}{2}(\pm e_1 \pm e_2 \cdots - e_7 + e_8) : \text{four minus signs in total}\}$
∴ E_7 has $8(7)+\binom{8}{4} = 126$ roots, E_6 has $6(5)+2+\binom{6}{3}+\binom{6}{3} = 72$ roots

B_n has roots $\{\pm e_i\} \cup \{\pm e_i \pm e_j : i \neq j\} \subseteq \mathbb{Z}^n$. This is a root system:
$\langle \pm e_i \pm e_j, e_k^\vee \rangle = 2\langle \pm e_i \pm e_j^\vee, e_k \rangle = 2(\pm\delta_{ik} \pm \delta_{jk}) \in \mathbb{Z}$
$s_{e_i - e_j}$ = transposition of i,j co-ordinates
$s_{e_i + e_j}$ = transposition and sign-change of i,j co-ordinates (c.f. D_n)
s_{e_i} = sign-change of i co-ordinate
all these preserve R_{B_n}. Number of roots $= 2n + 2n(n-1) = 2n^2$
∴ associated Weyl group = generated by transpositions and sign changes $= (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$

Define C_n to have roots $R^\vee_{B_n} = \{\pm 2e_i\} \cup \{\pm e_i \pm e_j : i \neq j\} \subseteq \mathbb{Z}^n$
∴ it has the same Weyl group as B_n, and again $2n^2$ roots

Let $Q = \{(k_1,\cdots,k_4) : \text{all } k_i \in \mathbb{Z}, \text{ all } k_i \in \frac{1}{2}+\mathbb{Z}\} \subseteq \mathbb{Z}[\frac{1}{2}]^4$
F_4 has roots $\{\alpha \in Q : \langle \alpha,\alpha \rangle = 1 \text{ or } 2\} = \{\pm e_i\} \cup \{\pm e_i \pm e_j : i \neq j\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$
∴ $8 + 4\binom{4}{2} + 2^4 = 48$ roots.

Recall G_2, with its 12 roots being the vertices of ✡, and Weyl group D_{12}.

Choose a linear function $f : V \to \mathbb{R}$ such that $f(\alpha) \neq 0\ \forall \alpha \in R$
ie f measures the height of $v \in V$ from a hyperplane which does not contain any of the roots.
Let $R^+ = \{\alpha \in R : f(\alpha) > 0\}$, $R^- = \{\alpha \in R : f(\alpha) < 0\}$ ∴ $R = R^+ \cup R^-$, $R^+ \cap R^- = \emptyset$
ie R^+, R^- are roots lying on one side and the other side of the hyperplane.

$\alpha \in R$ is simple if $\alpha \in R^+$ and it is not the sum of 2 other positive roots.
Let Π = set of simple roots ∴ $\Pi \subseteq R^+$, Π is not unique

Example: A_n: set $f(e_i) = n+2-i$, $f(e_i - e_j) = j - i$ ∴ $R^+ = \{e_i - e_j : i < j\}$ (recall $1 \leq i \leq n+1$)
This is the root system of sl_n ∴ positive weight spaces of sl_n are span A_{ij} where $i<j$
ie upper triangular matrices.
$f(R) \subseteq \mathbb{Z}$ ∴ if $f(\alpha) = 1$, α must be simple $\Rightarrow \{e_1 - e_2, e_2 - e_3, \cdots e_{n-1} - e_n\} \subseteq \Pi$.
Since the $\mathbb{Z}^+$ span of these cover R^+, this is all of Π.

B_n: $f(e_i)=n+1-i \Rightarrow R^+=\{e_i\}\cup\{e_i+e_j\}\cup\{e_i-e_j : i<j\}$
This is the root system of so_{2n+1}. positive weight spaces = span of
$\{A_{ij} : 2n+2-j<i\leq n$ or $n+2\leq i<j$ or $i=n+1<j\}$ = upper triangular matrices
Again, $f(R)\subseteq \mathbb{Z}$ $\therefore \{e_1-e_2, e_2-e_3, \cdots e_{n-1}-e_n, e_n\}=f^{-1}(1)\cap R\subseteq \Pi$.
$\mathbb{Z}^+$ span of this set $=R^+$ $\therefore$ this is all of Π.

C_n: $f(e_i)=n+1-i \Rightarrow R^+=\{2e_i\}\cup\{e_i+e_j\}\cup\{e_i-e_j : i<j\}$
root system of sp_{2n}. positive weight spaces are span of $\{A_{ij} : 2n+1-j<i\leq n\}$
and $\{A_{ij} : n<i\leq j\}$ and $\{A_{i} : i>n\}$ = upper triangular matrices
$f(R)\subseteq \mathbb{Z}$ $\therefore \{e_1-e_2, e_2-e_3, \cdots e_{n-1}-e_n\}=f^{-1}(1)\cap R\subseteq \Pi$.
$\mathbb{Z}^+$ span of this set $\cup\, 2e_n = R^+$, and $2e_n \neq$ sum of 2 positive roots.
$\therefore \Pi=\{e_1-e_2, e_2-e_3, \cdots e_{n-1}-e_n, 2e_n\}$.

D_n: $f(e_i)=n+1-i \Rightarrow R^+=\{e_i+e_j\}\cup\{e_i-e_j : i<j\}$
root system of so_{2n}. positive weight spaces $=\{A_{ij} : n<i<j\}\cup\{A_{ij} : 2n+1-j<i<n\}$
= upper triangular matrices
$f(R)\subseteq \mathbb{Z} \Rightarrow f^{-1}(1)\cap R=\{e_1-e_2, e_2-e_3, \cdots e_{n-1}-e_n\}\subseteq \Pi$.
$e_{n-1}+e_n \neq$ sum of 2 positive roots; $\mathbb{Z}^+$ span of $f^{-1}(1)\cap R\cup e_{n-1}+e_n=R^+$
$\therefore \Pi=\{e_1-e_2, e_2-e_3, \cdots, e_{n-1}-e_n, e_{n-1}+e_n\}$

E_8: $f(e_1)=28$, $f(e_i)=9-i$ for $2\leq i\leq 8$. $f(R)\subseteq \mathbb{Z}$.
$\therefore f^{-1}(1)\cap R=\{e_2-e_3, e_3-e_4, \cdots e_7-e_8, \tfrac{1}{2}(e_1-e_2-e_3\cdots-e_7+e_8)\}\subseteq \Pi$
$f^{-1}(2)\cap R=\{e_2-e_4, e_3-e_5, \cdots e_6-e_8, \tfrac{1}{2}(e_1-e_2-e_3\cdots-e_6+e_7-e_8)\}$
and these are all in $\mathbb{Z}^+$ span of $f^{-1}(1)\cap R$ $\therefore$ not in Π.
$f^{-1}(3)\cap R=\{e_2-e_5, e_3-e_6, \cdots, e_5-e_8, \tfrac{1}{2}(e_1-e_2-e_3\cdots-e_5+e_6-e_7-e_8)\}\cup\{e_7+e_8\}$
the first set lie in $\mathbb{Z}^+$ span of $f^{-1}(1)\cap R$ $\therefore$ not simple. But e_7+e_8 does not
$\therefore e_7+e_8\in\Pi$.
$R^+=\{e_i+e_j\}\cup\{e_i-e_j : i<j\}\cup\{\tfrac{1}{2}(e_1\pm e_2\pm\cdots\pm e_8)$: even number of minus signs$\}$
$\subseteq \mathbb{Z}^+$ span of $f^{-1}(1)\cap R\cup\{e_7+e_8\}$ $\therefore$ this is all of Π.

E_7: just restrict f. Now $R^+=\{e_i-e_j : i<j\}\cup\{\tfrac{1}{2}(e_1\pm e_2\pm\cdots\pm e_8)$: 4 minus signs$\}$
$f^{-1}(1)\cap R=\{e_2-e_3, e_3-e_4, \cdots e_7-e_8\}\subseteq\Pi$
$f^{-1}(\{2,3,4,5\})\cap R\subseteq \mathbb{Z}^+$ span of $f^{-1}(1)\cap R$ $\therefore$ these $\notin\Pi$.
$f^{-1}(6)\cap R=\{e_2-e_8, \tfrac{1}{2}(e_1-e_2-e_3\cdots-e_5+e_6+e_7+e_8)\}$, $e_2-e_8\in$ span of $f^{-1}(1)\cap R$
and $\tfrac{1}{2}(e_1-e_2-e_3\cdots-e_5+e_6+e_7+e_8)\notin$ span of $f^{-1}(1)\cap R$
$\therefore \tfrac{1}{2}(e_1-e_2-e_3\cdots-e_5+e_6+e_7+e_8)\in\Pi$
These 7 vectors $\subseteq\Pi$ span R^+ (over $\mathbb{Z}^+$) $\therefore$ this is all of Π.

E_6: restrict f again. Now $R^+=\{e_i-e_j : i<j\leq 6\}\cup\{e_7-e_8\}$
$\cup\{\tfrac{1}{2}(e_1\pm e_2\pm\cdots\pm e_6+e_7-e_8)$: 4 minus signs$\}$
$\cup\{\tfrac{1}{2}(e_1\pm e_2\pm\cdots\pm e_6-e_7+e_8)$: 4 minus signs$\}$.
$f^{-1}(1)\cap R=\{e_2-e_3, \cdots e_5-e_6, e_7-e_8\}\subseteq\Pi$

$f^{-1}(\{2,3,4,5,6,7\}) \cap R \subseteq \{e_i - e_j\} \subseteq$ span of $f^{-1}(1) \cap R$.
$f^{-1}(8) \cap R = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 + e_5 + e_6 - e_7 + e_8) \notin$ span of $f^{-1}(1) \cap R$
$\therefore \frac{1}{2}(e_1 - e_2 - e_3 - e_4 + e_5 + e_6 - e_7 + e_8) \in \Pi$
and these 6 vectors span R^+ (over $\mathbb{Z}^+$) $\therefore$ this is all of Π.

F_4: $f(e_1) = 8$, $f(e_i) = 5 - i$ for $2 \le i \le 4$ $\therefore f(R) \subseteq \mathbb{Z}$.
$R^+ = \{e_i\} \cup \{e_i - e_j : i < j\} \cup \{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$
$f^{-1}(1) \cap R = \{e_2 - e_3, e_3 - e_4, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\} \subseteq \Pi$ and this is all of Π since their $\mathbb{Z}^+$ span $\supseteq$ all of R^+

G_2: the labelled vertices are R^+. $\Pi = \{\alpha, \beta\}$ by checking definition of Π directly.

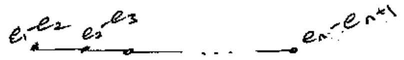
Properties of simple roots:

1. if $\alpha, \beta \in \Pi$, then $\alpha - \beta \notin R$: if $\alpha - \beta \in R^+$, then $\alpha = \alpha - \beta + \beta$ = sum of 2 positive roots
   if $\alpha - \beta \in R^-$, then $\beta - \alpha \in R^+$, $\beta = \beta - \alpha + \alpha$ = sum of 2 positive roots.
2. if $\alpha \ne \beta$, $\alpha, \beta \in \Pi$, then $\langle \alpha, \beta^\vee \rangle \le 0$: $s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta \in R$.
   since $\alpha - \beta \notin R$, the β-string-through-α stops at α $\therefore \langle \alpha, \beta^\vee \rangle \le 0$
3. $\forall \alpha \in R^+$, $\alpha = \sum_{\alpha_i \in \Pi} k_i \alpha_i$ with $k_i \in \mathbb{Z}^+$: if $\alpha \notin \Pi$, $\alpha = \beta + \gamma$ with $\beta, \gamma \in R^+$. Repeat this for β, γ.
   this process must terminate because, by discreteness of R^+, $\exists x = \min\{f(\alpha) : \alpha \in R^+\}$, and α is the sum of at most $\lfloor f(\alpha)/x \rfloor$ positive roots
4. simple roots form a basis for V: by above, they span R
   suppose $\sum_i a_i \alpha_i = 0$. Write this as $\sum_i a_i \alpha_i = \sum_j a_j \alpha_j$ where $a_i, a_j > 0$.
   then $\langle \sum_i a_i \alpha_i, \sum_j a_j \alpha_j \rangle = \sum_{i,j} a_i a_j \langle \alpha_i, \alpha_j \rangle \le 0$ which can only hold if $a_i = 0$ $\forall i$ (length ≥ 0) $\therefore \alpha_i$ are linearly independent.
5. if $\alpha \in R^+ \setminus \Pi$, then $\exists \alpha_i \in \Pi$ with $\alpha - \alpha_i \in R^+$: $0 < \langle \alpha, \alpha \rangle = \sum k_i \langle \alpha, \alpha_i \rangle$ $\therefore \exists \alpha_i$ with $\langle \alpha, \alpha_i \rangle > 0$.
   then $s_{\alpha_i}(\alpha) = \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i \in R$. Since strings are uninterrupted, $\alpha - \alpha_i \in R$. $\alpha - \alpha_i$ has at least one positive α_j coefficient $\therefore \in R^+$.
6. R is a decomposable root system $\Leftrightarrow \Pi = \Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$.
   if $R = R_1 \sqcup R_2$, take Π_i to be the simple roots of R_i. $\therefore \Pi_1 \perp \Pi_2$, Π_1, Π_2 are disjoint.
   As R_1, R_2 are orthogonal and disjoint, any decomposition of positive roots into sums of positive roots happen entirely inside R_1 or R_2 $\therefore \Pi_1 \sqcup \Pi_2 = \Pi$.
   if $\Pi = \Pi_1 \sqcup \Pi_2$, $\Pi_1 \perp \Pi_2$, then take $R_i = \text{span}\, \Pi_i \cap R$. Then $R_1 \perp R_2$.
   $\forall \alpha \in R$, $\alpha = \sum k_i \alpha_i + \sum k_j \alpha_j$ with $\alpha_i \in \Pi_1$, $\alpha_j \in \Pi_2$. Wlog, $\alpha \in R^+$ (otherwise use $-\alpha$) $\Rightarrow k_i, k_j \ge 0$.
   Take $\alpha_1 \in \Pi_1$ with $k_1 \ne 0$. $s_{\alpha_1}(\alpha) = -k_1 \alpha_1 + \sum_{i \ne 1} (k_i - 2\langle \alpha_i, \alpha_1^\vee \rangle) \alpha_i + \sum k_j \alpha_j$
   If k_j are not all zero, these coefficients have mixed signs, which cannot happen $\therefore R = R_1 \sqcup R_2$.

The Cartan matrix of a root system R is a $\dim V \times \dim V$ matrix with ij entry $\langle \alpha_i, \alpha_j^\vee \rangle$ where $\{\alpha_i\} = \Pi$.
Observe that: $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, $a_{ij} \le 0$ if $i \ne j$, $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$, and $\det A > 0$ $\quad (*)$
as $A = \begin{pmatrix} \frac{2}{\langle \alpha_1, \alpha_1 \rangle} & & 0 \\ & \ddots & \\ 0 & & \frac{2}{\langle \alpha_n, \alpha_n \rangle} \end{pmatrix} (\langle \alpha_i, \alpha_j \rangle)$ and the second matrix describes a +ve definite inner product, so has all eigenvalues positive.

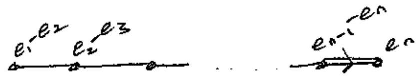
Each Cartan matrix can be represented visually by a Dynkin diagram:
each vertex corresponds to a simple root
α_i, α_j are joined by $a_{ij}a_{ji}$ edges. If α_i, α_j are roots of different lengths, put an arrow in the direction of the shorter root.

Example: A_n has Cartan matrix $\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & \ddots & \ddots \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}$



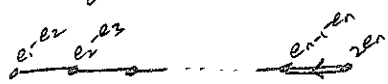
by induction and column expansion, we see determinant $= n+1$

B_n: $\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & \ddots & -1 \\ & & \ddots & \ddots & -2 \\ & & & -1 & 2 \end{pmatrix}$



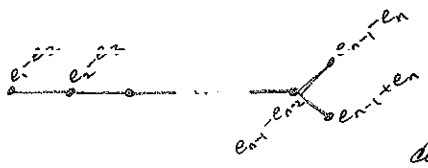
determinant $= 2$ (using A_n result)

Dualising the root system (rescaling α_i to $\alpha_i^\vee$) transposes the Cartan matrix.
$\therefore a_{ij}a_{ji}$ is unchanged $\Rightarrow$ same Dynkin diagram, but with arrows reversed since the scaling is reciprocal
$\therefore C_n$ has Dynkin diagram



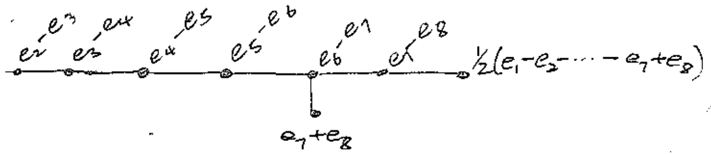
and same determinant

D_n: $\begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & & -1 & -1 \\ & & -1 & 2 & 0 \\ & & -1 & 0 & 2 \end{pmatrix}$



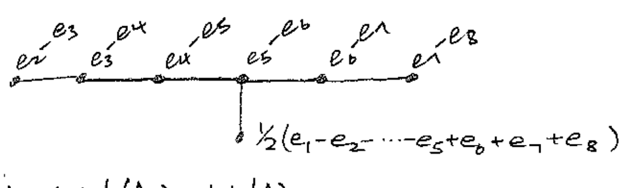
determinant $= 4$ (use A_n result)

E_8: $\begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & & -1 & \\ & & -1 & 2 & 0 \\ & & -1 & 0 & 0 & 2 \end{pmatrix}$



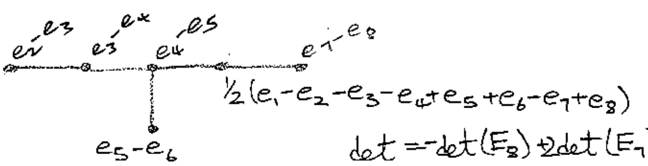
determinant $= 2\det(A_7) - 4\det(A_4) + \det(A_4) = 1$

E_7: $\begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & -1 \\ & \ddots & & -1 & 0 \\ & & -1 & 2 & 0 \\ & -1 & 0 & 0 & 2 \end{pmatrix}$



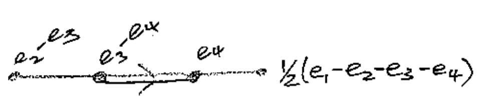
determinant $= 2\det(A_6) - 4\det(A_3) + \det(A_3) = 2$

E_6: $\begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & -1 & -1 \\ & & -1 & 2 & 0 \\ & -1 & 0 & 0 & 2 \end{pmatrix}$



$\det = -\det(E_8) + 2\det(E_7) = 3$

F_4: $\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -2 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$



determinant $= 1$

G_2: $\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}$ α β determinant $= 1$

Define an abstract Cartan matrix to be those satisfying $*$, and related abstract Dynkin diagrams.
Observe that any principal subminor (throw away some rows and corresponding columns) of a Cartan matrix from a root system remains an abstract Cartan matrix
$\therefore$ the subgraph of a root system Dynkin diagram is an abstract Dynkin diagram. (possibly disconnected)

Define a Cartan matrix to be indecomposable if it cannot be put into block form $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$ after some rearrangement of rows and corresponding columns. This corresponds to connected Dynkin diagrams and irreducible root systems.

The following discussion sketches a proof that the above list exhausts all irreducible root systems, by classifying connected Dynkin diagrams.

1. A 2×2 Cartan matrix has the form $\begin{pmatrix}2 & a\\ b & 2\end{pmatrix}$ with $a, b \le 0$, determinant $= 4 - ab \ge 1$ (as det $\in \mathbb{Z}$)
   $\therefore (a,b)$ is $(0,0)$
   $(-1,-1)$ (these roots have equal length as $\frac{a_{ij}}{a_{ji}} = \frac{\langle\alpha_i,\alpha_i\rangle}{\langle\alpha_j,\alpha_j\rangle}$)
   $(-2,-1)$ or $(-1,-2)$
   $(-3,-1)$ or $(-1,-3)$

2. Dynkin diagrams of root systems have no cycles:
   For any $\alpha_1, \alpha_2, \cdots \alpha_n \in \Pi$, let $\alpha = \sum_{i=1}^n \alpha_i / \sqrt{\langle\alpha_i,\alpha_i\rangle}$
   $0 < \langle\alpha,\alpha\rangle = n + \sum_{i<j} 2\langle\alpha_i,\alpha_j\rangle / \sqrt{\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle}$ $\langle\alpha_i,\alpha_j\rangle \le 0$, and $\frac{4\langle\alpha_i,\alpha_j\rangle^2}{\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle} = a_{ij}a_{ji}$
   $= n - \sum_{i<j} \sqrt{a_{ij}a_{ji}}$
   If a cycle existed on $\alpha_1, \alpha_2, \cdots \alpha_n$, then for n pairs (i,j), $a_{ij}a_{ji} \ge 1 \Rightarrow \sqrt{a_{ij}a_{ji}} \ge 1 \Rightarrow \sum_{i<j}\sqrt{a_{ij}a_{ji}} \ge n$, a contradiction.

3. To each of the Dynkin diagrams in the above list, we can associate an extended or affine Dynkin diagram by adding one vertex and some edges from it. These represent matrices with determinant 0 (same structure as Cartan matrix, but with linearly dependent α_i). $\therefore$ these cannot be subgraphs of Dynkin diagrams.

4. Suppose we have a simply laced connected Dynkin diagram.
   if it is not A_n, it has a branch point.
   $\tilde{D}_4 =$ is prohibited, so the branch point has degree 3
   $\tilde{D}_n =$ is prohibited $\therefore$ only one branch point.
   $\therefore$ it looks like p q r with $p \ge q \ge r$ (wlog) nodes on each branch (including branch point)
   $\tilde{E}_6 =$ is prohibited $\therefore r = 2$
   $\tilde{E}_7 =$ is prohibited $\therefore q = 2$ or 3. if $q = 2$, we get D_n.
   $\tilde{E}_8 =$ is prohibited $\therefore$ if $q \ne 2$, $p \le 5$ $\Rightarrow E_6, E_7, E_8$.

5. Suppose our Dynkin diagram contains
   $= \tilde{G}_2$ is prohibited
   is prohibited as its associated matrix has determinant 0. Same with $\therefore$ this is G_2.

6. Finally, consider the case where we have .

$\tilde{C}_n =$ is prohibited $\therefore$ there is a single copy of

$\tilde{B}_n =$ is prohibited $\therefore$ there is no branching

$\tilde{F}_4 =$ is prohibited $\therefore$ either we have F_4, or occurs at the end of the chain $\Rightarrow B_n$ or C_n.

While allocating Cartan matrices to Lie algebras, we have made two choices:

choice of maximal torus – all such are conjugate ie $\exists g \in \mathrm{Aut}\,\mathfrak{g}$ with $g(\mathfrak{t}_1) = \mathfrak{t}_2$

$\therefore$ if $\mathfrak{g}_\alpha$ is an eigenspace of $\mathfrak{t}_1$, $g(\mathfrak{g}_\alpha)$ is an eigenspace of $\mathfrak{t}_2$

$\therefore$ we obtain same roots irrespective of $\mathfrak{t}$.

choice of positive roots – all such are conjugate ie $\exists w \in$ Weyl group with $w(R_1^+) = R_2^+$

then $w(\Pi_1) = \Pi_2$, so for $\alpha_i, \alpha_j \in \Pi_1$, $\langle \alpha_i, \alpha_j \rangle = \langle w\alpha_i, w\alpha_j \rangle$

$\therefore$ we obtain same Cartan matrix irrespective of R^+.

In fact, if $\mathfrak{g}_1, \mathfrak{g}_2$ are semisimple Lie algebras with same Cartan matrix (up to reordering of rows and corresponding columns), then $\exists \Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ a Lie-algebra isomorphism identifying the maximal tori and the positive root spaces ie $\Phi(\mathfrak{t}_1) = \mathfrak{t}_2$, $\Phi(\mathfrak{g}_{\alpha_i}) = \mathfrak{g}_{\alpha_i(2)}$

Another way to obtain this uniqueness result:

Given a generalised Cartan matrix (relax the $\det > 0$ condition) of dimension $n \times n$,

define a Lie algebra $\tilde{\mathfrak{g}}$ to have generators E_i, F_i, H_i $(1 \le i \le n)$

where $[H_i, H_j] = 0$, $[H_i, E_j] = a_{ij} E_j$, $[H_i, F_j] = -a_{ij} F_j$, $[E_i, F_j] = \delta_{ij} H_i$ (ie glue together $\mathfrak{sl}_2$ s)

quotient $\tilde{\mathfrak{g}}$ by the Serre relations (due to Harish-Chandra, Chevalley) to obtain $\bar{\mathfrak{g}}$:

$(\mathrm{ad}\, E_i)^{1-a_{ij}} E_j = 0$, $(\mathrm{ad}\, F_i)^{1-a_{ij}} F_j = 0$, $\forall i \ne j$ (these define Kacs-Moody algebras)

$\bar{\mathfrak{g}}$ turns out to be the quotient of $\tilde{\mathfrak{g}}$ by its unique maximal ideal $\Rightarrow \bar{\mathfrak{g}}$ is simple.

If $\mathfrak{g}$ is any finite-dimensional simple Lie-algebra with this Cartan matrix, then mapping $E_i \to e_{\alpha_i}$, $F_i \to e_{-\alpha_i}$, $H_i \to h_{\alpha_i}$ is surjective $\bar{\mathfrak{g}} \to \mathfrak{g}$. As $\mathfrak{g}$ is simple, we must have $\bar{\mathfrak{g}} = \mathfrak{g}$.

A similar argument for semisimple Lie algebras then give uniqueness.

Existence of a Lie algebra for every Cartan matrix displayed above also follows from this result, by generalising the representation theory of finite dimensional Lie algebras to the infinite dimensional Kacs-Moody algebras.

For a finite dimensional semi-simple Lie algebra, with $E_i \in \mathfrak{g}_{\alpha_i}$, $\mathrm{ad}\, E_i (\mathfrak{g}_{\alpha_j}) = \mathfrak{g}_{\alpha_j + \alpha_i}$.

$\therefore$ Serre relations. $\Leftrightarrow \alpha_j + (\alpha_i - a_{ij}\alpha_i), -\alpha_j - (\alpha_i - a_{ij}\alpha_i)$ are not roots

$\Leftrightarrow s_{\alpha_i}(\alpha_j - \alpha_i), s_{\alpha_i}(\alpha_i - \alpha_j)$ are not roots

$\Leftrightarrow \alpha_j - \alpha_i, \alpha_i - \alpha_j$ are not roots

The analogous theorem for W is that $W = \langle s_{\alpha_1}, s_{\alpha_2}, \dots s_{\alpha_n} : s_{\alpha_i}^2 = \mathrm{id}, (s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = \mathrm{id} \rangle$

$$m_{ij} = \begin{cases} 2 & \\ 3 & \text{if } a_{ij}a_{ji} = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \\ 4 & \\ 6 & \end{cases}$$

ie W is generated by simple reflections subject to the braid relations.

Representation Theory of Semisimple Lie Algebras

Let $\mathfrak{g}$ be a semi-simple Lie algebra. $\mathfrak{g} = \mathfrak{t} \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$. Fix a choice of R^+ and hence $\Pi = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$
Let V be a representation of $\mathfrak{g}$. Define $V_\lambda = \{v\in V : t(v) = \lambda(t)v \ \forall \lambda\in \mathfrak{t}^*\}$, the λ-weight-space.

Proposition: Let V be a finite dimensional representation of $\mathfrak{g}$.
$V = \bigoplus_{\lambda\in\mathfrak{t}^*} V_\lambda$ (ie each $t\in\mathfrak{t}$ acts diagonally)
$V_\lambda \neq 0 \Rightarrow \lambda(h_\alpha)\in\mathbb{Z} \quad \forall\alpha\in R$.

Proof: For each $\alpha\in R$, h_α acts diagonally on V (from rep theory of sl_2), and h_α span $\mathfrak{t}$
$\therefore$ by previous lemma, $\mathfrak{t}$-action on V is simultaneously diagonalisable.
Take $v\in V_\lambda$. $h_\alpha(v) = \lambda(h_\alpha)v$. Considering this as m_α ($\cong sl_2$) action on V (which is still finite dimensional), we know the eigenvalues are integral $\therefore \lambda(h_\alpha)\in\mathbb{Z}$

Define $Q = \mathbb{Z}R$ = lattice of roots
$P = \{\gamma\in\mathbb{R}R : \langle\gamma,\alpha^\vee\rangle\in\mathbb{Z} \ \forall\alpha\in R\}$ = lattice of weights. This contains all possible weights of finite dimensional representations
Recall that, $\forall\beta\in R$, $\langle\beta,\alpha^\vee\rangle\in\mathbb{Z}$ $\therefore Q\subseteq P$.
Π spans R $\therefore$ every element of $\mathbb{R}R$ is uniquely determined by $\{\langle\cdot,\alpha_i^\vee\rangle : \alpha_i\in\Pi\}$.
$\therefore$ define ω_i such that $\langle\omega_i,\alpha_j^\vee\rangle = \delta_{ij} \ \forall\alpha_j\in\Pi$. $\therefore P = \oplus\mathbb{Z}\omega_i$. ($\omega_i$ is the dual basis, or fundamental weights).
$\forall\gamma\in P$, $\gamma = \sum_i \langle\gamma,\alpha_i^\vee\rangle\omega_i$.
In particular, $\alpha_i = \sum_j \langle\alpha_i,\alpha_j^\vee\rangle\omega_j$ = Cartan matrix (ω_i) ie Cartan matrix describes the change of basis.
$\therefore$ det of Cartan matrix $= \dfrac{\text{volume of } \alpha_i\text{-parallelopiped}}{\text{volume of } \omega_i\text{-parallelopiped}} = |Q/P|$

The character of the representation is $\chi(V) = \sum_\lambda \dim V_\lambda\, e^\lambda$ where e^λ is just notation
(this way we distinguish between V_λ of dimension 2 and $V_{2\lambda}$ of dimension 1)
(This agrees with what we did with sl_2 before, if we set $z = e^{\alpha/2}$)
Since the $\mathbb{Z}$-span of ω_i is P, we can express $\chi(V)$ completely in terms of e^{ω_i}.

Let $v\in V_\lambda$ $\therefore t(v) = \lambda(t)v$
$\Rightarrow t(e_\alpha)v = e_\alpha t v + [t,e_\alpha]v = e_\alpha\lambda(t)v + \alpha(t)e_\alpha v = (\lambda+\alpha)(t)e_\alpha v$
$\therefore e_\alpha(V_\lambda)\subseteq V_{\lambda+\alpha}$. $\forall$ possible weights λ

Fix a root α and consider V as an $m_\alpha\cong sl_2$ representation.
For any weight λ, m_α can only move V_λ to weight spaces of the form $V_{\lambda+k\alpha}$. $\therefore \bigoplus_k V_{\lambda+k\alpha}$ is a subrep.
V_λ is a h_α eigenspace with eigenvalue $\langle\lambda,\alpha^\vee\rangle$
From rep theory of sl_2, we know $\exists$ an h_α-eigenspace of eigenvalue $-\langle\lambda,\alpha^\vee\rangle$ in this subrep, of the same dimension. This is $V_{\lambda+k\alpha}$ where $\langle\lambda+k\alpha,\alpha^\vee\rangle = -\langle\lambda,\alpha^\vee\rangle \Rightarrow k = -\frac{2\langle\lambda,\alpha^\vee\rangle}{\langle\alpha,\alpha^\vee\rangle} = -\langle\lambda,\alpha^\vee\rangle$
ie if V_λ is a weight space, $V_{s_\alpha(\lambda)}$ is a weight space of the same dimension.
Applying this to each $\alpha\in R$, we see that $V_{w\lambda}$ is a weight space of the same dimension $\forall w\in W$.
ie $\dim V_\lambda = \dim V_{w\lambda}$, or $\chi(V)$ is invariant under W (acting on the exponents)
ie if we draw V by colouring points of P according to their multiplicities in V, the picture is symmetric.
(This really is a consequence of $W = N(T)/T$, where T is the maximal torus of the associated Lie algebras)

Define a partial ordering on t^*:
$\lambda \le \mu$ if $\mu - \lambda = \sum_i k_i \alpha_i$, with $k_i \in \mathbb{N}$
$\therefore \{\lambda \in P : \lambda \le \mu\}$ are lattice points in an obtuse cone.

For a representation V, $\mu \in P$ is a highest weight for V if $V_\mu \neq 0$, and $\forall \lambda$ with $V_\lambda \neq 0$, $\lambda \le \mu$.
For $\gamma \in P$, $v \in V_\gamma$ is a singular vector if $v \neq 0$, and $e_\alpha v = 0 \ \forall \alpha \in R^+$
If μ is a highest weight, $\mu \le \mu + \alpha \ \forall \alpha \in R$ (since $\alpha = \sum k_i \alpha_i$ with $k_i \in \mathbb{N}$)
$\therefore \mu + \alpha$ is not a weight $\Rightarrow$ highest weight vectors are singular.

$\mu \in P$ is an extremal weight if $w\mu$ is a highest weight for some $w \in W$.

The cone of dominant weights, $P^+ = \{\lambda \in P : \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in R^+\}$
$= \{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ \forall \alpha_i \in \Pi\} = \{\lambda \in t^* : \lambda(h_i) \in \mathbb{N} \ \forall i\}$

The discussion below will show that, for semisimple Lie algebra g:
all finite dimensional representations are a direct sum of irreducible representations
irreducible representations are labelled by weights in P^+
more precisely: any singular vector of a finite dimensional irreducible representation $\in$ weight space of dimension 1, weight $\in P^+$ (ie all extremal weight spaces have dim 1)
if two irreducible representations have singular vectors v, w with same highest weight λ, then $\exists$ isomorphism between them sending v to w.
$\forall \lambda \in P^+$, $\exists$ an irreducible representation $L(\lambda)$ with λ its unique highest weight.

Corollary: a finite dimensional representation is uniquely determined by its character.
Proof: For $\lambda \in P^+$, set $m_\lambda = \sum_{\gamma \in W\lambda} e^\gamma$ (ie orbit of λ under W)
Since characters are invariant under W, $\chi(L(\lambda)) = m_\lambda + \sum_{\mu < \lambda} a_{\mu\lambda} e^\mu$ for some $a_{\mu\lambda}$.
(we must have $\mu < \lambda$ for all other weights μ, otherwise highest weight is not unique)
It can be shown that $\mathbb{R}P^+ = \{x \in \mathbb{R}R : \langle x, \alpha_i^\vee \rangle \geq 0 \ \forall i\}$ is a fundamental domain for the W-action $\therefore \{m_\gamma : \gamma \in P^+\}$ is a basis of W-invariants in $\mathbb{Z}[P]$
$\Rightarrow \{\chi(L(\lambda)) : \lambda \in P^+\}$ " " " " " "
Hence $\chi(V)$ = linear combination of $\chi(L(\lambda))$ in exactly one way, which is given by complete reducibility. Hence if V, W are distinct representations, their characters have different basis expansions $\Rightarrow$ characters are distinct.

Given a representation V, define its dual: $\forall v \in V, f \in V^*, x \in g$, $(xf)v = -f(xv)$
and, if V, W are representations, we have a representation on $Hom(V,W)$; $[x\theta](v) = x(\theta v) - \theta(xv)$
Then we can identify $W \otimes V^* \cong Hom(V,W)$: $w \otimes f \simeq f(\cdot)w$, since
$x(w \otimes f) = xw \otimes f + w \otimes xf = xw \otimes f - w \otimes fx \cong f(\cdot)xw - fx(\cdot)w = x f(\cdot)w - fx(\cdot)w = x[f(\cdot)w]$

The set $\{\lambda, V_\lambda \neq 0\}$ is invariant under $W \Rightarrow \sum_{V_\lambda \neq 0} \lambda$ is invariant under $W \Rightarrow$ it is invariant under s_{α_i} $\forall i$
$\Rightarrow \langle \sum_{V_\lambda \neq 0} \lambda, \alpha_i \rangle = 0 \ \forall i \Rightarrow \sum_{V_\lambda \neq 0} \lambda = 0$.
$\therefore$ choose a basis $v_i \in V_{\lambda_i}$. The dual basis $v_i^* \in V_{-\lambda_i}$. Identify $V^* \simeq \Lambda^{\dim V - 1} V$, $v_i^* \simeq - v_1 \wedge v_2 \wedge \cdots v_{i-1} \wedge v_n \wedge v_{i+1} \wedge \cdots$

Example: Work with the adjoint representation, which is irreducible if g is simple (every irreducible component is an ideal).

$g = t \oplus_{\alpha\in R} g_\alpha$ and $g_\alpha = V_\alpha$, $t = V_0$ $\therefore \chi(ad) = \dim t + \sum_{\alpha\in R} e^\alpha$.

The highest weight here is the highest root θ. $\therefore$ ad rep $= L(\theta)$.

e.g. highest root of $A_n = \varepsilon_1 - \varepsilon_{n+1} = \omega_1 + \omega_n$ (by evaluating $\langle\theta, \alpha_i^\vee\rangle$)

highest root of $B_n = \varepsilon_1 + \varepsilon_2 = \omega_2$

highest root of $C_n = 2\varepsilon_1 = 2\omega_1$

highest root of $D_n = \varepsilon_1 + \varepsilon_2 = \omega_2$

highest root of $E_8 = \varepsilon_1 + \varepsilon_2 = \omega_1$

highest root of $E_7 = \varepsilon_1 + \varepsilon_2 = \omega_1$

highest root of $E_6 = \varepsilon_1 + \varepsilon_2 = \omega_1$

highest root of $F_4 = \varepsilon_1 + \varepsilon_2 = \omega_1$

highest root of $G_2 = 3\alpha + 2\beta = 3\omega_\beta$

Example: Standard sl_n representation. Let the basis vectors be denoted by $v_1, v_2, \cdots v_n$.

Then $v_i \in \varepsilon_i$-weight space

$\Rightarrow \chi = e^{\varepsilon_1} + e^{\varepsilon_2} \cdots e^{\varepsilon_n}$.

$\forall i > 1$, $\varepsilon_i = \varepsilon_1 - (\varepsilon_1 - \varepsilon_2) - (\varepsilon_2 - \varepsilon_3) - \cdots (\varepsilon_{i-1} - \varepsilon_i)$ $\therefore \varepsilon_1$ is the highest weight.

By analysis below, $\cdot$ this representation is irreducible. $= L(\omega_1)$

Let V be above representation.

$\Lambda^k V$ has basis $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \in \varepsilon_{i_1} + \varepsilon_{i_2} + \cdots \varepsilon_{i_k}$-weight space $(i_1 < i_2 \cdots < i_k)$

(recall $g(v \otimes w) = gv \otimes w + v \otimes gw$) (this gives character, all have multiplicity 1)

The matrix with 1 in the $i, i+1$ th entry is in $g_{\varepsilon_i - \varepsilon_{i+1}}$. This sends v_{i+1} to v_i, all other v_j to 0.

$\therefore v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ is sent to 0 if there is no v_{i+1} component, or both v_i, v_{i+1} appear.

$\therefore v_1 \wedge v_2 \wedge \cdots \wedge v_k$ is the only subspace sent to 0 by all matrices in g_{α_i} (as all weight spaces 1D)

$\therefore$ singular subspace is 1-dimensional, this representation is irreducible. $\therefore$ is $L(\omega_k)$

$S^k V$ has basis $v_{i_1} \cdot v_{i_2} \cdot \cdots \cdot v_{i_k} \in \varepsilon_{i_1} + \varepsilon_{i_2} + \cdots + \varepsilon_{i_k}$-weight space $(i_1 \le i_2 \le \cdots \le i_k)$

Now matrices in g_{α_i} annihilate $v_{i_1} \cdot v_{i_2} \cdots v_{i_k}$ if there is no v_{i+1} component

$\therefore$ the only such subspace is v_1^k $\therefore$ singular vectors are 1-dimensional, with weight $k\omega_1$

$\therefore$ this is the irreducible representation $L(k\omega_1)$

Using the identification $V^* \cong \Lambda^{n-1} V = L(\omega_{n-1})$, we see $V \otimes V^* = L(\omega_1) \otimes L(\omega_{n-1})$

Taking highest weight vectors v, w in the components, (which, by above, are singular),

we see $v \otimes w$ is singular of weight $\omega_1 + \omega_{n-1}$

$\therefore V \otimes V^*$ contains a copy of $L(\omega_1 + \omega_{n-1}) =$ adjoint representation.

$V \otimes V^*$ has dimension n^2, adjoint representation has dimension $n^2 - 1$

$\therefore V \otimes V^*$ must be adjoint representation and trivial representation

(Observe that, $g(\mathrm{Hom}_g(V,W)) = 0$, so $V \otimes V^*$ always contains a copy of the trivial representation)

(If V is 1-dimensional, then $[g,g]V = 0$ $\therefore$ if g is simple, V is the trivial representation

Example: Standard so_{2n} representation, basis $= v_1, v_2, \cdots v_{2n}$.

$1 \le i \le n$: $v_i \in \varepsilon_i$-weight space, $v_{2n+1-i} \in -\varepsilon_i$-weight space

The matrix corresponding to α_i has 1 in entry $i, i+1$

-1 in entry $2n-i, 2n+1-i$ $(i<n)$

matrix corresponding to α_n has 1 in entry $n-1, n+1$

-1 in entry $n \quad n+2$

$\therefore$ these send $v_{i+1} \to v_i$, $v_{2n+1-i} \to -v_{2n-i}$, $v_{n+1} \to v_{n-1}$, $v_{n+2} \to -v_n$.

$\therefore v_1$ is the only singular vector $\therefore V$ is irreducible with highest weight ε_1.

Identify V with V^*, $x \to x^T K$. (inner product)

Then $v_i^* = v_{2n-i}$, v_i^*, v_{2n-i} have the same weight $\therefore V \cong V^*$ as representations.

$\therefore V \otimes V$ contains trivial representation.

$v_1 \wedge v_2$ is singular in $\Lambda^2 V$, with weight $\varepsilon_1 + \varepsilon_2$ $\therefore \Lambda^2 V \supseteq L(\varepsilon_1 + \varepsilon_2)$

$L(\varepsilon_1+\varepsilon_2)$ is the adjoint representation, which has dimension $n+2n(n-1) = \dim \Lambda^2 V$

$\therefore \Lambda^2 V = L(\varepsilon_1 + \varepsilon_2)$.

v_1^2 is singular in $S^2 V$, with weight $2\varepsilon_1$.

No other $v_i v_j$ term is singular $\therefore$ other singular vectors must be linear combinations of $v_i v_j$ in the same weight space $\therefore$ they are $\sum a_i v_i v_{2n+1-i}$.

Matrix corresponding to α_i sends this to $a_{i+1} v_i v_{2n-i} - a_i v_i v_{2n-i}$ $\therefore a_i = a_{i+1}$ $\forall i<n$

α_n sends this to $a_n v_n v_{n-1} - a_{n-1} v_{n-1} v_n$ $\therefore a_{n-1} = a_n$.

$\therefore \sum_i v_i v_{2n+1-i}$ describes the only other singular subspace, which has weight 0

$\therefore S^2 V = L(2\varepsilon_1) \oplus$ trivial representation

Let g be a semi-simple Lie algebra, and V a representation of g.

Choose a basis $u_1, u_2, \cdots u_l$ of t, and a basis x_α of g_α.

Denote the dual basis by $u^1, u^2, \cdots u^l$, $x^\alpha \cdots$, with respect to the Killing form

Define the Casimir operator $\Omega = \sum_{i=1}^{l} u_i u^i + \sum_\alpha x_\alpha x^\alpha \quad : V \to V$

(this belongs in the universal enveloping algebra Ug, see appendix)

This definition is in fact basis independent: suppose y_i, z_j are both bases of g.

$y_i = \sum_j (y_i, z^j) z_j$, $z_j = \sum_k (z_j, y^k) y_k$ $\quad \therefore \sum_j (y_i, z^j)(z_j, y^k) = \delta_{ik}$

$y^i = \sum_j (y^i, z_j) z^j$ $\quad$ and similarly, $\sum_i (z_j, y^i)(y_i, z^k) = \delta_{jk}$

$$\therefore \sum_i y_i y^i = \sum_{j,k,i} (y_i, z^j) z_j (y^i, z_k) z^k = \sum_{j,k} \delta_{jk} z_j z^k = \sum_j z_j z^j$$

Lemma: $x\Omega = \Omega x$ $\forall x \in g$

Proof:
$$[\Omega, x] = \sum_i [y_i y^i, x] = \sum_i y_i [y^i, x] + \sum_i [y_i, x] y^i$$
$$= \sum_i y_i ([y^i, x], y_i) y^i + \sum_i ([y_i, x], y^i) y_i y^i$$
$$= \sum_i y_i ([x, y_i], y^i) y^i + \sum_i ([y_i, x], y^i) y_i y^i = 0$$

by invariance of $(,)$.

Lemma: Suppose $v \in V$ is singular with weight λ
Then $\Omega v = (|\lambda+\rho|^2 - |\rho|^2)v$, where $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$.

Proof: Normalise the basis x_α so that $(x_\alpha, x_{-\alpha}) = 1$ $\therefore x^\alpha = x_{-\alpha}$, $[x_\alpha, x_{-\alpha}] = t_\alpha$

$$\begin{aligned}
\therefore \Omega v &= \left(\sum_{i=1}^{l} u_i u^i + \sum_{\alpha\in R^+} x_\alpha x^\alpha + x_{-\alpha} x^{-\alpha}\right)v \\
&= \left(\sum_{i=1}^{l} u_i u^i + \sum_{\alpha\in R^+} x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha\right)v \\
&= \sum_{i=1}^{l} \langle\lambda, u_i\rangle\langle\lambda, u^i\rangle v + \sum_{\alpha\in R}[x_\alpha, x_{-\alpha}]v \qquad \text{as } x_\alpha v = 0 \ \forall \alpha\in R^+ \\
&= \sum_{i=1}^{l} \langle\lambda, u_i\rangle\langle\lambda, u^i\rangle v + \sum_{\alpha\in R} t_\alpha(v) \\
&= \langle\lambda,\lambda\rangle v + \sum_{\alpha\in R}\langle\alpha,\lambda\rangle v = (\langle\lambda,\lambda\rangle + 2\langle\rho,\lambda\rangle)v.
\end{aligned}$$

If V is irreducible, then, by Schur, Ω acts as multiplication by $(|\lambda+\rho|^2 - |\rho|^2)$ on all of V

Example: Let V be the adjoint representation (which is irreducible for simple $\mathfrak{g}$)
Then
$$\begin{aligned}
\operatorname{tr}\Omega &= \sum_i \operatorname{tr}(x_i x^i) \\
&= \sum_i (x_i, x^i)_{ad} \\
&= \dim\mathfrak{g}
\end{aligned}$$
Ω is a scalar matrix $\therefore$ the scalar is $\frac{\dim\mathfrak{g}}{\dim\mathfrak{g}} = 1$.

Now let V be any representation of $\mathfrak{g}$ ($\mathfrak{g}$ semisimple)
$v \in V_\lambda$ is *singular* if $\mathfrak{n}^+ v = 0$ (ie $\forall \alpha\in R^+$, $\alpha v = 0$)
Then the $\mathfrak{g}$-submodule of V generated by v ($= U\mathfrak{g}\, v$) is in fact generated by repeated applications of $\mathfrak{n}_-$:
given any expression $x_1 x_2 \cdots x_n \in U\mathfrak{g}$, with x_i the basis elements of $\mathfrak{g}$, use the commutation relations to move the x_i not in $\mathfrak{n}_-$ to the end: these send v to 0 or a multiple of v.
(induction on n)

This submodule is a *highest weight module* with highest weight λ
e.g. if V is finite-dimensional and irreducible, it is a highest weight module.

Proposition: Let V be a highest weight module with highest weight λ, singular vector v_λ:
- i $\mathfrak{t}$ acts diagonally on V, $V = \bigoplus_{\mu\le\lambda} V_\mu$
- ii V_λ is 1-dimensional, and all weight spaces are finite dimensional
- iii V is irreducible $\Leftrightarrow$ all singular vectors $\in V_\lambda$
- iv Ω acts as scalar multiplication by $|\lambda+\rho|^2 - |\rho|^2$ on all of V
- v if v_μ is another singular vector in V, then $|\mu+\rho|^2 = |\lambda+\rho|^2$
- vi if $\lambda\in\mathbb{R}R$, then $\exists$ only finite many μ with singular v_μ.
- vii V contains a unique maximal proper submodule $I = \bigoplus_\mu I\cap V_\mu$

Proof: $V = U(\mathfrak{n}^-)v_\lambda$ = spanned by $e_{-\beta_1}e_{-\beta_2}\cdots e_{-\beta_r}v_\lambda$ (across all possible sequences of $\beta_i\in R^+$)
$e_{-\beta_1}e_{-\beta_2}\cdots e_{-\beta_r}v_\lambda \in V_{\lambda-\beta_1-\beta_2-\cdots-\beta_r}$ $\therefore$ these vectors are 'eigenvectors' for $\mathfrak{t}$
$\therefore$ using a basis of these vectors, we see $\mathfrak{t}$ acts diagonally.
$V = \bigoplus V_\mu$ where $\mu = \lambda-\beta_1-\beta_2-\cdots-\beta_r \Rightarrow \mu\le\lambda$.

$\lambda-\beta_1-\beta_2-\cdots-\beta_r \ne \lambda$ $\therefore V_\lambda$ is 1-dimensional.
if $V_\mu \ne 0$, then $\mu = \lambda - \sum a_i\alpha_i$ (α_i simple) with a_i unique $\therefore \mu = \lambda-\beta_1-\cdots-\beta_r$ in finitely many ways.

If v_μ is another singular vector, then $U\mathfrak{n}^- v_\mu$ is a submodule W.
By (i), $W = \bigoplus_{\nu \leq \mu} W_\nu \subseteq \bigoplus_{\nu \leq \mu} V_\nu \Rightarrow V_\lambda \notin W$ $\therefore$ W is a proper submodule $\therefore$ V is
$\therefore$ V not irreducible.

Conversely, if $W \subseteq V$ is a proper submodule, then $W_\mu \subseteq V_\mu$ for each weight μ of W.
if $v_\lambda \in W$, then, by invariance of W, $V \subseteq W$, a contradiction
$\therefore$ $\forall$ weights $\mu \in W$, $\mu = \lambda - \sum k_i \alpha_i$ with $\sum k_i > 0$
Take μ with $\sum k_i$ minimal $\Rightarrow \forall \alpha \in R^+$, $\mu + \alpha \neq$ a root of W
$\therefore$ v_μ is a singular vector, and $v_\mu \notin V_\lambda$.

We have $\Omega v_\lambda = (|\lambda+\rho|^2 - |\rho|^2) v_\lambda$.
and $\Omega(e_{-\beta_1} e_{-\beta_2} \cdots e_{-\beta_r}) v_\lambda = e_{-\beta_1} e_{-\beta_2} \cdots e_{-\beta_r} \Omega v_\lambda = (|\lambda+\rho|^2 - |\rho|^2) e_{-\beta_1} e_{-\beta_2} \cdots e_{-\beta_r} v_\lambda$
and vectors of this form form a basis. v. is then immediate.

$|\mu+\rho|^2 = |\lambda+\rho|^2$, for fixed λ, ρ, describes a sphere in $\mathbb{R}R$. (if $\lambda \in \mathbb{R}R$)
This is a compact set $\therefore$ the discrete set $\{\mu : \mu \leq \lambda\}$ can only meet it at finitely many points.

Any proper submodule $W \subseteq V$ has the form $\bigoplus_{\mu < \lambda} W_\mu$
The sum of all such has this same form $\therefore$ is unique maximal proper submodule.

Let $\lambda \in \mathfrak{t}^*$. The Verma module $M(\lambda)$ has highest weight λ and satisfies the universal property that, if $\bar{V}$ is a highest weight module with highest weight λ and highest vector $\bar{v}_\lambda$, then $\exists$ a unique map of $\mathfrak{g}$-modules $M(\lambda) \to V$ with $v_\lambda \to \bar{v}_\lambda$, where v_λ is a highest weight vector of $M(\lambda)$. (ie $M(\lambda)_\mu \to V_\mu$ for all weights μ)

Proposition: given $\lambda \in \mathfrak{t}^*$, $M(\lambda)$ and $L(\lambda)$ exist and are unique.
Verma module $M(\lambda)$ = largest highest weight module with highest weight λ
Irreducible rep $L(\lambda)$ = smallest " " " " " " "

Proof: uniqueness of $M(\lambda)$ follows from universal property.

To construct $M(\lambda)$, take $\mathbb{C}_\lambda$, the one-dimensional representation of $\mathfrak{n}^+ \oplus \mathfrak{t}$
where $\mathfrak{n}^+(1) = 0$, $\mathfrak{t}(1) = \lambda(t)$.
Now set $M(\lambda) = \mathrm{Ind}_{\mathfrak{n}^+ \oplus \mathfrak{t}}^{\mathfrak{g}} \mathbb{C}_\lambda = U\mathfrak{g} \otimes_{U\mathfrak{n} \otimes U\mathfrak{t}} \mathbb{C}_\lambda$ (just as $\mathrm{Ind}_H^G V = \mathbb{C}G \otimes_{\mathbb{C}H} V$)
Ind, Res are adjoint functors
$\therefore \mathrm{Hom}_\mathfrak{g}(U\mathfrak{g} \otimes_{U\mathfrak{n} \otimes U\mathfrak{t}} \mathbb{C}_\lambda, \bar{V}) = \mathrm{Hom}_\mathfrak{g}(\mathbb{C}_\lambda, \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}^+ \oplus \mathfrak{t}} \bar{V})$ which is completely determined by $\mathrm{im}(1) \in \{v \in \bar{V} : \mathfrak{n}^+ v = 0, tv = \lambda(t) v \ \forall t \in \mathfrak{t}\} = \bar{V}_\lambda$.
(this is Frobenius reciprocity)

$L(\lambda) = M(\lambda) /$ unique maximal proper submodule, since by universal property, every $\bar{V}$ with highest weight λ has the form $M(\lambda)/$ some submodule, and irreducibility of L means the submodule must be maximal.

Corollary: let $R^+=\{\beta_1,\beta_2,\cdots\beta_s\}$. Then $e_{-\beta_1}^{k_1}e_{-\beta_2}^{k_2}\cdots e_{-\beta_s}^{k_s}v_\lambda$ is a basis of $M(\lambda)$. (these are non-zero and linearly independent by PBW).

By uniqueness of $L(\lambda)$, any finite dimensional irreducible representation is isomorphic to $L(\lambda)$ for a unique λ, with $\lambda(h_i)\in\mathbb{N} \Rightarrow \lambda\in P^+$. The converse, that, for $\lambda\in P^+$, $L(\lambda)$ is finite-dimensional is Cartan's theorem (an explicit construction, which we will not give)

Example: $g=sl_2$ $\quad M(\lambda)=\{v_\lambda, f(v_\lambda), f^2(v_\lambda),\cdots\}$
recall that, if $\lambda(h)\in\mathbb{N}$, $e(f^{\lambda(h)+1}v_\lambda)=0$ $\quad\therefore f^{\lambda(h)+1}v_\lambda$ is singular
$\therefore$ if $\lambda(h)\in\mathbb{N}$, $M(\lambda)$ is not irreducible.
If $M(\lambda)$ is not irreducible, $\exists k>0$ with $0=e(f^kv_\lambda)=k(\lambda(h)-k+1)f^{k-1}v_\lambda \Rightarrow \lambda(h)=k-1\in\mathbb{N}$.
since $f^{k-1}v_\lambda\neq 0$. This is the only other weight space with singular vectors $\therefore f^kv_\lambda$ generate the unique maximal proper submodule (whose quotient produces $L(\lambda)$).
$\therefore$ For sl_2, $M(\lambda)=L(\lambda)\Leftrightarrow \lambda(h)\notin\mathbb{N}$

In general, if $\lambda(h_i)\in\mathbb{N}$, then, from sl_2-theory, $e_i(f_i^{\lambda(h_i)+1}v_\lambda)=0$.
$e_\alpha(f_i^{\lambda(h_i)+1}v_\lambda)\in V_{\lambda+\alpha-(\lambda(h_i)+1)\alpha_i}$ $\quad(\alpha\neq\alpha_i$, so $\alpha=\sum k_i\alpha_i$ with some $k_j>0, j\neq i$, or $k_i>1)$
All weights $\leq\lambda$ $\quad\therefore \lambda+\alpha-(\lambda(h_i)+1)\alpha_i$ is not a weight $\Rightarrow e_j(f_i^{\lambda(h_i)+1}v_\lambda)=0\quad\forall j\neq i$
$\therefore f_i^{\lambda(h_i)+1}v_\lambda$ is a singular vector (there are many more)

Lemma: $s_{\alpha_i}(R^+\setminus\alpha_i)=R^+\setminus\alpha_i$
Proof: take any $\alpha\in R^+ \Rightarrow \alpha=\sum k_i\alpha_i$, $k_i\geq 0$.
$\therefore s_{\alpha_i}(\alpha)=\sum_{j\neq i}k_j\alpha_j+(\sum_{j\neq i}(\alpha_j,\alpha_i^\vee)k_j-k_i)\alpha_i$
if $\alpha\neq\alpha_i$, $\exists j$ with $k_j>0 \Rightarrow$ coefficient of α_j in $s_{\alpha_i}(\alpha)>0$.
As $R=R^+\cup R^-$, $s_{\alpha_i}(\alpha)$ must be in R^+.

Observe that, for any $w\in W$, $ws_\alpha w^{-1}$ fixes $w\alpha^\perp$ and sends $w\alpha$ to $-w\alpha$ $\quad\therefore ws_\alpha w^{-1}=s_{w\alpha}$

Lemma: $\rho=\frac{1}{2}\sum_{\alpha\in R^+}\alpha=\omega_1+\omega_2+\dots\omega_{\dim R}$
Proof: $s_{\alpha_i}\rho=s_{\alpha_i}(\frac{1}{2}\alpha_i+\sum_{\alpha\neq\alpha_i}\frac{1}{2}\alpha)=-\frac{1}{2}\alpha_i+\sum_{\alpha\neq\alpha_i}\frac{1}{2}\alpha$ $\quad$ by above lemma
$=\rho-\alpha_i$
Also $s_{\alpha_i}(\rho)=\rho-\langle\rho,\alpha_i^\vee\rangle\alpha_i \Rightarrow \langle\rho,\alpha_i^\vee\rangle=1$ and this holds $\forall$ simple α_i.

Key lemma: Suppose $\lambda\in P^+$, $\mu+\rho\in P^+$, $\lambda\geq\mu$ and $|\lambda+\rho|=|\mu+\rho|$. Then $\lambda=\mu$
Proof: $0=\langle\lambda+\rho,\lambda+\rho\rangle-\langle\mu+\rho,\mu+\rho\rangle$
$=\langle\lambda-\mu,\lambda+\rho+(\mu+\rho)\rangle$ $\qquad$ let $\lambda-\mu=\sum_i k_i\alpha_i$, $k_i\geq 0$.
$=\sum k_i\langle\alpha_i,\lambda+\rho+(\mu+\rho)\rangle$
$\lambda\in P^+$, $\mu+\rho\in P^+$ $\therefore \langle\alpha_i,\lambda\rangle\geq 0$, $\langle\alpha_i,\mu+\rho\rangle\geq 0$, and, by above, $\langle\rho,\alpha_i\rangle>0$ $\forall i$.
$\therefore$ above sum can only be 0 if all $k_i=0$ ie $\lambda=\mu$.

Weyl's theorem of complete reducibility:
Let V^{n_+} denote $\{x \in V : n_+ x = 0\}$. By Engel, $V^{n_+} \neq 0$ $\therefore V^{n_+} = \bigoplus_{\mu \in P} V_\mu^{n_+}$
Take $v_\mu \in V_\mu^{n_+}$, a singular vector (if $v \in V^{n_+}$, $v = \sum v_i$, $v_i \in V_{\lambda_i}$, then $n_+(v_i) = 0$ $\forall i$, since $n_+(v_i) \in$ different V_λ)
The g-submodule generated by v_μ is a highest weight module with highest weight μ.
Suppose v_γ is another singular vector in this submodule
$\Rightarrow \gamma \leq \mu$, $|\gamma+\rho| = |\mu+\rho|$ and $\gamma, \mu \in P^+$ (from sl_2 theory, $\gamma(h_i) \geq 0$ $\forall i$)
$\Rightarrow \gamma = \mu$ from key lemma
$\therefore$ this g-submodule is irreducible, and contains only 1 1-dimensional subspace of V^{n_+}.

$\therefore V' = U g V^{n_+} = \bigoplus_\mu L(\mu)$ by uniqueness of $L(\mu)$ (by their irreducibility, they cannot intersect)
where we sum over μ with $V_\mu^{n_+} \subseteq V^{n_+}$, with correct multiplicities.
If $V' \neq V$, then let $U = V/V'$. Take v_λ singular in U, using same construction as above.
U is finite dimensional so $\lambda \in P^+$
Let $\bar{v}_\lambda \in V_\lambda$ be a lift of v_λ. Then $e_i \bar{v}_\lambda \in V_{\lambda+\alpha_i}$
$\bar{v}_\lambda$ is not singular (otherwise it is in $V^{n_+} \subseteq V'$) $\therefore \exists i$ with $e_i \bar{v}_\lambda \neq 0$.
But $e_i \bar{v}_\lambda \in V'$ $\therefore e_i \bar{v}_\lambda \in L(\mu)$ for some μ $\Rightarrow \lambda+\alpha_i$ is a weight in $L(\mu)$ $\therefore \lambda+\alpha_i \leq \mu$
$\therefore \Omega(e_i \bar{v}_\lambda) = e_i \Omega(\bar{v}_\lambda) = (|\mu+\rho|^2 - |\rho|^2) e_i \bar{v}_\lambda \Rightarrow \Omega(e_i v_\lambda) = (|\mu+\rho|^2 - |\rho|^2) e_i v_\lambda$ (pass to U)
v_λ is singular in U $\therefore \Omega(e_i v_\lambda) = (|\lambda+\rho|^2 - |\rho|^2) e_i v_\lambda$
$\therefore |\mu+\rho| = |\lambda+\rho|$, $\lambda, \mu \in P^+$, $\lambda+\alpha_i \leq \mu \Rightarrow \lambda < \mu$. By key lemma, this cannot occur.

Lemma: $\chi(M(\lambda)) = e^\lambda / \prod_{\alpha \in R^+}(1-e^{-\alpha}) = e^\lambda/\Delta$ where $\Delta = \prod_{\alpha \in R^+}(1-e^{-\alpha})$
Proof: Let $R^+ = \{\beta_1, \beta_2, \cdots \beta_s\}$.
By PBW, $e_{-\beta_1}^{k_1} e_{-\beta_2}^{k_2} \cdots e_{-\beta_s}^{k_s} v_\lambda \in M(\lambda)_{\lambda - k_1\beta_1 - k_2\beta_2 - \cdots - k_s\beta_s}$ is a basis for $M(\lambda)$.
$\therefore \dim M(\lambda)_{\lambda-\beta} = \#$ ways which β can be written as the sum of β_i, with coefficients ≥ 0.
$=$ coefficient of $e^{-\beta}$ in $\prod_{\alpha \in R^+}(1 + e^{-\alpha} + e^{-2\alpha} + \cdots)$

Lemma: $w(e^\rho \Delta) = \det w \, (e^\rho \Delta)$ (where $w(e^\lambda) = e^{w\lambda}$) $w \in W$.
Proof: it suffices to show this for $w = s_{\alpha_i}$, since these generate W (and det is multiplicative)
$$s_{\alpha_i}(e^\rho \Delta) = s_{\alpha_i}\left(e^\rho (1-e^{-\alpha_i}) \prod_{\alpha \neq \alpha_i}(1-e^{\alpha})\right)$$
$$= e^{\rho-\alpha_i}(1-e^{\alpha_i}) \prod_{\alpha \neq \alpha_i}(1-e^\alpha) \quad \text{as } s_{\alpha_i}(\rho) = \rho - \alpha_i,\ s_{\alpha_i}(R^+ \setminus \alpha_i) = R^+ \setminus \alpha_i.$$
$$= -e^\rho \Delta$$

Weyl character formula: $\chi(L(\lambda)) = \frac{1}{\Delta} \sum_{w \in W} \det w \, e^{w(\lambda+\rho)-\rho} = \sum_{w \in W} \det w \, \chi M(w(\lambda+\rho)-\rho)$ if $\lambda \in P^+$

1. Let $V(\lambda)$ be a highest weight module with highest weight $\lambda \in \mathbb{R}R$
If $V(\lambda)$ is irreducible, then, by uniqueness of irreducible highest weight modules, $V(\lambda) = L(\lambda)$
$\therefore \chi(V(\lambda)) = \chi(L(\lambda))$
If $V(\lambda)$ is not irreducible, $\exists$ singular vector v_μ with $\mu \leq \lambda$, $|\mu+\rho| = |\lambda+\rho|$.
$\{\mu : \mu \leq \lambda, |\mu+\rho| = |\lambda+\rho|\}$ is a finite set $\therefore \exists \mu$ such that $V(\lambda)_{\mu'}$ contains no singular vectors $\mu' < \mu$ ie v_μ generate $L(\mu)$.
Let $\overline{V(\lambda)}$ denote the quotient $V(\lambda)/L(\mu)$. The standard basis for $V(\lambda)$ = basis of $L(\mu) \amalg$ basis of $\overline{V(\lambda)}$
$\therefore \chi(V(\lambda)) = \chi(\overline{V(\lambda)}) + \chi(L(\mu))$ and $\overline{V(\lambda)}$ has highest weight λ.
Continue peeling off irreducible subrepresentations (induction) we see that $\chi(V(\lambda)) = \sum_{\substack{\mu<\lambda \\ |\mu+\rho| = |\lambda+\rho|}} a_\mu \chi(L(\mu)) + \chi(L(\lambda))$

for some $a_\mu \in \mathbb{N}$.

2\. Fix r. We can view $\chi(M(\lambda)) = \sum_{\substack{\mu < \lambda \\ |\mu+\rho| = |\lambda+\rho|}} a_\mu \chi(L(\mu)) + \chi(L(\lambda))$ as a system of equations, for $\lambda \le r$, $|\lambda+\rho| = |r+\rho|$ (which is a finite set)
Totally order this set, extending the partial ordering.
In this basis, the equations are represented by the matrix $\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$ with integer coefficients,

which has an inverse, also with integer entries, and 1s along the diagonal.
Take the $\chi(L(r))$ entry: $\chi(L(r)) = \sum_{\substack{\mu < r \\ |\mu+\rho| = |r+\rho|}} b_\mu \chi(M(\mu)) + \chi(M(r))$ with $b_\mu \in \mathbb{N}$

3\. $\chi(L(r)) = \sum b_\mu e^\mu/\Delta + e^r/\Delta \Rightarrow e^\rho \Delta \chi(L(r)) = \sum b_\mu e^{\mu+\rho} + e^{r+\rho}$ $*$
Since $\chi(L(r))$ is W-invariant, and $e^\rho\Delta$ is W-anti-invariant, the RHS is W-anti-invariant.
As $\mu+\rho$ ranges over P^+, $\sum_w \det w\, e^{w(\mu+\rho)}$ is a basis of anti-invariant elements of $\mathbb{Z}[P]$
$\therefore$ if $r \in P$, $\chi(L(r)) \in \mathbb{Z}[P]$, so, by comparison with $*$,
$e^\rho \Delta \chi(L(r)) = \sum c_\mu \sum_w \det w\, e^{w(\mu+\rho)} + \sum_w \det w\, e^{w(r+\rho)}$
where the first sum ranges over $\mu+\rho \in P^+$, $\mu < r$, $|\mu+\rho| = |r+\rho|$ — by key lemma, this set contains no elements. Now divide by $e^\rho\Delta$ to get result.

Observe that $\chi(L(\lambda)) = \sum_{w\in W} (\det w\, e^\rho \Delta)^{-1} e^{w(\lambda+\rho)} = \sum_{w\in W} [w(e^\rho\Delta)]^{-1} e^{w(\lambda+\rho)} = \sum_{w\in W} w(e^\lambda/\Delta)$
which suggests that the representation came from cohomology of a coherent sheaf.

Take $\lambda = 0$ $\therefore L(0)$ is the 1-dimensional trivial representation
$\therefore 1 = e^0 = \frac{1}{\Delta} \sum_{w\in W} \det w\, e^{w\rho - \rho}$ — this is the Weyl denominator identity

This can be thought of as a generalisation of the Vandermonde determinant formula: ε_i
in A_n $\rho = \omega_1 + \omega_2 + \dots + \omega_n = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots \varepsilon_n$
$\Rightarrow \sum_{w\in W} \det w\, e^{w\rho} = \det\left((e^{\varepsilon_{n+1-j}})^{i-1}\right)$ $\quad (\det w\, e^{w\rho} = \det w \prod_j (e^{\varepsilon_{w(n+1-j)}})^{j-1})$ $(i, j \le n+1)$
Now set $x_i = e^{\varepsilon_{n+1-i}}$
$e^\rho \Delta = e^{n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots \varepsilon_n} \prod_{j<i} (1 - e^{-\varepsilon_j + \varepsilon_i}) = \prod_{j<i} (e^{\varepsilon_j} - e^{\varepsilon_i}) = \prod_{i<j} (x_i - x_j)$
$\therefore$ we have $\det(x_j^{i-1}) = \prod_{i<j} (x_i - x_j)$

Define $\rho^\vee = \frac{1}{2} \sum_{\alpha\in R^+} \alpha_i^\vee$. Working in $R^\vee$, we see that $\langle \rho^\vee, \alpha_i \rangle = 1$
Consider a homomorphism $F_\mu : \mathbb{C}[P] \to \mathbb{C}(q)$, $F_\mu(e^\lambda) = q^{-\langle \lambda, \mu \rangle}$
$\therefore F_0(e^\lambda) = 1 \Rightarrow F_0(\chi(L\lambda)) = \dim L(\lambda)$ — but this is not useful since it gives $0/0$.

q-dimension formula: $F_{\rho^\vee}(\chi(L\lambda)) = \sum_\mu \dim L(\lambda)_\mu q^{-\langle \mu, \rho^\vee \rangle} = q^{-\langle \lambda, \rho^\vee \rangle} \prod_{\alpha\in R^+} \frac{(1 - q^{\langle \alpha^\vee, \lambda+\rho \rangle})}{(1 - q^{\langle \alpha^\vee, \rho \rangle})}$

Proof: Weyl denominator identity says $\Delta e^\rho = \sum_{w\in W} \det w\, e^{w\rho}$
Apply F_μ to both sides: $q^{-\langle \rho, \mu \rangle} \prod_{\alpha\in R^+} (1 - q^{\langle \alpha, \mu \rangle}) = \sum_w \det w\, q^{-\langle w\rho, \mu \rangle}$ $*$
(as $\det w = \det w^{-1}$) $\quad = \sum_w \det w\, q^{-\langle \rho, w^{-1}\mu \rangle} = \sum_w \det w\, q^{-\langle \rho, w\mu \rangle}$
$\therefore F_{\rho^\vee}(\chi(L\lambda)) = \dfrac{\sum_w \det w\, q^{-\langle w(\lambda+\rho), \rho^\vee \rangle}}{\sum_w \det w\, q^{\langle w\rho, \rho^\vee \rangle}} = \dfrac{q^{-\langle \rho^\vee, \lambda+\rho \rangle} \prod_{\alpha\in R^+} (1 - q^{\langle \alpha^\vee, \lambda+\rho \rangle})}{q^{-\langle \rho^\vee, \rho \rangle} \prod_{\alpha\in R^+} (1 - q^{\langle \alpha^\vee, \rho \rangle})}$ $\quad$ [set $\mu = \lambda+\rho$, $\mu = \rho$]
using $R^\vee$ version of $*$.

Taking $q \to 1$ (e.g. using L'Hopital rule) gives the Weyl dimension formula:

$$\dim L(\lambda) = \prod_{\alpha \in R^+} \frac{\langle \alpha^\vee, \lambda+\rho \rangle}{\langle \alpha^\vee, \rho \rangle} = \prod_{\alpha \in R^+} \frac{\langle \alpha, \lambda+\rho \rangle}{\langle \alpha, \rho \rangle}$$

recall $\langle \alpha, \rho^\vee \rangle$ = # simple roots in expansion of α = height of α.

e.g. For sl_3, $R^+ = \{\alpha, \alpha+\beta, \beta\}$ and $\lambda \in P^+$ has the form $m_1\omega_1 + m_2\omega_2$.

$\langle \alpha, \rho \rangle = 1$ $\quad$ $\langle \alpha+\beta, \rho \rangle = 2$ $\quad$ $\langle \beta, \rho \rangle = 1$

$\langle \alpha, \lambda+\rho \rangle = m_1+1$ $\quad$ $\langle \alpha+\beta, \lambda+\rho \rangle = m_1+m_2+2$ $\quad$ $\langle \beta, \lambda+\rho \rangle = m_2+1$

$\therefore \dim L(\lambda) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$

For so_5, $R^+ = \{\alpha, 2\alpha+\beta, \alpha+\beta, \beta\}$, and let $\lambda = m_1\omega_1 + m_2\omega_2$ again.

$\langle \alpha, \rho \rangle = 1$ $\quad$ $\langle 2\alpha+\beta, \rho \rangle = 3$ $\quad$ $\langle \alpha+\beta, \rho \rangle = 2$ $\quad$ $\langle \beta, \rho \rangle = 1$

$\langle \alpha, \lambda+\rho \rangle = m_1+1$ $\quad$ $\langle 2\alpha+\beta, \lambda+\rho \rangle = 2m_1+m_2+3$ $\quad$ $\langle \alpha+\beta, \lambda+\rho \rangle = m_1+m_2+2$ $\quad$ $\langle \beta, \lambda+\rho \rangle = m_2+1$

$\therefore \dim L(\lambda) = \frac{1}{6}(m_1+1)(2m_1+m_2+3)(m_1+m_2+2)(m_2+1)$

Observe that the q-dimension formula has the form $\sum_n \dim L(\lambda)_n q^n$, where we consider $L(\lambda)$ as a representation of $2t_{\rho^\vee} \in \mathfrak{t}$ (eigenspace decomposition). (take $z^2 = e^{\alpha/2} = q$)

h_i form a basis for $\mathfrak{t}$ $\therefore t_{\rho^\vee} = \sum c_i h_i$ for some $c_i \in \mathbb{R}$.

Set $E = \sum e_i$, $F = \sum c_i f_i$.

Then $[E, F] = \sum c_i [e_j, f_i] = \sum c_i \delta_{ij} h_j = t_\rho$ $\quad$ ($[e_j, f_i] = \delta_{ij} h_i$ as $\alpha_j - \alpha_i$ not a root)

$[2t_{\rho^\vee}, E] = \sum 2\alpha_i(t_{\rho^\vee}) e_i = \sum \langle \rho^\vee, \alpha_i \rangle e_i = \sum 2e_i = 2E$

$[2t_{\rho^\vee}, F] = \sum -2\alpha_i(t_{\rho^\vee}) c_i f_i = \sum 2\langle \rho^\vee, -\alpha_i \rangle c_i f_i = \sum -2c_i f_i = -2F$

$\therefore \sqrt{2}E, \sqrt{2}F$ and $2t_\rho$ form a principal sl_2 in $\mathfrak{g}$.

This shows that the q-dimension formula is symmetric (coefficient of q^n = coefficient of q^{-n}) with integer coefficients. Since $\langle \rho^\vee, \mu \rangle = \frac{1}{2}$ eigenvalue of $2t_{\rho^\vee}$ action, the exponents are half-integral or integral – in fact they can only be one or the other as difference between exponents $= \langle \rho^\vee$, difference between eigenvalues$\rangle = \langle \rho^\vee$, sum of simple roots$\rangle \in \mathbb{Z}$. $\therefore$ formula is unimodal

Example: sl_n has $n-1$ simple roots; $n-2$ positive roots which are sums of 2 simple roots

$n-3$ positive roots which are sums of 3 simple roots

$\vdots$ $\qquad$ $\vdots$

1 positive root which is the sum of all $n-1$ simple roots

$$\therefore \text{q-dimension of } L(\lambda) = \frac{q^{-\langle \rho^\vee, \lambda \rangle} \prod_{\alpha \in R^+} (1 - q^{\langle \alpha, \lambda+\rho \rangle})}{(1-q)^{n-1}(1-q^2)^{n-2} \cdots (1-q^{n-1})}$$

For the adjoint representation, $\lambda = \theta = \omega_1 + \omega_{n-1}$

$$\therefore \text{q-dimension of adjoint representation} = \frac{q^{-2}(1-q^{n-1})(1-q^{n+1})}{(1-q)^2}$$

For $\mathbb{C}^n$ representation, $\lambda = \omega_1$ $\therefore$ q-dimension $= \dfrac{q^{-?}(1-q^n)}{(1-q)}$

For $S^k\mathbb{C}^{n+1}$ representation, $\lambda = k\omega_1$ $\therefore$ q-dimension $= \dfrac{q^{-k}(1-q)^{k+1}(1-q)^{k+2} \cdots (1-q)^{k+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \begin{bmatrix} k+n \\ k \end{bmatrix} q^{-k}$

where $[n]! = \dfrac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q)\cdots(1-q)}$ $\quad$ $\therefore \begin{bmatrix} k+n \\ k \end{bmatrix} q^{-k}$ is symmetric and unimodal.

It can be shown that the principal sl_2 representation is a sum of exactly $\dim R$ irreducibles.
ie q-dimension $\cong \sum_{i=1}^{\dim R} \chi(L(2e_i))$ for some integral e_i.
Then $|W| = \prod (e_i+1)$.

Crystals (due to Kashiwara)

A crystal for a semisimple Lie algebra $\mathfrak{g}$ is a set B with functions $wt: B \to P$

$\tilde{e}_i : B \to B \amalg \{0\}$ $\qquad \varepsilon_i(b) = \max\{n \geq 0 : \tilde{e}_i^n b \neq 0\}$

$\tilde{f}_i : B \to B \amalg \{0\}$ $\qquad \varphi_i(b) = \max\{n \geq 0 : \tilde{f}_i^n b \neq 0\}$

such that: if $\tilde{e}_i(b) \neq 0$ then $wt(\tilde{e}_i b) = wt(b) - \alpha_i$?? shouldn't this be +.

if $f_i(b) \neq 0$ then $wt(\tilde{f}_i b) = wt(b) + \alpha_i$

$\tilde{e}_i b = b' \Leftrightarrow \tilde{f}_i b' = b$ $\qquad \forall b, b$

$\varphi_i(b) - \varepsilon_i(b) = \langle wt(b), \alpha_i^\vee \rangle$

This can be described by a crystal graph: let the vertices denote points of B, and colour an edge $b \to b'$ with colour i if $\tilde{e}_i(b') = b$ ie $\tilde{f}_i(b) = b'$
e.g an sl_2-string for an irreducible finite-dimensional representation: here, $\varphi_i(b) + \varepsilon_i(b)$ = length of string.

If B_1, B_2 are crystals, $B_1 \otimes B_2$ is the set $B_1 \times B_2$ where $wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$

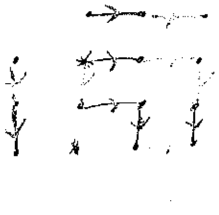
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

hence we must have
$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

e.g. if B_1, B_2 both have graphs
then $B_1 \otimes B_2$ has graph:

(vertical copy is B_2, horizontal copy is B_1)
(do each colour separately)

e.g.



this is $B_1^\vee \otimes B_2$, if we define $B^\vee$ to be B with arrows reversed:
ie $B^\vee = \{b^\vee : b \in B\}$, $wt(b^\vee) = -wt(b)$

$\tilde{e}_i(b^\vee) = (f_i b)^\vee \qquad \tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee$

$\varepsilon_i(b^\vee) = \varphi_i(b) \qquad \varphi_i(b^\vee) = \varepsilon_i(b)$

Theorem: let $L(\lambda)$ be an irreducible highest weight module for $\mathfrak{g}$, with highest weight λ.
Then there is a crystal $B(\lambda)$ whose vertices bijectively correspond to a basis of $L(\lambda)$
such that $\chi(L(\lambda)) = \sum_{b \in B(\lambda)} e^{wt(b)}$
the decomposition of $L(\lambda)$ as sl_2-modules for each simple sl_2 is given by strings of each colour in $B(\lambda)$

the connected components of $B(\lambda)\otimes B(\mu)$ describe the irreducible components of $L(\lambda)\otimes \lambda(\mu)$.

Hence the above examples show $V\otimes V = S^2V \oplus \Lambda^2 V$ with $\Lambda^2 V = V^*$
$V\otimes V^* =$ adjoint $\oplus$ trivial
where V is the 3-dimensional representation of $sl_3 = L(\omega_1)$

A crystal is integrable if it describes a finite dimensional representation.
The tensor product operation is commutative for integrable crystals, but not in general.
However, tensoring is always associative.

Call $b\in B$ a highest weight vector if $\tilde{e}_i(b)=0 \ \forall i$.
By representation theory of lie algebras, all integrable crystals contain highest weight vectors, one in each connected component (denoted $*$), and arrows leaving $*$ "generate" the whole component.

Lemma: $\tilde{e}_i(b_1\otimes b_2)=0 \Leftrightarrow \tilde{e}_i(b_1)=0$ and $\varepsilon_i(b_2)\leq \phi_i(b_1)$
Proof: $\Leftarrow$: straight from formula for $\tilde{e}_i(b_1\otimes b_2)$
$\Rightarrow$: $\tilde{e}_i(b_1\otimes b_2)=0 \Rightarrow \tilde{e}_i(b_1)=0$ and $\varepsilon_i(b_2)\leq\phi_i(b_1)$
or $\tilde{e}_i(b_2)=0$ and $\varepsilon_i(b_2)>\phi_i(b_1)$.
but $\tilde{e}_i(b_2)=0 \Rightarrow \varepsilon_i(b_2)=0$, so second alternative cannot hold.

Corollary: $b\otimes b' \in B(\lambda)\otimes B(\mu)$ is a highest weight vector
$\Leftrightarrow$ b is the highest weight vector of $B(\lambda)$, and $\varepsilon_i(b')\leq\langle\lambda,\alpha_i^\vee\rangle \ \forall i$
since $\langle\lambda,\alpha_i^\vee\rangle$ = length of i-string through b, since $\phi_i(b)=\phi_i(b)-\varepsilon_i(b)=\langle \mathrm{wt}\, b,\alpha_i^\vee\rangle$.

This gives us a rule for finding all the highest weights in $B(\lambda)\otimes B(\mu) \Rightarrow$ we can decompose any tensor product into the sum of irreducibles. We can also read off from the crystal graph the multiplicity of each weight space in any finite dimensional representation. (ie weights and corresponding multiplicities can be found completely combinatorially).
This is a miraculous result - there is no reason why a representation should come with a canonical choice of basis (which is what the vertices of the crystal graph represent).
(It was discovered via quantum groups)

The standard representations: $(1\ 2\ \cdots\ n-2\ n-1\ n)$

sl_n (A_{n-1}): $\overset{*}{1} \to 2 \to \cdots \to n-1 \to n$

so_{2n+1} (B_n): $\overset{*}{1} \to 2 \to \cdots \to n \to n+1 \to n+2 \to \cdots \to 2n \to 2n+1$

so_{2n} (D_n):

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & & n \arrow[dr] & & & \\
\overset{*}{1} \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[ur] \arrow[dr] & & n+2 \arrow[r] & \cdots \arrow[r] & 2n-1 \arrow[r] & 2n \\
 & & & & n+1 \arrow[ur] & & &
\end{tikzcd}$$

sp_{2n} (C_n): $\overset{*}{1} \to 2 \to \cdots \to n \to n+1 \to \cdots \to 2n-1 \to 2n$

Example: $\mathfrak{g}=\mathfrak{sl}_n$, $V=\mathbb{C}^n$ = standard representation.
Recall that $\Lambda^i\mathbb{C}^n$ is the irreducible representation with weight $\omega_i=\varepsilon_1+\varepsilon_2+\cdots\varepsilon_i$. $\therefore \Lambda^i\mathbb{C}^n=L(\omega_i)$
Let $\lambda=k_1\omega_1+\cdots+k_{n-1}\omega_{n-1}$, $k_i\geq 0$.
Then $L(\lambda)$ is a summand of $L(\omega_1)^{\otimes k_1}\otimes L(\omega_2)^{\otimes k_2}\otimes\cdots\otimes L(\omega_{n-1})^{\otimes k_{n-1}}$
(if v_i is highest weight vector in $L(\omega_i)$, then $v_1^{\otimes k_1}\otimes v_2^{\otimes k_2}\cdots\otimes v_{n-1}^{\otimes k_{n-1}}$ is highest weight vector of weight λ)
But each $L(\omega_i)$ is a summand of $\mathbb{C}^{n\otimes i}$
$\therefore$ every finite dimensional representation of $\mathfrak{sl}_n$ occurs as a summand of $\mathbb{C}^{n\otimes N}$, some $N\geq 0$
(ie starting with the standard crystal, we get all integrable crystals by tensoring and decomposing)

In the language of crystals this says we can embed $B(\lambda)$ into $B(\omega_1)^{\otimes k_1}\otimes\cdots\otimes B(\omega_{n-1})^{\otimes k_{n-1}}$ by sending the highest weight vector to $b_{\omega_1}^{\otimes k_1}\otimes b_{\omega_2}^{\otimes k_2}\cdots\otimes b_{\omega_{n-1}}^{\otimes k_{n-1}}$
(b_{ω_i} = highest weight vector of $B(\omega_i)$)
Vertices of $B(\omega_1)^{\otimes i}$ are described by i-tuples of $\{1,2,\cdots n-1\}$ — write this as a column vector
The highest weight vector corresponding to ω_i is $\begin{pmatrix}1\\2\\\vdots\\i\end{pmatrix}$:
· generalising the corollary by induction, we see that $b_1\otimes b_2\otimes\cdots\otimes b_r$ is a highest weight vector $\Leftrightarrow$ b_1 is a highest weight vector, $\varepsilon_i(b_j)\leq\langle\lambda_1+\lambda_2+\cdots\lambda_{j-1},\alpha_i^\vee\rangle$ $\forall i$, where $\lambda_j=wt(b_j)$ $\forall j>1$, $j\leq r$. (observe this always holds if all the b_js are highest weight vectors, as $\lambda_j\in P^+$)
· in $\mathfrak{sl}_n$, $\varepsilon_i(j)=\delta_{i,j-1}$, $wt(j)=\omega_j-\omega_{j-1}$ $\forall j>1$. Hence $\begin{pmatrix}1\\2\\\vdots\\i\end{pmatrix}$ satisfies the required conditions.
$\therefore B(\omega_i)$ spanned by $\begin{pmatrix}1\\2\\\vdots\\i\end{pmatrix}$ under $\tilde{f}_i$ action.
On $B(\omega_1)^{\otimes i}$, $\tilde{f}_i$ sends a vector to 0, or changes some i components to $i+1$.
Suppose the components are strictly increasing, and i, $i+1$ appear (in necessarily adjacent positions). By the action of $\tilde{f}_i$ on tensor products, $\tilde{f}_i$ can only send the i component to $i+1$ if $\varphi_i(i)>\varepsilon_i(i+1)$, which is not true
$\therefore B(\omega_i)\subseteq$ strictly increasing vectors
Inductively, we show that all strictly increasing vectors $\in B(\omega_i)$:
given any strictly increasing vector, let the last component be $r\Rightarrow r\geq i$. (if $r=i$, trivial)
$\therefore \forall j\in(i,r)$, $\tilde{f}_j\begin{pmatrix}1\\2\\\vdots\\i-1\end{pmatrix}=0\Rightarrow\varphi_j\begin{pmatrix}1\\2\\\vdots\\i-1\end{pmatrix}=0\Rightarrow f_j\begin{pmatrix}1\\2\\\vdots\\i-1\\j\end{pmatrix}=\begin{pmatrix}1\\2\\\vdots\\i-1\\j+1\end{pmatrix}$ $\therefore f_{r-1}f_{r-2}\cdots f_i\begin{pmatrix}1\\2\\\vdots\\i\end{pmatrix}=\begin{pmatrix}1\\2\\\vdots\\i-1\\r\end{pmatrix}$

We do the same thing to move the $i-1^{th}$ component — since we only apply f_j for $i-1\leq j<r$, the last r doesn't affect the f_j action. ($\varphi_j\begin{pmatrix}1\\2\\\vdots\\j-1\end{pmatrix}=1\geq\varepsilon_j(r)$, $\therefore \tilde{f}_j\begin{pmatrix}1\\2\\\vdots\\j-1\\r\end{pmatrix}=\tilde{f}_j\begin{pmatrix}1\\2\\\vdots\\j-1\end{pmatrix}\otimes r$)

??? $\Rightarrow$ $\therefore B(\omega_1)^{\otimes k_1}\otimes\cdots\otimes B(\omega_{n-1})^{\otimes k_{n-1}}$ can be represented by tableaux with k_i columns of length i, where each column is strictly increasing. (We usually work with $B(\omega_{n-1})^{\otimes k_{n-1}}\otimes\cdots\otimes B(\omega_1)^{\otimes k_1}$.
$\therefore$ $\begin{matrix}1&1&\cdots&1&1&\cdots&1&1\cdots1\\2&2&\cdots&2&2&\cdots&2&\\\vdots&\vdots&&&&&&\\n-1&n-1&&&&&&\end{matrix}$ is a highest weight vector of weight $\sum k_i\omega_i$ $\therefore$ it spans $B(\sum k_i\omega_i)$ under $\tilde{f}_i$ action.
It is clear that, under $\tilde{f}_i$ action, each column stays strictly increasing. In fact, the rows stay decreasing (such a tableaux is called semi-standard tableaux)
suppose the r left-most columns do not contain $i\Rightarrow\varphi_i$(any such column)$=0$.
by induction, we can show that φ_i(all these columns together)$=0$
$\therefore \tilde{f}_i$(left-most $r+1$ columns) = change $r+1^{th}$ column (which contains i), leave others.
$\tilde{f}_i$(.. .. $r+2$ columns) $=\begin{cases}\text{change } r+1^{th} \text{ column} & \text{if } \varepsilon_i(r+2^{th}\text{ column})<1\\ \text{change } r+2^{th}\text{ column} & \text{if } \varepsilon_i(r+2^{th}\text{ column})=1\end{cases}$

The second condition $\Leftrightarrow$ $r+2^{th}$ column contains $i+1$ but not i $\Rightarrow f_i(r+2^{th}\text{ column})=0$.
Under the first condition, this part of the tableaux stays decreasing. If no further columns contain $i+1$ but not i, this change in column $r+1$ is all that happens.
If $\exists$ a column with $i+1$ but not i, then the column which changes is the next column containing i to the left of the rightmost such column.
$\therefore$ on any row where i is repeated, any change $i \to i+1$ on that row must occur for the leftmost i.
And we can move the highest weight vector to any semi-standard tableaux by creating the n entries, then the $n-1$ entries, etc, from left to right e.g.:

$$\begin{tikzcd}
\begin{array}{llll}1&1&1&1\\2&2&2\\3\end{array} \arrow[r, "\tilde f_3"] & \begin{array}{llll}1&1&1&1\\2&2&2\\4\end{array} \arrow[r, "\tilde f_4"] & \begin{array}{llll}1&1&1&1\\2&2&2\\5\end{array} \arrow[r, "\tilde f_2"] & \begin{array}{llll}1&1&1&1\\3&2&2\\5\end{array} \arrow[d, "\tilde f_3"] \\
\begin{array}{llll}2&2&1&1\\4&3&2\\5\end{array} & \begin{array}{llll}2&1&1&1\\4&3&2\\5\end{array} \arrow[l, "\tilde f_1"] & \begin{array}{llll}1&1&1&1\\4&3&2\\5\end{array} \arrow[l, "\tilde f_1"] & \begin{array}{llll}1&1&1&1\\4&2&2\\5\end{array} \arrow[l, "\tilde f_2"]
\end{tikzcd}$$

So the connected component of $B(\lambda)$ has vertices in bijection with semi-standard tableaux.

Not every representation of B_n can be generated by subrepresentations of ${\mathbb{C}^n}^{\otimes N}$, we also require the spin representations:
$B(\omega_n) = \{(i_1, i_2, \cdots i_n) : i_k = 1 \text{ or } -1\}$ $\quad wt(i_1, \cdots i_n) = \frac{1}{2}\sum_k i_k \varepsilon_k$

$$\tilde e_j = \begin{cases} \text{swap } i_j \text{ and } i_{j+1} & \text{if } i_j=-1, i_{j+1}=+1 \\ 0 & \text{otherwise}\end{cases} \qquad \tilde e_n = \begin{cases} \text{change } i_n \text{ to } +1 & \text{if } i_n=-1 \\ 0 & \text{otherwise}\end{cases}$$

Observe that the highest weight vector is $(1,1,1,\cdots 1)$.

Similarly, for D_n, we require the half spin representations:
$L(\omega_{n-1})$ corresponds to $B^- = \{(i_1, i_2, \cdots i_n) : i_k = 1 \text{ or } -1, \prod i_j = -1\}$
$L(\omega_n)$ corresponds to $B^+ = \{(i_1, i_2, \cdots i_n) : i_k = 1 \text{ or } -1, \prod i_j = +1\}$

$$\tilde e_j = \begin{cases} \text{swap } i_j \text{ and } i_{j+1} & \text{if } i_j=-1, i_{j+1}=+1 \\ 0 & \text{otherwise}\end{cases} \qquad \tilde e_n = \begin{cases} \text{change } i_{n-1}, i_n \text{ both to } +1 & \text{if } i_{n-1}=i_n=-1 \\ 0 & \text{otherwise.}\end{cases}$$

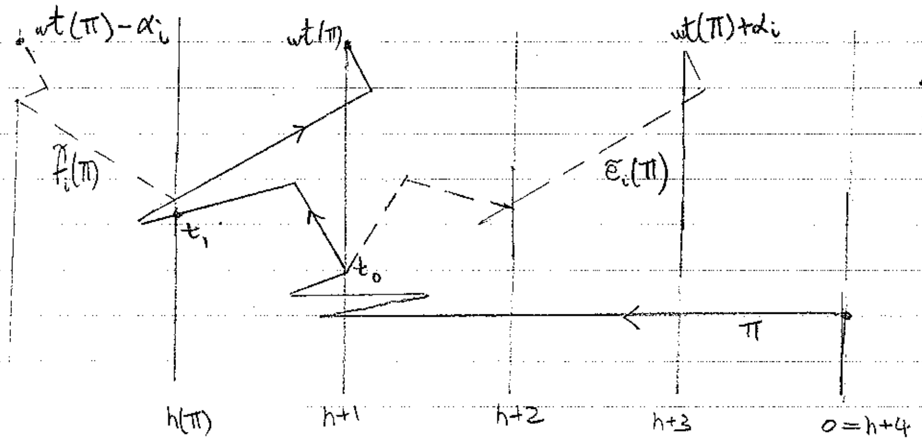
Littelmann paths are explicit constructions of crystals.
Consider paths = piecewise-linear continuous maps: $[0,1] \to \mathbb{R}P$, up to reparametrisation
Restrict to paths π where $\pi(0)=0$, $\pi(1)\in P$, and set $wt(\pi)=\pi(1)$
Let $h = \min \{0\} \cup \{\mathbb{Z} \cap \langle \pi(t), \alpha_i^\vee \rangle : 0 \le t \le 1\}$
= smallest integer in $\langle \pi[0,1], \alpha_i^\vee \rangle$, (including endpoints)
If $h_i = 0$, set $\tilde e_i(\pi) = 0$
If $h_i > 0$, set T_i = first time it crosses $\langle x, \alpha_i^\vee \rangle = h$ $\quad = \min\{t : \langle \pi(t), \alpha_i^\vee \rangle = h\}$
t_i = last time it crosses $\langle x, \alpha_i^\vee \rangle = h+1$ before T_i $= \max\{t < T_i : \langle \pi(t), \alpha_i^\vee \rangle = h+1\}$

$$\text{Then } \tilde e_i(\pi) = \begin{cases} \pi(t) & 0 \le t \le t_i \\ \pi(t_i) + s_{\alpha_i}(\pi(t) - \pi(t_i)) & t_i \le t \le T_i \quad \text{(reflect)} \\ \pi(t) + \alpha_i & T_i \le t \le 1 \quad \text{(translate)} \end{cases}$$

and $\tilde f_i(\pi) = [\tilde e_i(\pi^\vee)]^\vee$, where $\pi^\vee$ is π reversed (and translated to begin at the origin)

Hence $\pi^\vee(t)=\pi(1-t)-\pi(1) \Rightarrow \langle \pi^\vee(t), \alpha_i^\vee \rangle = \langle \pi(1-t), \alpha_i^\vee \rangle - \langle wt(\pi), \alpha_i^\vee \rangle$ $\quad \therefore wt(\pi^\vee) = -wt(\pi)$

$\Rightarrow \quad h_i(\pi^\vee) = h_i(\pi) - \langle wt(\pi), \alpha_i^\vee \rangle$



From this diagram it is clear that $\tilde{e}_i b = b' \Leftrightarrow \tilde{f}_i(b') = b$.

Observe $h_i(\tilde{e}_i(\pi)) = h_i(\pi)+1$ (if $\tilde{e}_i(\pi) \neq 0$) $\quad \therefore \varepsilon_i(\pi) = -h_i(\pi)$

$h_i(\tilde{f}_i(\pi)) = h_i(\pi)+1 - \langle wt\,\pi, \alpha_i^\vee \rangle - \langle wt\, e_i \pi_i^\vee, \alpha_i^\vee \rangle$ $\quad$ if $h_i(\pi) \neq \langle wt(\pi), \alpha_i^\vee \rangle$

$= h_i(\pi)+1 - \langle wt\,\pi, \alpha_i^\vee \rangle - \langle -wt\,\pi + \alpha_i, \alpha_i^\vee \rangle = h_i(\pi) - 1$

$\tilde{f}_i(\pi) = 0 \Leftrightarrow \tilde{e}_i(\pi^\vee) = 0 \Leftrightarrow h_i(\pi) = \langle wt(\pi), \alpha_i^\vee \rangle$

$h_i(\pi) \leq \langle wt(\pi), \alpha_i^\vee \rangle$, $h_i(\pi)$ falls by 1 and $\langle wt(\pi), \alpha_i^\vee \rangle$ falls by 2 when we apply $\tilde{f}_i$

$\therefore \varphi_i(\pi) = \langle wt(\pi), \alpha_i^\vee \rangle - h_i(\pi)$ $\quad \therefore \varphi_i(\pi) - \varepsilon_i(\pi) = \langle wt(\pi), \alpha_i^\vee \rangle$

$\therefore$ this is indeed a crystal.

Observe that, if $\pi[0,1] \subseteq \mathbb{R}^+ P$, $\tilde{e}_i(\pi) = 0 \ \forall i$

Define B_π = crystal generated by π $= \{\tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r}(\pi)\}$

Theorem: for $\pi, \pi' \subseteq \mathbb{R}^+ P$, $B_\pi \simeq B_{\pi'} \Leftrightarrow \pi(1) = \pi'(1)$, and $B_\pi \simeq B(\pi(1))$ (as defined earlier)

$\therefore B(\lambda)$ can be defined without using $L(\lambda)$, and this proves directly the Weyl character formula. (Littelmann constructed $B(\lambda)$ explicitly, generated by $\pi(t) = t\lambda$)

Example: $\mathfrak{g} = \mathfrak{sl}_2$ $\quad B(\alpha/2)$: $\tilde{f}(\overrightarrow{0 \ \ \alpha/2}) = (\overleftarrow{-\alpha/2 \ \ 0})$, $\tilde{f}(\overleftarrow{-\alpha/2 \ \ 0}) = 0$

$B(\alpha)$: $\tilde{f}(\overrightarrow{0 \ \ \alpha}) = (\overleftrightarrow{-\alpha/2 \ \ 0})$, $\tilde{f}(\overleftrightarrow{-\alpha/2 \ \ 0}) = (\overleftarrow{-\alpha \ \ 0})$

$\tilde{f}(\overleftarrow{-\alpha \ \ 0}) = 0$

For paths π_1, π_2, define $\pi_1 * \pi_2$ to be their concatenation: $\pi_1 * \pi_2(t) = \begin{cases} \pi_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \pi_1(1) + \pi_2(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$

$h_i(\pi_1 * \pi_2) = \min\{h_i(\pi_1), h_i(\pi_2) + \langle wt\,\pi_1, \alpha_i^\vee \rangle\}$

If $h_i(\pi_1) \leq h_i(\pi_2) + \langle wt\,\pi_1, \alpha_i^\vee \rangle \Leftrightarrow \varepsilon_i(\pi_2) \leq \varphi_i(\pi_1)$, then $T_i \leq \frac{1}{2} \Rightarrow t_i \leq \frac{1}{2}$.

$\therefore$ part of π_1 reflected, part of π_1 and all of π_2 translated $\Rightarrow \tilde{e}_i(\pi_1 * \pi_2) = \tilde{e}_i \pi_1 * \pi_2$.

If $h_i(\pi_1) > h_i(\pi_2) + \langle wt\,\pi_1, \alpha_i^\vee \rangle \Leftrightarrow \varepsilon_i(\pi_2) > \varphi_i(\pi_1)$, then $T_i > \frac{1}{2} \Rightarrow t_i > \frac{1}{2}$

$\therefore$ part of π_2 reflected, part of π_2 translated, π_1 fixed $\Rightarrow \tilde{e}_i(\pi_1 * \pi_2) = \pi_1 * \tilde{e}_i \pi_2$

$\therefore$ concatenation describes the tensor product of crystals.

Example: $g = so_{2n}$. Choose a basis of GL_n so that the symmetric form is given by $\begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix} = K$.
Then $A \in GL_n$ preserves this form if $A^T K A = K$
Differentiating gives $so_{2n} = \{X \in gl_n : X^T K + KX = 0\}$

$$so_{2n} \cap \{\text{diagonal elements of } gl_{2n}\} = \left\{ \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_{2n} \end{pmatrix} : \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_{2n} \end{pmatrix} \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix} = - \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix} \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_{2n} \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_{2n} \end{pmatrix} : \begin{pmatrix} & & & b_1 \\ & & b_2 & \\ & ⋰ & & \\ b_{2n} & & & \end{pmatrix} = - \begin{pmatrix} & & b_{2n} \\ & ⋰ & \\ b_1 & & \end{pmatrix} \right\}$$

$= \text{span } B_i$, $B_i = -1$ in entry $2n+1-i, 2n+1-i$, $(i \leq n)$
1 in entry i,i, 0 elsewhere.

$\therefore$ right multiplication by B_i removes all but column i, and column $2n+1-i$ with reversed sign
left multiplication by B_i removes all but row i, and row $2n+1-i$ with reversed sign.
$\therefore$ only diagonal elements of so_{2n} can commute with all B_is.
$\therefore$ if ad B_i is diagonalisable $\forall i$, span B_i is a maximal torus.

$(X^T K + KX)_{ij} = X_{(2n+1)-j,i} + X_{(2n+1)-i,j}$ (symmetric in i,j; non-zero if $i=j$)
$\therefore$ a basis of so_{2n} is A_{ij} with 1 in entry $(2n+1)-j, i$ and 0 elsewhere
-1 in entry $(2n+1)-i, j$ for $i<j$. $(A_{ij} = -A_{ji})$
if $j = 2n+1-i$, then A_{ij} is diagonal - forget about these for now ($\because i, j, 2n+1-i, 2n+1-j$ distinct)
$[B_i, A_{ij}] = -A_{ij}$ if $i \leq n$ $\quad [B_{2n+1-i}, A_{ij}] = A_{ij}$ if $i > n$
$[B_j, A_{ij}] = -A_{ij}$ if $j \leq n$ $\quad [B_{2n+1-j}, A_{ij}] = A_{ij}$ if $j > n$
and $[B_k, A_{ij}] = 0$ for all other k. $\therefore A_{ij}$ are 1-dimensional eigenspaces of ad action of span B_i.
$\therefore$ if $i<j\leq n$, $\underline{x}(A_{ij}) = (-x_i - x_j)(A_{ij})$
if $i \leq n < j$ $\quad \underline{x}(A_{ij}) = (-x_i + x_{2n+1-j})(A_{ij})$
if $n < i < j$ $\quad \underline{x}(A_{ij}) = (x_{2n+1-i} + x_{2n+1-j})(A_{ij})$ $\quad \therefore$ the eigenvalues have the form $x_i \pm x_j$

$\therefore$ root space decomposition: $so_{2n} = t \oplus_{i\neq j, 1\leq i,j\leq n} g_{\pm\epsilon_i \pm \epsilon_j}$

so_{2n} is simple for $n \geq 3$:
By adapting sl_n proof, we need only show:

1. $\exists B \in t$ with all $\alpha(B)$ nonzero: set $b_i = 3^i$ $1 \leq i \leq n$, consider the highest power of 3 dividing them

2. Adjoint action on weight spaces is transitive:
first, we move A_{ij} to A_{ik}. There are 2 possibilities:
$k < 2n+1-j$: $[A_{ij}, A_{k\, 2n+1-j}] = A_{ik}$
$2n+1-j < k$: $[A_{ij}, A_{2n+1-j\, k}] = -A_{ik}$
(use 2 moves if $k = 2n+1-j$ - assume $n \geq 3$)
now, move A_{ik} to A_{lk}. Again, 2 possibilities:
$l < 2n+1-i$: $[A_{ik}, A_{l\, 2n+1-i}] = A_{lk}$
$2n+1-i < l$: $[A_{ik}, A_{2n+1-i\, l}] = -A_{lk}$ (use 2 moves if $l = 2n+1-i$)

3. every B_i can be obtained from some A_{ij} by adjoint action:

$[A_{ij}, A_{2n+1-j}] = -B_i - B_j$
$\therefore \forall i \;\; B_i = \frac{1}{2}(B_i+B_j) + \frac{1}{2}(B_i+B_k) - \frac{1}{2}(B_j+B_k) \in$ an arbitrary ideal.

4. given any two roots α, β, $\exists B \in \mathfrak{t}$ with $\alpha(B) \neq \beta(B)$:
for every root $\pm\varepsilon_i \pm \varepsilon_j$, $\pm B_i \pm B_j$ is sent to zero by no root except this one and its negative. B_i would distinguish between $\varepsilon_i - \varepsilon_j$ and $\varepsilon_j - \varepsilon_i$.

Example: $\mathfrak{g} = so_{2n+1}$
Define diagonal elements B_i as before (no representative of middle co-ordinate)
Again, only diagonal elements can commute with all B_i
$\Rightarrow B_i$ spans a maximal abelian sub-algebra

Basis of $so_{2n+1} = \{B_i : 1 \leq i \leq n\} \cup \{A_{ij} : i<j ; j \neq 2n+2-i\}$
where A_{ij} has 1 in entry $2n+2-j, i$
-1 in entry $2n+2-i, j$
So, as before: $[B_i, A_{ij}] = -A_{ij}$ $\quad i \leq n \qquad [B_{2n+2-i}, A_{ij}] = A_{ij} \quad i \geq n+2$
$[B_j, A_{ij}] = -A_{ij} \quad j \leq n \qquad [B_{2n+2-j}, A_{ij}] = A_{ij} \quad j \geq n+2$
and B_k commutes with A_{ij} if $k \neq i, j, 2n+2-i, 2n+2-j$
$\therefore A_{ij} \in -\varepsilon_i - \varepsilon_j$ weight space $\quad i<j \leq n$
$A_{ij} \in \varepsilon_{2n+2-j} - \varepsilon_i$ weight space $\quad i \leq n, n+2 \leq j$
$A_{ij} \in \varepsilon_{2n+2-j} + \varepsilon_{2n+2-i}$ weight space $\quad n+2 \leq i < j$
$A_{ij} \in -\varepsilon_i$ weight space $\quad i < j = n+1$
$A_{ij} \in \varepsilon_{2n+2-j}$ weight space $\quad i = n+1 < j$
This is the root system B_n.

so_{2n+1} is simple
1. Again, take $b_i = 3^i \quad 1 \leq i \leq n \Rightarrow \alpha(B) \neq 0 \quad \forall$ roots α.
2, 3: above method generalises (for all n)
4. To distinguish $\pm\varepsilon_i \pm \varepsilon_j$ from something else, follow above procedure.
Easy to distinguish $\pm\varepsilon_i$ from each other $- \sum a_i B_i$ with all $a_i > 0$ and distinct.

Standard representation is $2n+1$-dimensional
i^{th}-basis vector has weight $\begin{cases} \varepsilon_i & i \leq n \\ -\varepsilon_{2n+2-i} & n+2 \leq i \\ 0 & i = n+1 \end{cases}$ $\Rightarrow$ highest weight is ε_1

$A_{i\,2n+2-j}$ or $A_{2n+2-j\,i}$ sends v_i to $\pm v_j$ for any $i \neq j \Rightarrow$ this is irreducible.

$\Lambda^2 V$ has weights $\varepsilon_i, -\varepsilon_i$ $(1 \leq i \leq n)$ $\pm\varepsilon_i \pm \varepsilon_j$ $(i \neq j)$, all with multiplicity one, and 0 with multiplicity n.
The decomposition into irreducibles = adjoint representation

$S^2 V$ has weights $\pm\varepsilon_i$, $\pm\varepsilon_i \pm \varepsilon_j$ $(i \neq j)$, $\pm 2\varepsilon_i$, all with multiplicity one, and 0 with multiplicity $n+1$ $\Rightarrow$ highest weight is $2\varepsilon_1$. Weights of an irreducible representation are invariant under Weyl group, and weights form uninterrupted strings

$\Rightarrow$ all non-zero weights also occur in the representation generated by $2\varepsilon_1$.
Other irreducible summands = copies of trivial.
The zero weight space has basis $\{v_i \times v_{2n+2-i} : 1 \le i \le n\}$
For $1 \le j \le n-1$, $A_{j+1\; 2n+2-j}(v_i \times v_{2n+2-i}) = \delta_{i\,j+1} v_j \times v_{2n+2-i} - \delta_{ij} v_i \times v_{2n+1-j}$
$\therefore \ker A_{j+1\; 2n+2-j}$ is spanned by $\{v_i \times v_{2n+2-i} : i \ne j, j+1\} \cup \{v_j \times v_{2n+2-j} + v_{j+1} \times v_{2n+1-j}\}$
$\therefore \ker \{A_{j+1\; 2n+2-j} : 1 \le j \le n-1\}$ = spanned by $\sum_{j=1}^{n-1} v_j \times v_{2n+2-j} + v_{j+1} \times v_{2n+1-j}$

(As with so_{2n}, $A_{j+1\; 2n+2-j}$ corresponds to the first $n-1$ simple roots – these are the root-matrices with entries on the super-diagonal $\left(\begin{smallmatrix} & & \ddots \\ & -1 & \end{smallmatrix}\right)$. The last simple root is on a "higher" diagonal)

$A_{n+2\; n+3}(v_n \times v_{n+2}) = v_n \times v_{n-1}$, $A_{n+2\; n+3}(v_{n-1} \times v_{n+3}) = -v_{n-1} \times v_n$
and all other vectors of the form $v_i \times v_{2n+2-i}$ are sent to 0
$\therefore \ker \{A_{j+1\; 2n+2-j} : 1 \le j \le n-1\} \subseteq \ker A_{n+2\; n+3} \Rightarrow$ space of highest weight vectors of weight 0 has dimension $1 \Rightarrow 1$ copy of trivial representation in $S^2 V$.

Example: $g = sp_{2n}$
$SP_{2n} = \{A : A^T J A = J\}$ where $J = \left(\begin{smallmatrix} & & & 1 \\ & & \iddots & \\ & -1 & & \\ \iddots & & & \end{smallmatrix}\right)$ ie A preserves a skew-symmetric form
$\therefore sp_{2n} = \{X : X^T J + JX = 0\}$
$(X^T J + JX)_{ij} = \pm X_{2n+1-j\; i} \pm X_{2n+1-i\; j}$
first sign is $-$ if $j > n$, second sign is $-$ if $i \le n$.
$\therefore$ basis: $A_{ij} = \begin{cases} 1 \text{ in entry } 2n+1-j, i & \text{if } j > n \\ -1 \text{ in entry } 2n+1-j, i & \text{if } j \le n \end{cases}$ $\begin{cases} -1 \text{ in position } 2n+1-i, j & \text{if } i \le n \\ 1 \text{ in position } 2n+1-i, j & \text{if } i > n \end{cases}$ $(i < j)$
and $A_i = 1$ in entry $2n+1-i, i$, $1 \le i \le 2n$.
write B_i for $A_{i\; 2n+1-i}$ — these are diagonal: 1 in entry i,i; -1 in entry $2n+1-i, 2n+1-i$
same argument as before shows only diagonal matrices can commute with all B_i.

$[B_i, A_{ij}] = -A_{ij}$ $\quad i \le n$ $\qquad [B_{2n+1-i}, A_{ij}] = A_{ij}$ $\quad i > n$ $\qquad [B_i, A_i] = -2A_i$ $\quad i \le n$
$[B_j, A_{ij}] = -A_{ij}$ $\quad j \le n$ $\qquad [B_{2n+1-j}, A_{ij}] = A_{ij}$ $\quad j > n$ $\qquad [B_{2n+1-i}, A_i] = 2A_i$ $\quad i > n$
$\therefore A_{ij}$ has weight: $-\varepsilon_i - \varepsilon_j$ $\quad i < j \le n$
$\varepsilon_{2n+1-j} - \varepsilon_i$ $\quad i \le n < j$
$\varepsilon_{2n+1-j} + \varepsilon_{2n+1-i}$ $\quad n < i \le j$

A_i has weight: $-2\varepsilon_i$ $\quad i \le n$
$2\varepsilon_{2n+1-i}$ $\quad i > n$.

$\therefore$ this is the root system C_n.

sp_{2n} is simple:
1. $b_i = b^i$ has $\pm\varepsilon_i \pm \varepsilon_j$, $\pm 2\varepsilon_i$ all distinct
2. $[A_{ij}, A_{k\; 2n+1-j}] = \pm A_{ik}$ ($-$ if $j \le n$)
$[A_{ij}, A_{2n+1-j\; k}] = \pm A_{ik}$ ($-$ if $j \le n$)
$[A_{ik}, A_{l\; 2n+1-i}] = \pm A_{lk}$ ($-$ if $i \le n$)
$[A_{ik}, A_{2n+1-i\; l}] = \pm A_{lk}$ ($-$ if $i \le n$)
$\therefore$ adjoint action is transitive on A_{ij}.

$[A_{j\,2n+1-i}, A_i] = A_{ij}$ or A_{ji}, whichever is valid ; $[A_{ij}, A_{2n}] = A_{j\,2n}$
$[A_{2n+1-j\;i}, A_{ij}] = \pm 2A_i$ ($-$ if $i \leq n < j$) ; $[A_{2n+1-j\;2n}, A_{j\,2n}] = \pm 2A_{2n}$ ($-$ if $j \leq n$)
$\therefore$ adjoint action transitive on $\{A_{ij}\} \cup \{A_i\}$

3. $[A_{2n+1-i}, A_i] = B_i$

4. Same as for B_n root system.

Standard representation : v_i has weight ε_i $\quad i \leq n$
$\qquad -\varepsilon_{2n+1-i}$ $\quad i > n$.
$\therefore$ highest weight $= \varepsilon_1$. This is only fundamental weight $\Rightarrow$ this representation is irreducible.
simple roots correspond to $A_{j+1\;2n+1-j} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ for $1 \leq j < n$, and $A_{n+1} = \begin{pmatrix} 1 & \\ & \ddots \end{pmatrix}$

Weights of $\Lambda^2 V = \{\pm\varepsilon_i \pm \varepsilon_j\}$ with multiplicity 1, 0 with multiplicity n
the only fundamental weights in this list are $\varepsilon_1 + \varepsilon_2$ and 0. $\therefore \Lambda^2 V = L(\varepsilon_1 + \varepsilon_2) \oplus$ some trivials
0-weight space has basis $v_i \wedge v_{2n+1-i}$, $1 \leq i \leq n$.
$A_{j+1\;2n+1-j}(v_i \wedge v_{2n+1-i}) = \delta_{i\,j+1} v_j \wedge v_{2n+1-i} - \delta_{ij} v_i \wedge v_{2n-j}$
$\therefore \ker\{A_{j+1\;2n+1-j} : 1 \leq j < n\} =$ multiple of $\sum_{j=1}^{n} v_j \wedge v_{2n+1-j}$
and $A_{n+1}(\sum_{j=1}^{n} v_j \wedge v_{2n+1-j}) = 0$ since $A_{n+1}(v_n \wedge v_{n+1}) = v_n \wedge v_n = 0$
$\therefore$ exactly 1 copy of trivial in $\Lambda^2 V$ (0 is singular weight of

Weights of $S^2 V = \{\pm 2\varepsilon_i\}, \{\pm\varepsilon_i \pm \varepsilon_j\}$ with multiplicity 1, 0 with multiplicity n
$\qquad =$ weights of adjoint representation
$\therefore S^2 V = L(2\varepsilon_1)$

PBW theorem

Let $\mathfrak{g}$ be any Lie algebra. Its universal enveloping algebra $U\mathfrak{g}$ is the associative algebra generated by $\mathfrak{g}$, with relation $x \otimes y - y \otimes x = [x,y]$
More formally, for any vector space V (over a field k), the tensor algebra of V is
$TV = k + V + V^{\otimes 2} + V^{\otimes 3} + \cdots$, where multiplication of monomials is given by the tensor product.
Then $U\mathfrak{g} = T\mathfrak{g}/J$ where J is the ideal generated by $x \otimes y - y \otimes x - [x,y]$.

In general, an enveloping algebra for $\mathfrak{g}$ is a linear map $\iota : \mathfrak{g} \to A$ where A is an associative algebra, such that $\iota x\, \iota y - \iota y\, \iota x = \iota[x,y]$
e.g. if V is a representation of $\mathfrak{g}$, $\mathrm{End}(V)$ is an enveloping algebra.

$U\mathfrak{g}$ satisfies a universal property:
given any enveloping algebra A, $\exists! f : U\mathfrak{g} \to A$ with $f(g) = \iota(g)\ \forall g \in \mathfrak{g}$.

$$\begin{tikzcd} & U\mathfrak{g} \arrow[d, dashed, "\exists! f"] \\ \mathfrak{g} \arrow[ur] \arrow[r, "\iota"] & A \end{tikzcd}$$

(we can always extend $\iota : \mathfrak{g} \to A$ to $T\mathfrak{g} \to A$; A is enveloping algebra so $J \subseteq \ker \iota \Rightarrow$ descends to $f : U\mathfrak{g} \to A$ an algebra homomorphism; f unique since algebra homomorphisms are completely defined by the image of their generators)
$\therefore$ taking $A = \mathrm{End}(V)$, this says every representation of $\mathfrak{g}$ extends uniquely to a representation of $U\mathfrak{g}$.

Observe that $T\mathfrak{g}$ is graded, but $x \otimes y - y \otimes x - [x,y]$ is not a homogeneous relation
$\therefore U\mathfrak{g}$ is filtered but not graded: let $U\mathfrak{g}_n$ = span of products of $\leq n$ elements of $\mathfrak{g}$.
$\therefore$ work with the associated graded algebra $\mathrm{gr}U\mathfrak{g} = \bigoplus_{n \geq 0} U\mathfrak{g}_{n+1}/U\mathfrak{g}_n$
Observe that, if $x \in U\mathfrak{g}_1$, $y \in U\mathfrak{g}_1$, $xy - yx = [x,y] \in U\mathfrak{g}_1$, and applying this inductively shows that, $\forall x \in U\mathfrak{g}_n, y \in U\mathfrak{g}_m$, $xy - yx \in U_{n+m-1}$ $\therefore U\mathfrak{g}$ is commutative.

$\therefore$ we can define $S\mathfrak{g} \to \mathrm{gr}U\mathfrak{g}$, which is clearly surjective. The theorem says that this is in fact an isomorphism ie given a basis $x_1, x_2, \cdots x_n$ of $\mathfrak{g}$, monomials $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ form a basis for $\mathrm{gr}U\mathfrak{g}$.