Introduction

Updated 16 October 2009: changed the intro a bit
Updated 2 April 2009: Fixed some typos
Updated 25 March 2009: I add a little about the consequences of a bijective antipode.

Here are some notes which go with my Cambridge Part III seminar talk (March 2009) on "Hopf Algebras and Representation Theory", repeated recently for Stanford's graduate student seminar (October 2009). I wrote this while I was learning the material, and not all of it made into my talk, plus the content was rearranged slightly for the talk. The last section on duality is meant to be a rough sketch only.

I apologise for the strange appearance of these notes - this was typed in Microsoft Word before I learnt to LaTeX.

I started a version of these notes which uses universal enveloping algebras of Lie algebras and quantum groups as examples instead of the group algebra, they're not quite complete, but you can email me for the draft: amypang@stanford.edu . Also please email me if you find errors, thanks.

Amy Pang, 16 October 2009

References / further reading:
M. Sweedler, Hopf Algebras - the standard text on Hopf algebras, takes a very category-theoretic approach.
J.C. Jantzen, Lectures on Quantum Groups, Chapter 3 - develops the axioms from the representation theory point of view, using quantum groups as the main example.
C. Kassel, Quantum Groups - haven't read this myself, but I'm told it's similar to Jantzen.
Wikipedia, Representations of Hopf Algebras - very short, but probably closest to what I do here.

Official abstract:
In this talk, I will use the properties of tensor and dual representations to motivate the definition of a Hopf algebra. Our main examples will be the group algebra CG and the vector space of functions:G->C; these two algebras are dual, and I will explain what that means. I may mention Lie algebras in passing, but most of the talk will be accessible to anyone with basic knowledge of the representation theory of groups.

A word on algebras

We work over $\mathbf{C}$ here (though all definitions can be made over arbitrary rings, see Sweedler's book). A $\mathbf{C}$-*algebra* A is simultaneously a vector space over $\mathbf{C}$ and a ring. These two structures are compatible in that, for $\lambda \in \mathbf{C}$, $a,b \in A$, $\lambda(ab) = (\lambda a)b = a(\lambda b)$. (We will assume that multiplication in A is associative - strictly speaking, A is an *associate algebra*.) As a ring, A has a unit, denoted 1; we will often consider scalar multiples of 1 as a copy of $\mathbf{C}$ inside A. For example, the set of n-by-n matrices is an algebra, its unit is the identity matrix. Observe that the multiplication in an algebra need not be commutative.

A representation of an algebra A is a map$: A \to End(V)$ for some vector space V which preserves all the structure of A - in other words, it is an *algebra homomorphism*. We will see that a Hopf algebra is an algebra whose representations behave in very nice ways.

The simplest example of a Hopf algebra is the group algebra $\mathbf{C}G$. For any group G, $\mathbf{C}G$ is the vector space whose basis are in bijection with the elements of G. We usually think of elements of $\mathbf{C}G$ as linear combinations of elements of G. We multiply basis elements of $\mathbf{C}G$ according to the multiplication in G, and this extends linearly to give a multiplication on all of $\mathbf{C}G$. Under this multiplication, the ring identity of $\mathbf{C}G$ is a single copy of the identity element of G.

Any $\mathbf{C}$-linear map on $\mathbf{C}G$ is completely defined by its values on G; conversely, any map from G to a $\mathbf{C}$-vector space can be uniquely extended to a $\mathbf{C}$-linear map on $\mathbf{C}G$. Since multiplication in $\mathbf{C}G$ is inherited from that of G, representations of $\mathbf{C}G$ are in bijection with representations of G. When discussing representations, we often work with $\mathbf{C}G$ because of the extra additive structure. Remember

Comultiplication and tensor representations

Given representations V, W of any $\mathbf{C}$-algebra A, we would like to define a related action on $V \otimes W$ (all tensor products are over $\mathbf{C}$). First observe that, given $f \in End(V)$, $g \in End(W)$, we can define

$$f \otimes g : V \otimes W \to V \otimes W, \ (f \otimes g)(v \otimes w) = fv \otimes gw \qquad (1)$$

(extended linearly to all of $V \otimes W$). The association $(f,g) \to f \otimes g$ is $\mathbf{C}$-bilinear, so it induces a map $End(V) \otimes End(W) \to End(V \otimes W)$. Hence, identifying A with its images in $End(V)$ and $End(W)$, we have a map $A \otimes A \to End(V \otimes W)$. So we might hope that a $\mathbf{C}$-linear *comultiplication* map $\Delta : A \to A \otimes A$ will allow us to 'multiply' two arbitrary representations. Following Sweedler's notation, we will write $\Delta(a)$ as $\sum a_{(1)} \otimes a_{(2)}$, for each $a \in A$.

Ideally, this new tensor multiplication should contain a unit - some trivial representation I with $V \otimes I$, $I \otimes V$, V all isomorphic as representations, for any representation V.

For finite-dimensional V and $V \otimes I$ to be isomorphic as vector spaces, the trivial representation must have dimension 1, ie the A-linear action on I is given by a *counit* map $\varepsilon : A \to End(\mathbf{C}) = \mathbf{C}$, which is again $\mathbf{C}$-linear. Fix a basis vector e of I. Then, as vector spaces, $V \otimes I \approx I \otimes V \approx V$ via the identification $v \otimes e \cong e \otimes v \cong v \ \ \forall v \in V$. For these to be isomorphisms of representations, we require

$$\mu(\iota \otimes \varepsilon)\Delta(a) = \mu(\varepsilon \otimes \iota)\Delta(a) = a \quad \Leftrightarrow \quad \sum \varepsilon(a_{(2)})a_{(1)} = \sum \varepsilon(a_{(1)})a_{(2)} = a \qquad (2)$$

where ι denotes the identity map on A, and μ is the multiplication of factors: $\mu(a \otimes b) = ab$. (Here, we consider ε as a map $: A \to A$, by identifying $\mathbf{C}$ with its embedding in A.) If A is non-commutative, then μ need not be an algebra homomorphism; however, direct computation shows that $\mu(\iota \otimes \varepsilon)$ and $\mu(\varepsilon \otimes \iota)$ are homomorphisms, since $\varepsilon(a) \in \mathbf{C}$ commutes with all of A.

Also, we are used to multiplication being associative, so let us impose *coassociativity*: $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$ for any representations V_1, V_2, V_3. In terms of comultiplication in our algebra A, this says:

$$(\Delta \otimes \iota)\Delta(a) = (\iota \otimes \Delta)\Delta(a) \quad \Leftrightarrow \quad \sum \Delta(a_{(1)}) \otimes a_{(2)} = \sum a_{(1)} \otimes \Delta(a_{(2)}) \qquad (3)$$

Definition A *coalgebra* is a $\mathbf{C}$-vector space with comultiplication and counit maps satisfying (2) and (3). Figure 4 shows these axioms as a diagram.

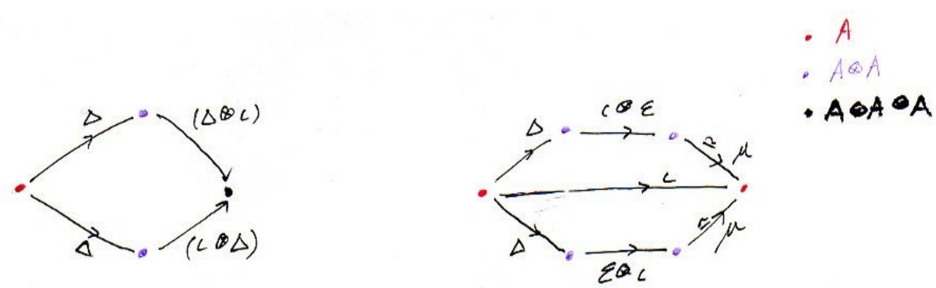

Figure 4 - Diagrammatic presentation of the coalgebra axioms (2) and (3)

Observe that (2) and (3) do not involve the multiplicative structure of A ($\mu(a \otimes b)$ is always defined if either a or b is a scalar); in general, coalgebras need not be algebras also.

Our axioms for comultiplication (2) and (3) are not sufficient to ensure that $V \otimes W$ is a representation - we need the comultiplication and multiplication to interact in a compatible way. Specifically, Δ and ε must be algebra homomorphisms, since, by definition, any representation of A is given by a homomorphism. In the definition below, we write out these conditions explicitly. The algebra structure on $A \otimes A$ is that imposed by our $A \otimes A$-action on $V \otimes W$ - namely, for any $a_1, b_1 \in End(V)$, $a_2, b_2 \in End(W)$,

$$\begin{aligned}(a_1 \otimes a_2)(b_1 \otimes b_2)(v \otimes w) &= (a_1 \otimes a_2)(b_1 v \otimes b_2 w) \\ &= (a_1 b_1 v \otimes a_2 b_2 w) \\ &= (a_1 b_1 \otimes a_2 b_2)(v \otimes w)\end{aligned} \tag{5}$$

So we should define multiplication on $A \otimes A$ by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1 \otimes a_2 b_2) \tag{6}$$

Definition A bialgebra is an algebra A that is also a coalgebra, with

$$\Delta(1) = 1 \otimes 1;\ \Delta(ab) = \Delta(a)\Delta(b) \quad \Leftrightarrow \quad \sum ab_{(1)} \otimes ab_{(2)} = \sum a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}$$

$$\varepsilon(1) = 1;\ \ \varepsilon(ab) = \varepsilon(a)\varepsilon(b)$$

In other words, a bialgebra structure allows the 'multiplication' of representations; technically speaking, it makes the category of A-representations into a monoid.

Recall that we are primarily interested in $\mathbf{C}G$, the group algebra, where:

$$\Delta(g) = g \otimes g\ ,\ \varepsilon(g) = 1\ \ \forall g \in G \tag{7}$$

and we extend this linearly to all of $\mathbf{C}G$. (Note that $\Delta(g+h) = g \otimes g + h \otimes h$, which is not $(g+h) \otimes (g+h)$. Δ, ε are clearly an algebra homomorphisms. To see that (7) defines a valid coalgebra structure on $\mathbf{C}G$, we compute, for arbitrary $g \in G$:

$$\sum \varepsilon(g_{(2)}) g_{(1)} = \varepsilon(g) g = g\ ;\ \sum \varepsilon(x_{(1)}) x_{(2)} = \varepsilon(1) x + \varepsilon(x) 1 = x + 0 \tag{8}$$

$$\sum \Delta(g_{(1)}) \otimes g_{(2)} = \Delta(g) \otimes g = g \otimes g \otimes g = g \otimes \Delta(g) = \sum g_{(1)} \otimes \Delta(g_{(2)})$$

(Since both sides of (2) and (3) are algebra homomorphisms, it suffices to check the axioms on the generators.)

Observe that, $\forall x \in \mathbf{C}G$, $\Delta(x)$ is invariant under the transposition of factors: $(a \otimes b) \to (b \otimes a)$. Coalgebras with this property are *cocommutative*; symbolically this condition says $\sigma\Delta = \Delta$, where $\sigma(a \otimes b) = (b \otimes a)$. We can define an analogous map σ_V on $V \otimes V$, where V is some vector space with an A-action. σ is clearly a homomorphism; for the same reason, $[\sigma(a \otimes b)][\sigma_V(v_1 \otimes v_2)] = \sigma_V(av_1 \otimes bv_2)$ for any $v_1 \otimes v_2 \in V \otimes V$. Cocommutativity of A is what allows us to take symmetric and exterior powers of representations, since in this case A-action commutes with σ_V:

$$\sigma_V(\Delta a)(v) = (\sigma\Delta a)(\sigma_V v) = (\Delta a)(\sigma_V v)\ \ \forall a \in A, v \in V \otimes V \tag{9}$$

In general, the subspace of $V \otimes V$ invariant under σ_V may not be preserved by the A-action.

Antipode and dual representations

Given an A-action on some vector space V, we would like A to act on V^*, the set of $\mathbf{C}$-linear maps $V \to \mathbf{C}$. The simplest way to achieve this is via a $\mathbf{C}$-linear *antipode* map $S : A \to A$; for all $f \in V^*$, $v \in V$, set

$$a(f)[v] = f[S(a)v] \tag{10}$$

ie a-action on V^* is precomposition with $S(a)$.

For this to be a valid action, we require

$$1(f) = f,\ (ab)(f) = a(b(f))\ \forall f \in V^* \quad \Leftrightarrow \quad S(1) = 1,\ S(ab) = S(b)S(a) \tag{11}$$

so S is not an algebra homomorphism. In fact, maps with the above property are often called *antimorphisms*.

The natural vector space map $V^* \otimes V \to \mathbf{C}$ given by evaluation: $f \otimes v \to f(v)$ suggests that A should act on $V^* \otimes V$ trivially:

$$a(f \otimes v) = \sum a_{(1)} f \otimes a_{(2)} v \approx \sum a_{(1)} f[a_{(2)} v] = \sum f[S(a_{(1)}) a_{(2)} v] \tag{12}$$

So we require:

$$\sum S(a_{(1)}) a_{(2)} = \varepsilon(a) \quad \Leftrightarrow \quad \mu(S \otimes \iota)\Delta = \varepsilon \tag{13}$$

(Again we view ε as a map $: A \to A$.) For symmetry, we also impose

$$\sum a_{(1)} S(a_{(2)}) = \varepsilon(a) \quad \Leftrightarrow \quad \mu(\iota \otimes S)\Delta = \varepsilon \tag{14}$$

but this does not in general make $V \otimes V^* \to \mathbf{C}$ (again through evaluation) into an A-homomorphism:

$$a(v \otimes f) = \sum a_{(1)} v \otimes a_{(2)} f \approx \sum a_{(2)} f[a_{(1)} v] = \sum f[S(a_{(2)}) a_{(1)} v] \neq \sum f[a_{(1)} S(a_{(2)}) v] \tag{15}$$

unless A is commutative. One more warning is in order here: $\mu(\iota \otimes S)$, $\mu(S \otimes \iota)$ are not algebra homomorphisms, but, if (13) and (14) holds for $a, b \in A$, then it holds for ab also, because

$$\sum_{ab} S(ab_{(2)}) ab_{(1)} = \sum_a \sum_b S(b_{(2)}) S(a_{(2)}) a_{(1)} b_{(1)} = \sum_b S(b_{(2)}) \left(\sum_a S(a_{(2)}) a_{(1)} \right) b_{(1)} = \varepsilon(a)\varepsilon(b) \tag{16}$$

and similarly for (14). Hence it suffices to check that (13) and (14) hold for the generators of A.

Definition A *Hopf algebra* is a bialgebra with $\mathbf{C}$-linear *antipode* map $S : A \to A$ satifying $\mu(\iota \otimes S)\Delta = \mu(S \otimes \iota)\Delta = \varepsilon$.

Aagin, we can illustrate this as a diagram, as in Figure 17. We have omitted the antimorphism axiom (11) as it is possible to derive it from the other axioms (13) and (14); see Proposition 4.0.1 in Sweedler's book. However, we will not rely on this result in this talk, instead treating antimorphism as a separate condition.

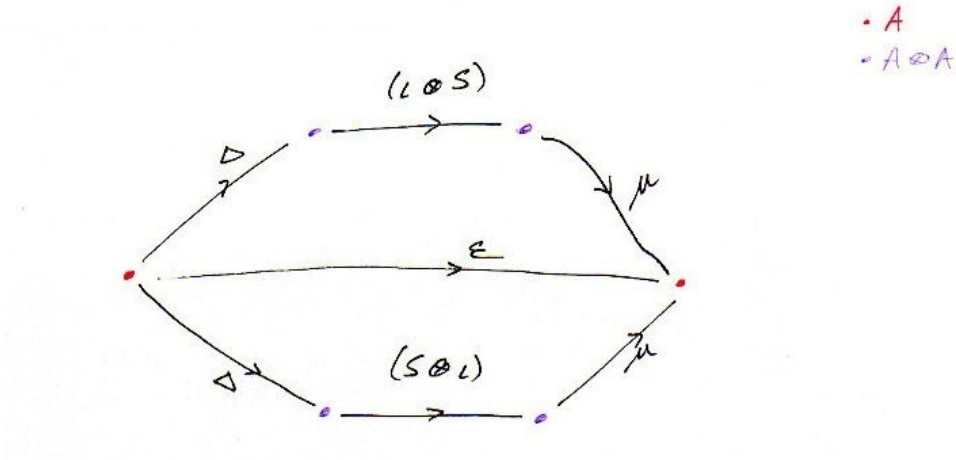

Figure 17 - Diagrammatic presentation of the Hopf algebra axiom (13) and (14)

The antipode on $\mathbf{C}G$ is defined by $S(g) = g^{-1}\ \forall g \in G$. Since $(gh)^{-1} = h^{-1}g^{-1}$, this indeed defines an antimorphism. We check (13) and (14):

$$\sum S(g_{(1)})g_{(2)} = S(g)g = g^{-1}g = \varepsilon(g) \tag{18}$$

$$\sum g_{(1)}S(g_{(2)}) = gS(g) = gg^{-1} = \varepsilon(g)$$

Here, S^2 is the identity map - this is true for commutative and cocommutative Hopf algebras (as proved in ?? of Sweedler's book), but need not hold in general. Hopf algebras with this property are called *involutive*.

Some authors require the antipode to be bijective (which is obviously the case for an involutive Hopf algebra like $\mathbf{C}G$). The advantage of a bijective antipode is that duals of irreducible representations remain irreducible: indeed, take an irreducible A-representation V and an A-invariant submodule $W^* \subseteq V^*$. Define $U = \{v \in V : f(v) = 0\ \forall v \in W^*\}$; from basic linear algebra, dim$U$ =dim$V$ -dimW^*. Observe

$$\forall a \in A, u \in U, f \in W^*\ \ f(au) = [S^{-1}(a)f](u) = 0 \tag{19}$$

since $S^{-1}(a)f \in W^*$ by A-invariance of W^*. Hence U is A-invariant also; by irreducibility of V, $U = \{0\}$ or $U = V$. These correspond to $W^* = V^*$ or $W^* = \{0\}$ respectively, from the dimension count above.

Finally, we remark that, if an algebra A admits a Hopf algebra structure, it is not usually unique: for example, given any Hopf algebra with comultiplication Δ, counit ε and bijective antipode S, we can define a new Hopf algebra structure with comultiplication $\sigma\Delta$ (recall that σ is the transposition map), counit ε and antipode S^{-1}. However, once Δ, ε are fixed, the axioms (13), (14) specifies S uniquely - this follows trivially if we use the alternative definition of S found in Sweedler's book.

Dualising a Hopf algebra

Most textbooks on Hopf algebras introduces the coalgebra axioms by dualising the axioms for algebras (draw the axiom diagrams and reverse all the arrows). Indeed, dualising a finite-dimensional algebra always gives a coalgebra, and dualising any coalgebra creates an algebra (finite dimensionality is required in the first case so we can identify $A^*\otimes A^*$ with $(A\otimes A)^*$). Hence dualising a finite dimensional vector space with both these structures produces another such object - indeed, it is another Hopf algebra because the axioms are "self dual" ie the diagrams are "symmetric" in some sense. Let us see how this works in the example of the dual vector space $\mathbf{C}G^*$ (for a finite group G). (We will denote the maps for $\mathbf{C}G^*$ by μ^*, ε^* etc to distinguish them from the maps for $\mathbf{C}G$.) Recall that $\mathbf{C}$-linear maps on $\mathbf{C}G$ are in bijection with maps $:G\to\mathbf{C}$, so we can think of $\mathbf{C}G^*$ as the space of maps $:G\to\mathbf{C}$ with pointwise addition and multiplication.

Observe that this pointwise multiplication map μ^* on $\mathbf{C}G^*$ is induced from the comultiplication map of $\mathbf{C}G$ in the following way: $\mu^*(\phi\otimes\varphi)=(\phi\otimes\varphi)\Delta$, with the two factors on the right hand side combined using multiplication in $\mathbf{C}$ ($\phi,\varphi\in\mathbf{C}G^*$). Inspired by this, we define $\Delta^*\phi=\phi\mu$. Viewing $\mathbf{C}G^*\otimes\mathbf{C}G^*$ as bilinear maps on $\mathbf{C}G$, or maps on $G\times G$ (again identifying the two image spaces, $\mathbf{C}\otimes\mathbf{C}$ and $\mathbf{C}$, via multiplication of factors in the first) this says $\Delta\phi(g,h)=\phi(gh)$. Note that cocommutativity of $\mathbf{C}G$ has appeared as commutativity of $\mathbf{C}G^*$; similarly, associativity of $\mathbf{C}G$ ($\mu\otimes(\iota\otimes\mu)=(\iota\otimes\mu)\otimes\mu$) ensures that $\mathbf{C}G^*$ is coassociative:

$$(\iota^*\otimes\Delta^*)\Delta^*\phi=(\iota^*\otimes\Delta^*)[\phi\mu]=\phi\mu(\iota\otimes\mu)=\phi(\iota\otimes\mu)\mu=\Delta^*[\phi(\iota\otimes\mu)]=\Delta^*(\iota^*\otimes\Delta^*)\phi \tag{20}$$

where we have used $\iota^*\phi=\phi\iota$.

The natural choice for ε^* is "evaluation at the identity" (maps from dual spaces to $\mathbf{C}$ are usually given by evaluation, and 1 is the only distinguished non-zero element of $\mathbf{C}G$). Identifying $\mathbf{C}$ with its copy in $\mathbf{C}G$, this says $\varepsilon^*\phi=\phi\varepsilon$, so

$$\mu^*(\iota\otimes\varepsilon^*)\Delta^*\phi=[(\iota\otimes\varepsilon^*)\Delta^*\phi]\Delta=[\Delta^*\phi](\iota\otimes\varepsilon)\Delta=\phi\mu(\iota\otimes\varepsilon)\Delta=\phi \tag{21}$$

using axiom (2) on $\mathbf{C}G$ for the last inequality. Explicitly for our example:

$$\mu^*(\iota\otimes\varepsilon^*)\Delta^*\phi(g)=[(\iota\otimes\varepsilon^*)\Delta^*\phi](g\otimes g)=[\Delta^*\phi](g\otimes 1)=\phi\mu(g\otimes 1)=\phi(g) \tag{22}$$

and $\mu^*(\varepsilon^*\otimes\iota)\Delta^*(\phi)=\phi$ can be checked similarly. Observe that this duality between counit and identity maps also runs "the other way": ε in $\mathbf{C}G$ is the constant function 1, the identity element of $\mathbf{C}G^*$.

What we have defined is a valid bialgebra structure on $\mathbf{C}G^*$:

$$\Delta^*(\phi\varphi)(g,h) = \phi\varphi(gh) = \phi(gh)\varphi(gh) = (\Delta^*\phi)(\Delta^*\varphi)(g,h) \tag{23}$$
$$\varepsilon^*(\phi\varphi) = \phi\varphi(1) = \phi(1)\varphi(1) = (\varepsilon^*\phi)(\varepsilon^*\varphi)$$

The case for general A is trickier to check (it involves writing the bialgebra axioms in terms of our multiplication map μ), so we omit it here.

Finally, we define the new antipode by $S^*(\phi) = \phi S$ - in our example of $\mathbf{C}G^*$, $(S^*\phi)(g) = \phi(g^{-1})$. We check that

$$\mu^*(\iota \otimes S^*)\Delta^*\phi = [(\iota \otimes S^*)\Delta^*\phi]\Delta = [\Delta^*\phi](\iota \otimes S)\Delta = \phi\mu(\iota \otimes S)\Delta = \phi\varepsilon = \varepsilon^*\phi \tag{24}$$

Some last thoughts

Instead of G, we could've performed all this analysis with a Lie algebra $\mathbf{g}$. The role of $\mathbf{C}G$ is then occupied by the universal enveloping algebra $U(\mathbf{g})$, and dualisation gives a Hopf algebra structure to the set of algebraic functions on the corresponding Lie group.