A Quick Introduction to Combinatorial Hopf algebras

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(The exposition below is adapted from [Pan14, Sec. 4.1]. I'm keen to keep this correct and reader-friendly; please let me know if anything is unclear.)

A graded, connected Hopf algebra is a graded vector space $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ equipped with two linear maps: a product $m : \mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_{i+j}$ and a *coproduct* $\Delta : \mathcal{H}_n \to \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$. The product is associative and has a unit which spans $\mathcal{H}_0$. The corresponding requirements on the coproduct are *coassociativity*: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ (where ι denotes the identity map) and the counit axiom: $\Delta(x) - 1 \otimes x - x \otimes 1 \in \bigoplus_{j=1}^{n-1} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$, for $x \in \mathcal{H}_n$. The product and coproduct satisfiy the compatibility axiom $\Delta(wz) = \Delta(w)\Delta(z)$, where multiplication on $\mathcal{H} \otimes \mathcal{H}$ is componentwise. This condition may be more transparent in *Sweedler notation*: writing $\sum_{(x)} x_{(1)} \otimes x_{(2)}$ for $\Delta(x)$, the axiom reads $\Delta(wz) = \sum_{(w),(z)} w_{(1)}z_{(1)} \otimes w_{(2)}z_{(2)}$. Below, there will be no more instances of Sweedler notation.

The definition of a general Hopf algebra, without the grading and connectedness assumptions, is slightly more complicated (it involves an extra *antipode* map, which is automatic in the graded case); the reader may consult [Swe69]. However, that reference (like many other introductions to Hopf algebras) concentrates on finite-dimensional Hopf algebras, which are useful in representation theory as generalisations of group algebras. These behave very differently from the infinite-dimensional Hopf algebras that one uses in combinatorics.

In a combinatorial Hopf algebra, the product and coproduct encode respectively how to combine and split combinatorial objects. A motivating example is:

Example 1 (Shuffle algebra)**.** The shuffle algebra $\mathcal{S}(N)$, as defined in [Ree58], has as its basis the set of all words in the letters $\{1, 2, \dots, N\}$. The number of letters N is usually unimportant, so we write this algebra simply as $\mathcal{S}$. These words are notated in parantheses to distinguish them from integers.

The product of two words is the sum of all their interleavings, with multiplicity. For example,

$$m((13) \otimes (52)) = (13)(52) = (1352) + (1532) + (1523) + (5132) + (5123) + (5213),$$

$$(12)(231) = 2(12231) + (12321) + (12312) + (21231) + (21321) + (21312) + (23121) + 2(23112).$$

[Reu93, Sec. 1.5] shows that deconcatenation is a compatible coproduct. For example,

$$\Delta((316)) = \emptyset \otimes (316) + (3) \otimes (16) + (31) \otimes (6) + (316) \otimes \emptyset.$$

(Here, $\emptyset$ denotes the empty word, which is the unit of $\mathcal{S}$.)

The idea of using Hopf algebras to study combinatorial structures was originally due to Joni and Rota [JR79]. The concept enjoyed increased popularity in the late 1990s, when [Kre98] linked a combinatorial Hopf algebra on trees to renormalisation in theoretical physics. Today, an abundance of combinatorial Hopf algebras exists; see the introduction of [Foi12] for a list of references to many examples. An instructive and entertaining overview of the basics and the history of the subject is in [Zab10]. [LR10] gives structure theorems for these algebras.

A particular triumph of this algebrisation of combinatorics is [ABS06, Th. 4.1], which claims that $QSym$, the algebra of quasisymmetric functions (Example 5 below) is the terminal object in the category of combinatorial Hopf algebras with a multiplicative linear functional called a *character*. Their explicit map from any such algebra to $QSym$ unifies many ways of assigning polynomial invariants to combinatorial objects, such as the chromatic polynomial of graphs and Ehrenboug's quasisymmetric function of a ranked poset.

There is no universal definition of a combinatorial Hopf algebra in the literature; each author considers Hopf algebras with slightly different axioms. What they do agree on is that such an algebra $\mathcal{H}$ should have a distinguished basis $\mathcal{B}$ indexed by "combinatorial objects", such as permutations, set partitions, or trees, and it should be graded by the "size" of these objects: $\mathcal{H} = \bigoplus_n \mathcal{H}_n$. The Hopf algebra is *connected* ($\dim(\mathcal{H}_0) = 1$) since the empty object is the only object of size 0. Many families of combinatorial objects have a single member of size 1, so $\mathcal{H}_1$ is often also one-dimensional.

For $x, y, z, w \in \mathcal{B}$, define *structure constants* ξ^y_{wz}, η^{wz}_x by

$$m(w \otimes z) = wz = \sum_{y \in \mathcal{B}} \xi^y_{wz} y, \quad \Delta^{[a]}(x) = \sum_{wz \in \mathcal{B}} \eta^{wz}_x w \otimes z.$$

In a combinatorial Hopf algebra, ξ^y_{wz} and η^{wz}_x should have interpretations respectively as the (possibly weighted) number of ways to combine w, z and obtain y, and the (possibly weighted) number of ways to break x into w, z. Then, the compatibility axiom $\Delta(wz) = \Delta(w)\Delta(z)$ translates roughly into the following: suppose y is one possible outcome when combining w and z; then every way of breaking y comes (bijectively) from a way of breaking w and z separately. The axioms $\deg(wz) = \deg(w) + \deg(z)$ and $\Delta(x) \in \bigoplus_{i=0}^{\deg(x)} \mathcal{H}_i \otimes \mathcal{H}_{\deg(x)-i}$ simply say that the "total size" of an object is conserved under breaking and combining.

These are the minimal conditions for a combinatorial Hopf algebra. A common additional hypothesis is the existence of an internal product $\mathcal{H}_n \otimes \mathcal{H}_n \to \mathcal{H}_n$, and perhaps also an internal coproduct. Note that commutativity of a combinatorial Hopf algebra indicates a symmetric assembling rule, and a symmetric breaking rule induces a cocommutative Hopf algebra.

The rest of this document is a whistle-stop tour of three sources of combinatorial Hopf algebras. A fourth important source is operads [Hol04], but that theory is too technical to cover in detail here.

1 Species-with-Restrictions

The theory of species originated in [Joy81], as an abstraction of common manipulations of generating functions. Loosely speaking, a species is a type of combinatorial structure which one can build on sets of "vertices". Important examples include (labelled) graphs, trees and permutations. The formal definition of a species is as a functor from the category of sets with bijections to the same category. In this categorical language, the species of graphs maps a set V to the set of all graphs whose vertices are indexed by V. There are operations on species which correspond to the multiplication, composition and differentiation of their associated generating functions; these are not so revelant to the present Hopf algebra construction, so the reader is referred to [BLL98] for further details.

Schmitt [Sch93] first makes the connection between species and Hopf algebras. He defines a species-with-restrictions, or R-species, to be a functor from sets with coinjections to the category of functions. (A *coinjection* is a partially-defined function whose restriction to where it's defined is a bijection; an example is $f : \{1, 2, 3, 4\} \to \{7, 8\}$ with $f(1) = 8$, $f(3) = 7$ and $f(2), f(4)$ undefined.) Intuitively, these are combinatorial structures with a notion of restriction to a subset of their vertex

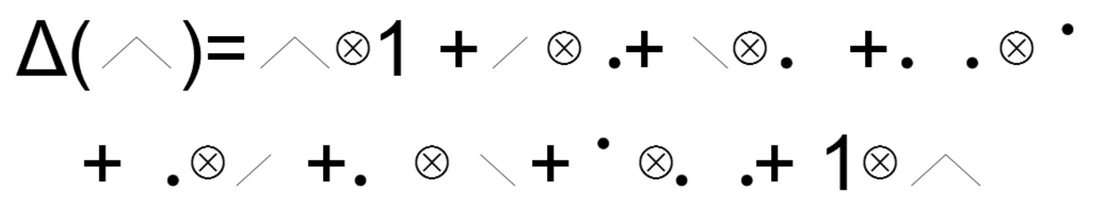

Figure 1.1: An example coproduct calculation in $\bar{\mathcal{G}}$, the Hopf algebra of graphs

set; for example, one can restrict a graph to a subset of its vertices by considering only the edges connected to this subset (usually known as the *induced subgraph*). Schmitt fashions from each such species a Hopf algebra which is both commutative and cocommutative; Example 2 below explains his construction via the species of graphs.

Example 2 (The Hopf algebra of graphs)**.** [Sch94, Sec. 12; Fis10, Sec. 3.2] Let $\bar{\mathcal{G}}$ be the vector space with basis the set of simple graphs (no loops or multiple edges). The vertices of such graphs are unlabelled, so these may be considered the isomorphism classes of graphs. Define the degree of a graph to be its number of vertices. The product of two graphs is their disjoint union, and the coproduct is

$$\Delta(G) = \sum G_S \otimes G_{S^C}$$

where the sum is over all subsets S of vertices of G, and G_S, G_{S^C} denote the subgraphs that G induces on the vertex set S and its complement. As an example, Figure 1.1 calculates the coproduct of P_3, the path of length 3. Writing P_2 for the path of length 2, and $\bullet$ for the unique graph on one vertex, this calculation shows that

$$\Delta(P_3) = P_3 \otimes 1 + 2P_2 \otimes \bullet + \bullet^2 \otimes \bullet + 2 \bullet \otimes P_2 + \bullet \otimes \bullet^2 + 1 \otimes P_3.$$

As mentioned above, this Hopf algebra, and analogous constructions from other species-with-restrictions, are both commutative and cocommutative.

Recently, Aguiar and Mahajan [AM10] extended vastly this construction to the concept of a *Hopf monoid in species*, which is a finer structure than a Hopf algebra. Their Chapter 15 gives two major pathways from a species to a Hopf algebra: the Bosonic Fock functor, which is essentially Schmitt's original idea, and the Full Fock functor. In addition there are decorated and coloured variants of these two constructions, which allow the input of parameters. Many popular combinatorial Hopf algebras, including all examples in this thesis, arise from Hopf monoids; perhaps this is an indication that the Hopf monoid is the "correct" setting to work in.

2 Representation rings of Towers of Algebras

The ideas of this construction date back to Zelevinsky [Zel81, Sec. 6], which the lecture notes [GR14, Sec. 4] retell in modern notation. The archetype is as follows:

Example 3 (Representations of symmetric groups)**.** Let $\mathcal{B}_n$ be the irreducible representations of the symmetric group $\mathfrak{S}_n$, so $\mathcal{H}_n$ is the vector space spanned by all representations of $\mathfrak{S}_n$. The product of representations w, z of $\mathfrak{S}_n$, $\mathfrak{S}_m$ respectively is defined using induction:

$$m(w \otimes z) = \operatorname{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} w \times z,$$

and the coproduct of x, a representation of $\mathfrak{S}_n$, is the sum of its restrictions:

$$\Delta(x) = \bigoplus_{i=0}^{n} \operatorname{Res}_{\mathfrak{S}_i \times \mathfrak{S}_{n-i}}^{\mathfrak{S}_n} x.$$

Mackey theory ensures these operations satisfy $\Delta(wz) = \Delta(w)\Delta(z)$. This Hopf algebra is both commutative and cocommutative, as $\mathfrak{S}_n \times \mathfrak{S}_m$ and $\mathfrak{S}_m \times \mathfrak{S}_n$ are conjugate in $\mathfrak{S}_{n+m}$; however, the general construction need not have either symmetry.

It's natural to attempt this construction with, instead of $\{\mathfrak{S}_n\}$, any series of algebras $\{A_n\}$ where an injection $A_n \otimes A_m \subseteq A_{n+m}$ allows this outer product of its modules. For the result to be a Hopf algebra, one needs some additional hypotheses on the algebras $\{A_n\}$; this leads to the definition of a *tower of algebras* in [BL09]. In general, two Hopf algebras can be built this way: one using the finitely-generated modules of each A_n, and one from the finitely-generated projective modules of each A_n. (For the above example of symmetric groups, these coincide, as all representations are semisimple.) These are dual as graded Hopf algebras. For example, [KT97, Sec. 5] takes A_n to be the 0-Hecke algebra, then the Hopf algebra of finitely-generated modules is $QSym$, the Hopf algebra of quasisymmetric functions. Example 5 below will present $QSym$ in a different guise that does not require knowledge of Hecke algebras. The Hopf algebra of finitely-generated projective modules of the 0-Hecke algebras is **Sym**, the algebra of noncommutative symmetric functions of [Gel+95]. Further developments regarding Hopf structures from representations of towers of algebras are in [BLL12].

Interestingly, it is sometimes possible to tell a similar story with the basis $\mathcal{B}_n$ being a set of reducible representations, possibly with slight tweaks to the definitions of product and coproduct. In [Agu+12; BV13; ABT13; And14], $\mathcal{B}_n$ is a *supercharacter* theory of various matrix groups over finite fields. This means that the matrix group can be partitioned into *superclasses*, which are each a union of conjugacy classes, such that each supercharacter (the characters of the representations in $\mathcal{B}_n$) is constant on each superclass, and each irreducible character of the matrix group is a consituent of exactly one supercharacter. [DI08] gives a unified method to build a supercharacter theory on many matrix groups; this is useful as the irreducible representations of these groups are extremely complicated.

3 Subalgebras of Power Series

The starting point for this approach is the algebra of symmetric functions, widely considered as the first combinatorial Hopf algebra in history, and possibly the most extensively studied. Thorough textbook introductions to its algebra structure and its various bases are [Mac95, Chap. 1] and [Sta99, Chap. 7].

Example 4 (Symmetric functions)**.** Work in the algebra $\mathbb{R}[[x_1, x_2, \dots]]$ of power series in infinitely-many commuting variables x_i, graded so $\deg(x_i) = 1$ for all i. The *algebra of symmetric functions* Λ is the subalgebra of power series of finite degree invariant under the action of the infinite symmetric group $\mathfrak{S}_\infty$ permuting the variables. (These elements are often called "polynomials" due to their finite degree, even though they contain infinitely-many monomial terms.)

An obvious basis of Λ is the sum of monomials in each $\mathfrak{S}_\infty$ orbit; these are the *monomial symmetric functions*:

$$m_\lambda := \sum_{\substack{(i_1,\dots,i_l) \\ i_j \text{ distinct}}} x_{i_1}^{\lambda_1} \dots x_{i_l}^{\lambda_l}.$$

Here, λ is a partition of $\deg(m_\lambda)$: $\lambda_1 + \cdots + \lambda_{l(\lambda)} = \deg(m_\lambda)$ with $\lambda_1 \geq \cdots \geq \lambda_{l(\lambda)}$. For example, the three monomial symmetric functions of degree three are:

$$\begin{aligned} m_{(3)} &= x_1^3 + x_2^3 + \dots; \\ m_{(2,1)} &= x_1^2 x_2 + x_1^2 x_3 + \cdots + x_2^2 x_1 + x_2^2 x_3 + x_2^2 x_4 + \dots; \\ m_{(1,1,1)} &= x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots + x_1 x_3 x_4 + x_1 x_3 x_5 + \cdots + x_2 x_3 x_4 + \dots. \end{aligned}$$

It turns out [Sta99, Th. 7.4.4, Cor. 7.6.2] that Λ is isomorphic to a polynomial ring in infinitely-many variables: $\Lambda = \mathbb{R}[h_{(1)}, h_{(2)}, \dots]$, where

$$h_{(n)} := \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \dots x_{i_n}.$$

(This is often denoted h_n, as it is standard to write the integer n for the partition (n) of single part.) For example,

$$h_{(2)} = x_1^2 + x_1 x_2 + x_1 x_3 + \cdots + x_2^2 + x_2 x_3 + \dots.$$

So, setting $h_\lambda := h_{(\lambda_1)} \dots h_{(\lambda_{l(\lambda)})}$ over all partitions λ gives another basis of Λ, the *complete symmetric functions*.

Two more bases are important: the *power sums* are $p_{(n)} := \sum_i x_i^n$, $p_\lambda := p_{(\lambda_1)} \dots p_{(\lambda_{l(\lambda)})}$; and the *Schur functions* $\{s_\lambda\}$ are the image of the irreducible representations under the *Frobenius characteristic isomorphism* from the representation rings of the symmetric groups (Example 3) to Λ [Sta99, Sec. 7.18]. This map is defined by sending the indicator function of an n-cycle of $\mathfrak{S}_n$ to the scaled power sum $\frac{p_{(n)}}{n}$. (I am omitting the elementary basis $\{e_\lambda\}$, as it has similar behaviour as $\{h_\lambda\}$.)

The coproduct on Λ comes from the "alphabet doubling trick". This relies on the isomorphism between the power series algebras $\mathbb{R}[[x_1, x_2, \dots, y_1, y_2, \dots]]$ and $\mathbb{R}[[x_1, x_2, \dots]] \otimes \mathbb{R}[[y_1, y_2, \dots]]$, which simply rewrites the monomial $x_{i_1} \dots x_{i_k} y_{j_1} \dots y_{j_l}$ as $x_{i_1} \dots x_{i_k} \otimes y_{j_1} \dots y_{j_l}$. To calculate the coproduct of a symmetric function f, first regard f as a power series in two sets of variables $x_1, x_2, \dots, y_1, y_2, \dots$; then $\Delta(f)$ is the image of $f(x_1, x_2, \dots y_1, y_2, \dots)$ in $\mathbb{R}[[x_1, x_2, \dots]] \otimes \mathbb{R}[[y_1, y_2, \dots]]$ under the above isomorphism. Because f is a symmetric function, the power series $f(x_1, x_2, \dots, y_1, y_2, \dots)$ is invariant under the permutation of the x_is and y_is separately, so $\Delta(f)$ is in fact in $\Lambda \otimes \Lambda$. For example,

$$\begin{aligned} h_{(2)}(x_1, x_2, \dots y_1, y_2 \dots) &= x_1^2 + x_1 x_2 + x_1 x_3 + \cdots + x_1 y_1 + x_1 y_2 + \dots \\ &\quad + x_2^2 + x_2 x_3 + \cdots + x_2 y_1 + x_2 y_2 + \dots \\ &\quad + \dots \\ &\quad + y_1^2 + y_1 y_2 + y_1 y_2 + \dots \\ &\quad + y_2^2 + y_2 y_3 + \dots \\ &\quad + \dots \\ &= h_{(2)}(x_1, x_2, \dots) + h_{(1)}(x_1, x_2, \dots) h_{(1)}(y_1, y_2, \dots) + h_{(2)}(y_1, y_2, \dots), \end{aligned}$$

so $\Delta(h_{(2)}) = h_{(2)} \otimes 1 + h_{(1)} \otimes h_{(1)} + 1 \otimes h_{(2)}$. In general, $\Delta(h_{(n)}) = \sum_{i=0}^n h_{(i)} \otimes h_{(n-i)}$, with the convention $h_{(0)} = 1$. (This is Geissenger's original definition of the coproduct [Gei77].) Note that $\Delta(p_{(n)}) = 1 \otimes p_{(n)} + p_{(n)} \otimes 1$; this property is the main reason for working with the power sum basis.

The generalisation of Λ is easier to see if the $\mathfrak{S}_\infty$ action is rephrased in terms of a function to a fundamental domain. Observe that each orbit of the monomials, under the action of the infinite symmetric group permuting the variables, contains precisely one term of the form $x_1^{\lambda_1} \dots x_l^{\lambda_l}$ for some partition λ. Hence the set $\mathcal{D} := \left\{ x_1^{\lambda_1} \dots x_l^{\lambda_l} | l, \lambda_i \in \mathbb{N}, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0 \right\}$ is a fundamental domain for this $\mathfrak{S}_\infty$ action. Define a function f sending a monomial to the element of $\mathcal{D}$ in its orbit; explicitly,

$$f\left(x_{j_1}^{i_1} \dots x_{j_l}^{i_l}\right) = x_1^{i_{\sigma(1)}} \dots x_l^{i_{\sigma(l)}},$$

where $\sigma \in \mathfrak{S}_l$ is such that $i_{\sigma(1)} \geq \dots \geq i_{\sigma(l)}$. For example, $f(x_1 x_3^2 x_4) = x_1^2 x_2 x_3$. It is clear that the monomial symmetric function m_λ, previously defined to be the sum over $\mathfrak{S}_\infty$orbits, is the sum over preimages of f:

$$m_\lambda := \sum_{f(x) = x^\lambda} x,$$

where x^λ is shorthand for $x_1^{\lambda_1} \dots x_l^{\lambda_l}$. Summing over preimages of other functions can give bases of other Hopf algebras. Again, the product is that of power series, and the coproduct comes from alphabet doubling. Example 5, essentially a simplified, commutative, version of [NT06, Sec. 2], builds the algebra of quasisymmetric functions using this recipe. This algebra is originally due to Gessel [Ges84], who defines it in terms of P-partitions.

Example 5 (Quasisymmetric functions)**.** Start again with $\mathbb{R}[[x_1, x_2, \dots]]$, the algebra of power series in infinitely-many commuting variables x_i. Let pack be the function sending a monomial $x_{j_1}^{i_1} \dots x_{j_l}^{i_l}$ (assuming $j_1 < \dots < j_l$) to its *packing* $x_1^{i_1} \dots x_l^{i_l}$. For example, $\text{pack}(x_1 x_3^2 x_4) = x_1 x_2^2 x_3$. A monomial is *packed* if it is its own packing, in other words, its constituent variables are consecutive starting from x_1. Let $\mathcal{D}$ be the set of packed monomials, so $\mathcal{D} := \left\{ x_1^{i_1} \dots x_l^{i_l} | l, i_j \in \mathbb{N} \right\}$. Writing I for the *composition* $(i_1, \dots, i_l)$ and x^I for $x_1^{i_1} \dots x_l^{i_l}$, define the *monomial quasisymmetric functions* to be:

$$M_I := \sum_{\text{pack}(x) = x^I} x = \sum_{j_1 < \dots < j_{l(I)}} x_{j_1}^{i_1} \dots x_{j_{l(I)}}^{i_{l(I)}}.$$

For example, the four monomial quasisymmetric functions of degree three are:

$$\begin{aligned}
M_{(3)} &= x_1^3 + x_2^3 + \dots; \\
M_{(2,1)} &= x_1^2 x_2 + x_1^2 x_3 + \dots + x_2^2 x_3 + x_2^2 x_4 + \dots + x_3^2 x_4 + \dots; \\
M_{(1,2)} &= x_1 x_2^2 + x_1 x_3^2 + \dots + x_2 x_3^2 + x_2 x_4^2 + \dots + x_3 x_4^2 + \dots; \\
M_{(1,1,1)} &= x_1 x_2 x_3 + x_1 x_2 x_4 + \dots + x_1 x_3 x_4 + x_1 x_3 x_5 + \dots + x_2 x_3 x_4 + \dots.
\end{aligned}$$

$QSym$, the *algebra of quasisymmetric functions*, is then the subalgebra of $\mathbb{R}[[x_1, x_2, \dots]]$ spanned by the M_I.

Note that the monomial symmetric function $m_{(2,1)}$ is $M_{(2,1)} + M_{(1,2)}$; in general, $m_\lambda = \sum M_I$ over all compositions I whose parts, when ordered decreasingly, are equal to λ. Thus Λ is a subalgebra of $QSym$.

The basis of $QSym$ with representation-theoretic significance, analogous to the Schur functions of Λ, are the fundamental quasisymmetric functions:

$$F_I = \sum_{J \geq I} M_J$$

where the sum runs over all compositions J refining I (i.e. I can be obtained by gluing together some adjacent parts of J). For example,

$$F_{(2,1)} = M_{(2,1)} + M_{(1,1,1)} = \sum_{j_1 \leq j_2 < j_3} x_{j_1} x_{j_2} x_{j_3}.$$

The fundamental quasisymmetric functions are sometimes denoted L_I or Q_I in the literature. They correspond to the irreducible modules of the 0-Hecke algebra [KT97, Sec. 5]. The analogue of power sums are more complex (as they natually live in **Sym**, the dual Hopf algebra to $QSym$), see [Gel+95, Sec. 3] for a full definition.

In the last decade, a community in Paris have dedicated themselves [DHT02; NT06; FNT11] to recasting familiar combinatorial Hopf algebras in this manner, a process they call *polynomial realisation*. They usually start with power series in noncommuting variables, so the resulting Hopf algebra is not constrained to be commutative. The least technical exposition is probably [Thi12], which also provides a list of examples. The simplest of these is **Sym**, a noncommutative analogue of the symmetric functions, see [NPT13, Sec. 2]. For a more interesting example, take M_T to be the sum of all noncommutative monomials with Q-tableau equal to T under the Robinson-Schensted-Knuth algorithm [Sta99, Sec. 7.11]; then their span is **FSym**, the Poirier-Reutenauer Hopf algebra of tableaux [PR95]. [Hiv07, Th. 31] and [Pri13, Th. 1] give sufficient conditions on the functions for this construction to produce a Hopf algebra. One motivation for this program is to bring to light various bases that are free (like h_λ), interact well with the coproduct (like p_λ) or are connected to representation theory (like s_λ), and to carry over some of the vast amount of machinery developed for the symmetric functions to analyse these combinatorial objects in new ways. Indeed, Joni and Rota anticipated in their original paper [JR79] that "many an interesting combinatorial problem can be formulated algebraically as that of transforming this basis into another basis with more desirable properties".

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