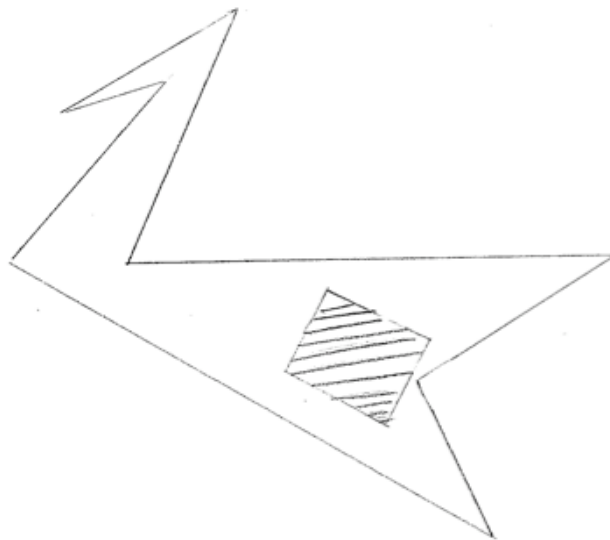


# The SUMO Speaker Series for Undergraduates

*(food from Pizza Chicago)*  
Wednesday, April 21  
4:40-5:30, room 380C

## The Art Gallery Problem

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Abstract: Imagine you are the curator of an art gallery with the above floor plan. How many (stationary) CCTV cameras do you need so they can see all points of the gallery, assuming they each have a 360 degree view? Chvatal proved that  $\lceil n/3 \rceil$  cameras are sufficient to guard a  $n$ -sided polygonal gallery without holes. I'll reproduce Fisk's proof of this result, using graph colourings, then discuss how to generalise it for galleries with holes like the one shown.

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Two caveats first:

I wrote this to help me prepare for my talk, to make sure that I can explain everything coherently; it's not really intended to be read on its own, but since I have it, I might as well share it. It makes much more sense if you watch me draw the diagrams as I talk about it, and then I can point to things.

This is very different from what I usually do (namely algebra). So there may well be mistakes, and I don't know much more than what's written here.

## 1. THE ORIGINAL ART GALLERY PROBLEM

The original version was posed by Victor Klee in 1973, actually here at Stanford. It's sometimes referred to as Chvatal's Watchmen problem, though nowadays the term watchmen sometimes refers to variants where the guards or cameras are allowed to move. According to Wolfram, it's appeared in NUMB3RS season 2 (2006).

**Problem.** What (in terms of  $n$ ) is the fewest number of cameras guaranteed to guard any simply-connected  $n$ -sided polygonal gallery?

Here, *polygonal* means the walls of the gallery are straight, and *simply-connected* means the gallery has no holes. A set of cameras is said to guard a gallery if every point in the gallery is within direct line of sight of a camera; that is, if each point of the gallery can be joined to some camera with a straight line lying entirely inside the polygon.

Chvatal gave the answer in 1975:

**Theorem.**  $\lfloor n/3 \rfloor$  cameras are sufficient to guard any simply-connected  $n$ -sided polygonal gallery.

( $\lfloor n/3 \rfloor$  means the integer you get by rounding down  $n/3$ ) First observe that, if we have no more information on the shape of our gallery,  $\lfloor n/3 \rfloor$  is the best we can do: In Figure 1 (which is Chvatal's original example), each camera sees at most one of the 4 "top" vertices, so 4 cameras are necessary. Specifically, given a point, we can find the area where the camera that sees that particular point must be located, and the sets corresponding to the 4 "top" vertices are disjoint. In fact, there's nothing special about the number 4 here, I can make a gallery with  $k$  spikes, which will have  $3k$  walls and require  $k$  guards.

I'll describe a proof by Fisk, 1978. It's simpler, but if you really want to explore similar problems then you should see Chvatal's proof too (eg in O'Rourke's book), because sometimes that generalises while Fisk's ideas doesn't. (Chvatal's proof doesn't use graphs).

Intuition says we should divide up the gallery into "simpler" pieces, and then make sure each piece is guarded. The simplest polygon is a triangle, and that's a good start because we understand how to guard triangles: they can be guarded by a camera at any one of its vertices. (Any *convex* polygon has this property. So what we're really using is that all triangles are convex). So our line of attack will be

- *triangulate* the polygon (figure 2 is an example of a triangulation).
- guard each triangle in the triangulation

Since we will guard each triangle “separately” at the end, it makes sense to split the gallery into as few triangular pieces as possible. This we do by finding ears:

**Definition.** An *ear* of a polygon is a triangle formed by three adjacent vertices of the polygon, that lies entirely inside the polygon.

Fig 2 has some examples and non-examples of ears. Once we know that any polygon has an ear, we’re done, by induction on the number of sides: remove the ear, by inductive hypothesis we can triangulate the remaining polygon, that together with the diagonal that “cut off” the ear gives a triangulation of the original polygon. (If you’re not familiar with proof by induction, think of it this way: draw a diagonal that separates an ear. Do the same to the part of the polygon that isn’t an ear, ie, find an ear within that polygon and separate that off. If you keep doing this, you’ll separate the original polygon into triangles.) Observe that in this process we never needed to add vertices, all we do is draw  $n - 3$  non-intersecting diagonals, which makes  $n - 3$  triangles. (From now on, all triangulations are assumed to be created from ears in this way.)

The proof that every polygon has an ear is non-constructive and quite technical, so you can skip this part if you want, although there’s one common graph theory phenomenon that’s worth discussing here. Most graph theory proofs are by induction, so it often happens that a stronger assertion is easier to prove, because one then has a stronger inductive hypothesis to work with. Try going through the following argument with just the weaker one-ear assumption, and you’ll see it doesn’t work.

**Lemma.** *Every polygon with at least 4 sides has two interior-disjoint ears (ie these two ears may share a vertex, or an edge that is a diagonal of the polygon, but no more).*

*Proof.* The case  $n = 4$  can be checked by hand. So take  $n \geq 5$ , and start by picking a concave vertex  $v$ , that is, a vertex with an acute or obtuse angle. (If you pick a vertex with a reflex angle, you’re guaranteed to fail, But this is not the only way you can fail, see figure 2). Look at its neighbouring vertices  $v_1$  and  $v_2$ . If this is an ear, great; the  $n - 1$ -gon that’s left when we remove the ear  $v_1vv_2$  has two ears, by inductive hypothesis, and since these are disjoint, one of them doesn’t involve the line  $v_1v_2$ , and therefore is an ear in the original  $n$ -gon. If  $v_1vv_2$  isn’t an ear, then there’s a vertex in this triangle (possibly more than one). Slide the line  $v_1v_2$  towards  $v$ , and let  $x$  be the last vertex this process hits. The triangle formed by  $v$  and the translated line must lie entirely in the polygon, so  $xv$  lies entirely inside the polygon. So  $xv$  divides the polygon into two, each with fewer vertices: by inductive hypothesis, they each are either triangles (hence an ear of the original polygon), or contain two ears. In the latter case, one of these ears doesn’t contain  $xv$  (by same argument as before), and is therefore an ear in the original  $n$ -gon. This applies to both of the smaller polygons, creating two ears as desired.  $\square$

Now that we’ve triangulated the polygon, we want to *3-colour* the triangulation graph.

**Definition.** A (*vertex*) *k-colouring* of a graph is an assignment of one of  $k$  colours to each vertex of a graph so that no two adjacent vertices (ie no two vertices that are connected by an edge) have the same colour. A graph on which we can define a *k-colouring* is called *k-colourable*.

So you can think of a  $k$ -colouring as a function from the set of vertices of the graph to the set  $\{1, 2, \dots, k\}$  satisfying the condition that no two adjacent vertices map to the same number. Figure 4 shows a 3-colouring Figure 2 (where R,G,B denote red, green and blue).

**Lemma.** *Any triangulation graph (of a simply-connected polygon, created from ears) is 3-colourable.*

I'll give two proofs which are essentially the same. The first is from more of a CS viewpoint, and requires more terminology than the second:

**Definition.** Given a triangulation graph, the *dual graph* is constructed as follows: take a vertex for every face of the triangulation, and join two vertices if the triangles in the triangulation that they represent share an edge.

So Figure 5 is the dual graph of Figure 4. (The numbers are for later).

**Definition.** A *tree* is a graph that is *connected* (there is a path between every pair of vertices) and *acyclic* (has no loops).

*Proof.* (1) I claim that the dual graph of the triangulation is a tree. Connectedness is easy: given two vertices of the dual graph, they represent two triangles in the polygon. As the polygon is connected, there's a path in the polygon linking these two triangles, and looking at which triangles this passes through give a path connecting the chosen vertices of the dual graph. For example, the dotted path in Figure 4 gives you the "bottom" part of Figure 5 as a line joining the leftmost and rightmost vertices. The dual graph is acyclic because the polygon has no holes.

Observe that any two vertices in a tree are joined by a unique shortest path (in fact, this is an equivalent definition for a tree). So, given a basepoint, there's a well-defined distance from any point to that basepoint, by counting the number of nodes on the unique path between them. I've marked these distances on Figure 5, where the left-most vertex is the basepoint. The basepoint triangle is clearly 3-colourable. Next, colour the vertices of triangles of distance 1, then those of distance 2, and so on. (I think CS people call this a depth first traversal?) Each triangle of distance  $n$  shares an edge with precisely one triangle of distance  $n - 1$ , (in other words, each distance  $n$  vertex of the dual graph is joined to precisely one distance  $n - 1$  vertex), so the vertices on this edge (call them  $x, y$ ) are already coloured, and there is one colour left for the vertex (call it  $z$ ) not on this edge. We can use this colour because all vertices adjacent to  $z$  that aren't  $x$  or  $y$  are in triangles of distance  $> n$ , so they're not coloured yet.  $\square$

If you didn't catch all that about dual graphs and trees, here's an alternative way of thinking through the same argument (if you unravel the induction, you'll see we're colouring the graph ear by ear, which is exactly the same as proceeding down the dual graph.):

*Proof.* (2) We use induction on the number of vertices, observing that a triangle is clearly 3-colourable. Recall that we started our triangulation by separating off an ear. Remove this ear; by inductive hypothesis, we can 3-colour the remaining graph. The removed vertex is connected to two vertices, which either have the same colour, or two different colours; in either case, there's a third colour free for it.  $\square$

This is the special case of a useful graph theory lemma: if  $H$  is a  $k$ -colourable subgraph of  $G$ , (a *subgraph* is created by removing some vertices and the edges going out from that vertex; we're not allowed to remove an edge and keep both endpoints,) and every vertex of  $G \setminus H$  is joined to  $\leq k - 1$  other vertices, then  $G$  is  $k$ -colourable.

Observe that a 3-colouring forces every triangle to have precisely one red vertex, one green vertex and one blue vertex. So if we put a camera at every green vertex, these cameras will cover every triangle in the triangulation. There is nothing special about the colour green here; indeed, in Figure 4, if we had chosen red instead, then one fewer camera will do. So we should pick the colour with the fewest vertices, and the number of these vertices, by pigeonhole principle, is then at most  $\lfloor n/3 \rfloor$ .

Computationally, the first step is the most difficult; once we have the triangulation, the dual graph allows us to colour the triangulation graph efficiently. In other words, the first step has a non-constructive proof, but the second step is constructive.

(You might think this result means we just need to place a guard on every third vertex: but this isn't true. O'Rourke's book has a nice counterexample. Also, this result doesn't mean the minimum number of cameras can always be achieved by putting cameras at the vertices: again, O'Rourke demonstrates a case where 2 cameras are needed if we place them at vertices, but 1 interior camera suffices.)

## 2. GALLERIES WITH HOLES

It might seem bizarre to build a gallery with holes, but sometimes structural pillars block the view of the CCTV cameras. Can you guess how to adapt our proof above to show this result of O'Rourke (1982)?

**Theorem.**  $\lfloor (n + 2h)/3 \rfloor$  cameras suffice to guard an  $n$ -sided polygonal gallery with  $h$  polygonal holes (where  $n$  includes the sides bordering the holes).

The idea is simple: if we build a wall from one hole to another, two holes merge into one. We can also build a wall from a hole to the outside, in which case we lose a hole. Either way, each wall decreases  $h$  by 1 and increases  $n$  by 2 (since both sides of the walls becomes new sides of the gallery). So, after building  $h$  walls, we have a holeless gallery, with  $2h$  additional sides, and applying the previous theorem gives the result.

The hard work is in showing that we can build these walls between the corners of the original gallery, so no existing side becomes split into two in this construction. Also, one needs to check that the resulting gallery is still connected, that it is one polygon rather than two or more.

Bjorling-Sachs and Souvaine managed even better in 1991:

**Theorem.**  $\lfloor (n + h)/3 \rfloor$  cameras suffice to guard an  $n$ -sided polygonal gallery with  $h$  polygonal holes.

This is the best possible; there are examples where no fewer number will do.

### 3. A LOWER BOUND FOR THIS EXAMPLE

Our starting example had 12 sides and 1 hole, so we know 4 cameras are definitely enough. We finish by showing that no fewer will do.

Remember how we proved that the  $k$ -spiked gallery needs  $k$  cameras: we found  $k$  points which must be guarded by different cameras. But there is no obvious choice of 4 points here whose corresponding “camera sets” are disjoint. So we’ll have to do something a bit cleverer.

Start by observing that no camera can see both the points marked  $a$  and  $b$  on Figure 6. So write  $A$  for the camera that sees  $a$ , and  $B$  for the camera guarding  $b$ .  $A$  cannot see  $c_1$  nor  $c_2$ , and  $B$  cannot see them both: in fact, no camera can see both  $c_1$  and  $c_2$  as they’re on parallel sides of the hole.

First, suppose  $B$  sees  $c_1$ , and let  $C$  be the camera watching  $c_2$ . Then none of  $A$ ,  $B$  or  $C$  can see  $d_2$ , so a fourth camera is needed. In the case where  $B$  sees  $c_2$  and  $C$  sees  $c_1$ , none of  $A$ ,  $B$  or  $C$  can see  $d_1$ , so we come to the same conclusion.

### GENERALISATIONS

They are collectively known as visibility problems. Some examples:

Orthogonal galleries (galleries whose sides are “vertical” and “horizontal”) need only  $\lfloor n/4 \rfloor$  guards.

What if we require every point to be guarded by at least 2 cameras?

Higher dimensions: In 3D, we can’t always triangulate a polygon without adding new vertices: see Schönhardt polyhedron. And there can be points of the interior that are not visible from any vertex. (Both constructions are given in chapter 10 of O’Rourke’s book). So much more complicated machinery is needed.

Number of lights required to illuminate every point of a room where all the walls are mirrors, under various restrictions.

Efficient algorithms for determining where to place cameras, in the various scenarios.

### REFERENCES:

My main example (and all of section 3) comes directly from an episode of Japanese pop-math show "Coma University of Mathematics". Unfortunately I can find no references in English on it.

A detailed book on the subject, with many more variations: <http://maven.smith.edu/~orourke/books/ArtGalleryTheorem.pdf>

A lot of section 1 is from: <http://cgm.cs.mcgill.ca/~godfried/teaching/cg-projects/97/Thierry/thierry507web>

<http://eric.gruver.net/ArtGalleryProblems.html>

<http://mathworld.wolfram.com/ArtGalleryTheorem.html>

[http://en.wikipedia.org/wiki/Art\\_gallery\\_problem](http://en.wikipedia.org/wiki/Art_gallery_problem)

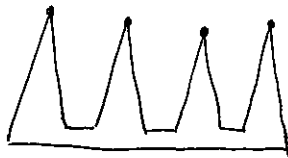


Fig 1

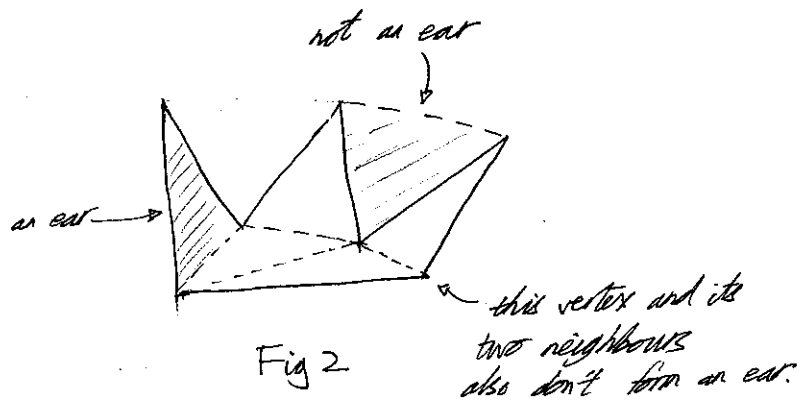


Fig 2

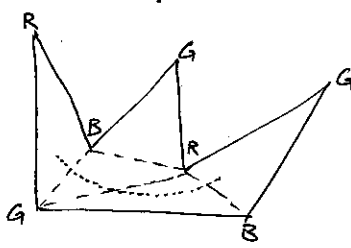


Fig 4

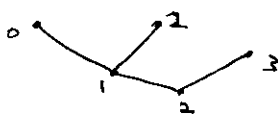


Fig 5

Figure 6

