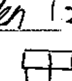
Algebraic Combinatorics

Symmetric function theory is starting to become a unifying theory of combinatorics: there are 5 common bases of symmetric functions, and the change-of-basis matrices count various things. In particular, the Schur basis is related to representation theory of the symmetric and unitary groups, so the combinatorics help us understand the representations and vice versa.

We will usually work over $\mathbb{Q}$, although the coefficient ring can be any commutative ring with unit. We work with infinitely many variables x_i.
A homogeneous symmetric function of deg n is $\sum_\alpha c_\alpha x^\alpha$, summing over all weak compositions α of n (ie all $\mathbb{N}$-sequences whose terms sum to n) such that $f(x_1, x_2, \dots) = f(x_{w(1)}, x_{w(2)}, \dots)$ for all bijections $w: \mathbb{N} \to \mathbb{N}$. (here, x^α denotes $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, and c_α lies in the coefficient ring) ie it is invariant under permutation.
The set of all such functions is Λ^n. Λ^n is closed under addition, and, for all $f \in \Lambda^m$, $g \in \Lambda^n$, $fg \in \Lambda^{m+n}$, so $\bigoplus_{n \in \mathbb{N}} \Lambda^n$, the symmetric functions, form a graded ring.

A composition λ is a partition if $\lambda_1 \geq \lambda_2 \geq \cdots$ (in particular, all the 0s lie at the end).
The length of λ is the number of non-zero entries.
We will often write partitions like a monomial: $(3,2,2,2,1,1,0,\dots)$ is written $1^2 2^3 3$.
We can also draw a Young diagram for each partition: $2,1,1$ has diagram .
The transpose λ^T of λ is obtained by reflecting the Young diagram along the main diagonal, so the columns and rows are switched: $(2,1,1)^T = 3,1$.

We will use 3 orderings on partitions / compositions:
- the natural partial ordering is dominance order: $\lambda \prec \mu$ if $\lambda_1 + \lambda_2 + \dots \lambda_i < \mu_1 + \mu_2 + \dots \mu_i \ \forall i$
  in terms of diagrams, $\lambda \prec \mu$ if we can move blocks of λ upwards to give μ.
- $\lambda \subseteq \mu$ if the diagram of λ lies entirely inside the diagram of μ, ie $\lambda_i < \mu_i \ \forall i$
- lexicographic ordering, which is a common refinement of both partial orders above:
  $\lambda \leq \mu$ if $\exists i$ such that $\lambda_1 = \mu_1, \lambda_2 = \mu_2, \dots \lambda_{i-1} = \mu_{i-1}, \lambda_i < \mu_i$.

The monomial symmetric functions are $m_\lambda(x_1, x_2 \dots) = \sum x^\alpha$ where α runs through all distinct permutations of λ ie x^α occurs if and only if $\alpha_1, \alpha_2, \dots = \lambda_{w(1)}, \lambda_{w(2)}, \dots$ for some bijection $\omega : \mathbb{N} \to \mathbb{N}$.

e.g. $m_1 = \sum x_i$

$m_{1,1} = \sum_{i<j} x_i x_j$

$m_{2,1} = \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2$.

It's clear that, across all partitions λ, m_λ are linearly independent.
Since every composition is a permutation of a unique partition, symmetry means that m_λ span Λ — so m_λ give a basis.

The elementary symmetric functions are $e_n(x_1, \dots) = \sum_{i_1 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$

$$e_\lambda(x_1, \dots) = e_{\lambda_1} e_{\lambda_2} \cdots = \prod_i e_i^{\# i \text{ in } \lambda}$$

we will show that e_λ is a basis.

Proposition: set $e_\lambda = \sum_\mu \overset{e}{m}_{\lambda\mu} m_\mu$.

then $\overset{e}{m}_{\lambda\mu}$ = # matrices whose entries are each 0 or 1 and whose i^{th} row sum to λ_i, whose j^{th} column sum to μ_j

Proof: consider the infinite matrix $\begin{bmatrix} x_1 & x_2 & \cdots \\ x_1 & x_2 \\ \vdots & \vdots \end{bmatrix}$.

a term in e_λ is obtained by choosing λ_1 entries from the first row, λ_2 from the second row and taking the product of these.
the product is x^α where α_j is the number of chosen entries in column j.
such a choice of entries is equivalent to filling the matrix with 1s in the chosen entries and 0s elsewhere
so the number of times x^α appears in e_λ is the number of 0-1 matrices with row sum λ and column sum α.
as e_λ is invariant under permutation, m_μ appears the same number of times x^μ does. m_μ is a basis of Λ, which gives the result.

Since transposing a matrix exchanges its row and column sums, $\overset{e}{m}_{\lambda\mu} = \overset{e}{m}_{\mu\lambda}$.
The $x^\lambda y^\mu$ term in $\prod_{i,j}(1 + x_i y_j)$ is the number of $\{(i,j)\}$ such that i occurs λ_i times and j occurs μ_j times — ie is $\overset{e}{m}_{\lambda\mu}$. By symmetry of $\prod_{i,j}(1+x_i y_j)$ in x, y separately, $\prod_{i,j}(1 + x_i y_j) = \sum_{\lambda,\mu} \overset{e}{m}_{\lambda\mu} m_\lambda(x) m_\mu(y) = \sum_{\lambda,\mu} m_\lambda(x) e_\mu(y)$
The left hand side is easy to calculate using the Fast Fourier Transform, so this calculates $\overset{e}{m}_{\lambda\mu}$ easily.

Observe that $e_n(x_1, x_2, \cdots)$ is the generating function for putting n balls into infinitely many boxes without repetition, known to physicists as Fermi-Dirac allocation.

and $\prod_{i=1}^{\infty}(1+x_i t) = 1 + e_1 t + e_2 t^2 + \cdots$

Theorem: e_λ is a vector space basis for Λ, so $e_1, e_2, \ldots$ are algebraically independent ie $\Lambda = \mathbb{Q}[e_1, e_2, \cdots]$ (this is the fundamental theorem of symmetric functions)

Proof: we first deduce half of the Gale-Ryser theorem.

Suppose $\overset{e}{m}_{\lambda\mu} \neq 0$ ie there is a 0-1 matrix A with row sum λ and column sum μ. Push all the 1s in A to the left, as far as they can go. The new column sum is λ^T. The first i columns now contain more 1s than they used to $\therefore \lambda^T \geq \mu$.

In other words, $\overset{e}{m}_{\lambda\mu} = 0$ if $\mu \geq \lambda^T$. Observe also that $\overset{e}{m}_{\lambda\lambda^T} = 1$.

Observe that, if $\lambda < \mu$, then blocks of λ move upwards to give μ, but because of the shape of partitions, these blocks must move rightwards. So $\lambda^T > \mu^T$ is equivalent to $\lambda < \mu$.

So, if the rows are ordered lexicographically increasing, the columns by the transpose of this ordering (so it is decreasing in dominance order), then the matrix $\overset{e}{m}$ would have 1s on the diagonal and be lower triangular (λ^T-row above μ-row $\Rightarrow \mu^T$-column to the left of λ-column $\therefore$ entries to the right of μ, μ^T are 0) so e_λ is a basis.

The homogeneous symmetric functions are $h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$

$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$

ie they are the versions of e_λ where terms are allowed to repeat.

The same proofs as before show that:

if $h_\lambda = \sum_\mu \overset{h}{m}_{\lambda\mu} m_\mu$, then $\overset{h}{m}_{\lambda\mu} = \#$ $\mathbb{N}$-matrices with row sum λ and column sum μ;

$\prod_{i,j}(1-x_i y_j)^{-1} = \sum_\lambda m_\lambda(x) h_\lambda(y)$;

$\prod_{i=1}^{\infty}(1-x_i t)^{-1} = 1 + h_1 t + h_2 t^2 + \cdots$

and $h_n(x_1, x_2, \cdots)$ is the generating function for Bose-Einstein, putting n balls into boxes possibly with repetition

Define an involution $\omega: \Lambda \to \Lambda$ by $\omega(e_n) = h_n$ (this is a well-defined algebra homomorphism as e_n are algebraically independent.

As $\prod_{i=1}^{\infty}(1+x_i(-t)) \prod_{i=1}^{\infty}(1-x_i t)^{-1} = 1$, we have $(1 - e_1 t + e_2 t^2 - \cdots)(1 + h_1 t + h_2 t^2 + \cdots) = 1$

Taking each coefficient of t: $0=\sum_{i=0}^{n}(-1)^i e_i h_{n-i}$ ie $h_n=\sum_{i=1}^{n}(-1)^{i+1}e_i h_{n-i}$,
$e_n=\sum_{i=0}^{n-1}(-1)^{i+1}e_i h_{n-i}$

This allows us to show inductively that $\omega(h_n)=e_n$, so ω is indeed an involution. So h_λ is the image of the basis e_λ under an algebra isomorphism, and is therefore a basis of Λ. Thus the h_n are also algebraically independent.

The power sums are $p_n=\sum_i x_i^n$, $p_\lambda=p_{\lambda_1}p_{\lambda_2}\cdots$

Set $z_\lambda=\prod_i i^{\#\text{ times } i \text{ occurs in } \lambda}(\#\text{ times } i \text{ occurs in } \lambda)!$

($n!z_\lambda^{-1}$ is the number of cycles of type λ in $S_{\lambda_1+\lambda_2+\dots}$)

Proposition: $\prod_{i,j}(1-x_iy_j)^{-1}=e^{\sum_{n=1}^{\infty}\frac{1}{n}p_n(x)p_n(y)}=\sum z_\lambda p_\lambda(x)p_\lambda(y)$

$\prod_{i,j}(1+x_iy_j)=e^{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}p_n(x)p_n(y)}=\sum z_\lambda^{-1}\operatorname{sgn}(\lambda)p_\lambda(x)p_\lambda(y)$

where $\operatorname{sgn}(\lambda)$ is the sign of a permutation of type λ, which is also $(-1)^{\#\text{squares}-\#\text{parts in }\lambda}$

Proof: $\prod_{i,j}(1-x_iy_j)^{-1}=e^{-\sum_{i,j}\log(1-x_iy_j)}$

$=e^{\sum_{i,j}\sum_n\frac{1}{n}(x_iy_j)^n}$

$=e^{\sum_n\frac{1}{n}(\sum_i x_i^n)(\sum_j y_j^n)}=e^{\sum_n\frac{1}{n}p_n(x)p_n(y)}$

By Polya's cycle index theorem, the coefficient of t^n in $e^{\sum_n\frac{x_nt^n}{n}}$ is $\sum_{\lambda\vdash n}\frac{1}{z_\lambda}\prod_i x_i^{\lambda_i}$. Set $t=p_n(x)p_n(y)$ and $x_i=1$ $\forall i$.

The second statement is proved analogously.

Applying ω in the y-variables to the first identity shows

$\sum z_\lambda^{-1}p_\lambda(x)\,\omega p_\lambda(y)=\omega\prod_{i,j}(1-x_iy_j)^{-1}=\sum_{\lambda,\mu}m_\lambda(x)\,\omega h_\mu(y)$

$=\sum_{\lambda,\mu}m_\lambda(x)e_\mu(y)$

$=\sum z_\lambda^{-1}\operatorname{sgn}(\lambda)p_\lambda(x)p_\lambda(y)$

so $\omega p_\lambda=\operatorname{sgn}(\lambda)p_\lambda$ since $p_\lambda(x)$ are linearly independent (use reverse lexicographic ordering to see this)

Define the Hall inner product on Λ: $\langle m_\lambda,h_\mu\rangle=\delta_{\lambda\mu}$ and extend linearly.

This is indeed symmetric: $\langle h_\lambda,h_\mu\rangle=\sum \overset{h}{m}_{\lambda\mu}=\sum\overset{h}{m}_{\mu\lambda}=\langle h_\mu,h_\lambda\rangle$.

($\overset{h}{m}_{\lambda\mu}=\overset{h}{m}_{\mu\lambda}$ by transposing the indexing matrices, as for $\overset{e}{m}$)

That this is positive definite will follow from the following lemma applied to $u_\lambda=p_\lambda$, $v_\lambda=z_\lambda p_\lambda$.

Proposition: $\{u_\lambda\}, \{v_\lambda\}$ are bases for Λ_n. These are dual (ie $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$) if and only if

$$\Pi_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda u_\lambda(x) v_\lambda(y)$$

Proof: Write $m_\lambda = \sum_\tau \overset{m}{u}_{\lambda\tau} u_\tau$, $h_\mu = \sum_\nu \overset{h}{v}_{\mu\nu} v_\nu$

so $\delta_{\lambda\mu} = \langle m_\lambda, h_\mu \rangle = \sum_{\tau,\nu} \overset{m}{u}_{\lambda\tau} \overset{h}{v}_{\mu\nu} \langle u_\tau, v_\nu \rangle$

and $\Pi_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda m_\lambda(x) h_\lambda(y) = \sum_{\tau,\nu} \left(\sum_\lambda \overset{m}{u}_{\lambda\tau} \overset{h}{v}_{\lambda\nu}\right) u_\tau(x) v_\nu(y)$

now $\{u\}, \{v\}$ are dual $\Rightarrow \delta_{\lambda\mu} = \sum_\tau \overset{m}{u}_{\lambda\tau} \overset{h}{v}_{\mu\tau} \Rightarrow \delta_{\tau\nu} = \sum_\lambda \overset{m}{u}_{\lambda\tau} \overset{h}{v}_{\lambda\tau}$ since a matrix left-inverse is also a right-inverse, and substituting into the previous line shows $\Pi_{i,j} (1 - x_i y_j)^{-1} = \sum_\tau u_\tau(x) v_\tau(y)$.

conversely, if $\Pi_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda u_\lambda(x) v_\lambda(y)$, then equating coefficients (as u_λ, v_λ are bases) show $\delta_{\tau\nu} = \sum_\lambda \overset{m}{u}_{\lambda\tau} \overset{h}{v}_{\lambda\nu}$ ie $\overset{m}{u}$ is the matrix inverse of $\overset{h}{v}$, so $\langle u_\tau, v_\nu \rangle$ must be the identity matrix.

The application $u_\lambda = p_\lambda$, $v_\lambda = z_\lambda p_\lambda$ shows that p_λ are orthogonal: $\langle p_\lambda, p_\lambda \rangle = z_\lambda^{-1}$

(if we are working over $\mathbb{R}$, we can rescale to get an orthonormal basis)

Also, the involution ω is a self-adjoint isometry: $\langle \omega p_\mu, \omega p_\lambda \rangle = \text{sgn}(\lambda)\,\text{sgn}(\mu)\,\delta_{\mu\lambda} = \delta_{\mu\lambda}$

and $\langle \omega p_\mu, p_\lambda \rangle = \text{sgn}(\mu)\delta_{\mu\lambda} = \text{sgn}(\lambda)\delta_{\mu\lambda} = \langle p_\mu, \omega p_\lambda \rangle$

The Hall inner product is really "the same" as the inner product on $L^2(U_n)$:

given $f \in \Lambda$, define $\tilde{f} \in L^2(U_n)$, $\tilde{f}(\text{a matrix}) = f(\text{its eigenvalues}, 0, 0, \dots)$

(since f is symmetric, the order of the eigenvalues does not matter)

now $\langle f, g \rangle = \int_{U_n} \tilde{f}(x) \overline{\tilde{g}(x)}\, d\mu$ where μ is Haar measure.

This will be "obvious" once we establish that Schur functions are orthonormal.

A semi-standard Young tableaux of shape λ is a placement of integers into the Young diagram of λ with weakly increasing rows and strictly increasing columns.

e.g. $\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \cline{1-2} 5 & 9 & \\ \cline{1-2} 17 & & \\ \cline{1-1} \end{array}$ is a semi-standard Young tableaux of shape $(3,2,2,1)$

The content of a semi-standard Young tableaux T is $\alpha(T) = (\#\text{ of }1\text{s}, \#\text{ of }2\text{s}, \dots)$

For $\mu \subseteq \lambda$, the skew diagram of shape $\lambda \backslash \mu$ is the diagram of λ with the diagram of μ removed. Skew diagrams need not be connected. A semi-standard skew tableaux is the analogous thing e.g. $\begin{array}{ccc} & & \boxed{5} \\ & \boxed{3} & \\ \boxed{4} & \boxed{4} & \\ \boxed{7} & & \end{array}$ is a semi-standard skew tableaux.

The Schur function $s_{\lambda\backslash\mu}(x) = \sum_T x^{\alpha(T)}$ where we sum over all semi-standard skew tableaux T of shape $\lambda\backslash\mu$. We usually write $s_\lambda(x)$ for $s_{\lambda\backslash\emptyset}(x)$ – ie we sum over all semi-standard Young tableaux of shape λ.

Proposition: $s_{\lambda\setminus\mu}$ is a symmetric function.

Proof: as the transposition $(i\ i{+}1)$ generate all permutations of $\mathbb{N}$, it suffices to construct a bijection, for each α,

$$\Phi_i \begin{Bmatrix} \text{semi-standard skew tableaux} \\ \text{of shape } \lambda\setminus\mu \text{ and content } \alpha \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{semi-standard skew tableaux} \\ \text{of shape } \lambda\setminus\mu \text{ and content } (\alpha_1 \cdots \alpha_{i+1}, \alpha_i, \dots) \end{Bmatrix}$$

Each row has a segment which looks like
$$\begin{array}{cccc} & & & i \cdots i \\ \cdots i \ \ i \cdots & \boxed{i \ \ i+1} & \cdots & i+1 \\ \cdots i+1 \cdots i+1 & & & \end{array}$$

Now, replace in the boxed part (the i's which do not have $i+1$'s below them, and the $i+1$'s which do not have i's above them — this is a continuous segment because the rows above and below this row must be weakly increasing) the i's with $i+1$ and the $i+1$'s with i's, and then rearrange so the row stays weakly increasing.
So some i's are now $i+1$'s and some $i+1$'s are now i's, but the columns stay strictly increasing since these columns only contained either i or $i+1$.
All i's which did not switch lie above an $i+1$, and all $i+1$'s which did not switch lie below an i $\therefore$ there is the same number of both, so the content changes in the desired way.

Define the Kostka numbers $K_{\lambda\setminus\mu,\nu}$ = # semi-standard skew tableaux of shape $\lambda\setminus\mu$ and content ν. So the coefficient of x^ν in $s_{\lambda\setminus\mu}$ is $K_{\lambda\setminus\mu,\nu}$.
Since $s_{\lambda\setminus\mu}$ is symmetric, this means $s_{\lambda\setminus\mu} = \sum_\nu K_{\lambda\setminus\mu,\nu}\, m_\nu$.

e.g. $\lambda = 2,1$, $\mu = \phi$: $\nu = 3$: no possible tableaux

$\nu = 2,1$: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}$ $\quad K_{(2,1),(2,1)} = 1$

$\nu = 1,1,1$: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$ $\quad K_{(2,1),(1,1,1)} = 2$

$\therefore s_{2,1} = m_{2,1} + 2m_{1,1,1}$

Proposition: if $K_{\lambda,\nu} = 0$, then $\lambda \geq \nu$

Proof: take a semi-standard Young tableaux with shape λ and content ν.

As its columns are strictly increasing, all i's must lie in the first i rows.
So $r_1 \le \lambda_1$, $r_1 + r_2 \le \lambda_1 + \lambda_2$ etc.

Observe that the above inequalities are all equalities if and only if the first row is filled with 1s, the second row with 2s etc. Hence $K_{\lambda\lambda} = 1$, so the change of basis matrix (from s_λ to m_λ) is triangular with 1s on the diagonal - ie s_λ give a basis for Λ.

There is no known algorithm to efficiently calculate $K_{\lambda\nu}$ except for some special values. If a semi-standard Young tableaux has content 1^n (ie the integers 1, 2, ... each appear once), then it is a standard Young tableaux.

Proposition: $K_{\lambda, 1^n}$ (where $\sum \lambda_i = n$) enumerate / the standard Young tableaux of shape λ
index: i saturated paths in the partition poset from ϕ to λ
ii ballot sequences of n votes, where the number of candidates is the number of parts of λ, and candidate i is always ahead of candidate i+1.
iii lattice paths in $\mathbb{R}^{\#\text{ parts in } \lambda}$ from 0 to λ, staying within the region $\{x_1 \ge x_2 \ge \cdots\}$

Proof: i the i^{th} element of the path is the partition given by the boxes filled with 1, 2, ... i. Conversely, a path describes an order to fill in the boxes with.
ii the i^{th} vote goes to the candidate whose row contains box i.
iii take the i^{th} step in the direction of the row containing i.

Consider the game of patience-sorting, which is played with the deck $\{1, 2, \cdots n\}$. The cards are turned up one at a time, and a lower card may be placed on a higher card. If we turn up a card which is higher than all cards showing, it starts a new pile. The object of the game is to finish with as few piles as possible.
e.g. If the deck is (in order) 5371264, then one possible play ends with $\begin{smallmatrix}1 & & \\ 3 & 2 & 4 \\ 5 & 7 & 6\end{smallmatrix}$.

Proposition: let w denote the string / permutation encoding the order of the cards in the deck. The number of piles at the end is bounded below by the length of the longest increasing subsequence (and this is achieved by the greedy algorithm, where we place each card on the smallest showing card that it can be placed on — the example above is played this way).

Proof: The cards belonging to an increasing subsequence must be placed on different piles. In a greedy game that ended with n piles:

the top card on pile n

the top card on pile $n-1$ when the above card was played

the top card on pile $n-2$ when the above card was played

⋮

give a decreasing sequence (because otherwise we would've placed it on the previous pile) going backwards in time — ie an increasing subsequence when read "forward" (in our example, this is 124).

So the greedy algorithm on patience sorting gives an efficient way to find the length of the longest increasing sequence — in fact, it is provably the fastest way. This is useful for computing the number of deletion-insertions necessary to transform one permutation into another. This motivates Ulam's problem: find the distribution of this length when we sample uniformly from S_n. The average length is $2\sqrt{n}$.

The original Robinson-Schensted-Knuth algorithm maps S_n bijectively to pairs of standard Young tableaux of the same shape with n-boxes, by greedy patience sorting on each level as follows: turn up a card and put it on the first level, replacing the smallest card higher than it if possible. Play the replaced card on the second level in the same way, and continue until a card does not replace anything. Now turn up the next card and repeat.

The end result after all cards are played is the first tableaux in the image. The second tableaux bookkeeps the process: put i in the slot where the last card is played when card i is turned up.

Example: $w = 67315429 8$ patience sorting game bookkeeping

6 1

(7>6 so can't replace 6)	67	12	
(3 replaces 6, which	37	12	
appears in level 2 now)	6	3	
(1 replaces 3 in level 1,	17	12	
3 replaces 6 in level 2,	3	3	
6 ends up in level 3)	6	4	
(5 replaces 7 in level 1,	15	12	
7 can't replace anything	37	35	
in level 2)	6	4	
	14	12	(existing numbers never
	35	35	move in the bookkeeping
	67	46	tableaux)
(note 5 changes	12	12	
columns here)	34	35	
	57	46	
	6	7	
	129	128	
	34	35	
	57	46	
	6	7	
this line is the	128	128	
image of the	349	359	
RSK map	57	46	
	6	7	

There is an extension which creates pairs of semi-standard Young tableaux: play patience-sorting with any sequence of natural numbers (we do not allow a card to replace a card of the same value, only if it is strictly higher) The bookkeeping uses, instead of $123 \cdots n$, a weakly increasing sequence, which increases strictly at positions where the deck has a descent.

e.g. deck $= 1332212$, which has descents at 3 and 5

$\therefore$ a possible bookkeeping sequence is 1112233.

then:

patience-sorting game	bookkeeping
$\boxed{1}$	1
$1\boxed{3}$	11
$13\boxed{3}$	111
$1\boxed{2}3$ $\boxed{3}$	111 2
$12\boxed{2}$ $3\boxed{3}$	111 22
$1\boxed{1}2$ $\boxed{2}3$ $\boxed{3}$	111 22 3
$112\boxed{2}$ 23 3	1113 22 3

The $\boxed{}$ above indicates the insertion paths: the cards that move when a card is turned up. Observe that insertion paths always go to the left as they descend (assuming that at each step we indeed have a semi-standard Young tableaux – we will show this inductively using this property of the insertion path in a minute): if r,s lies on the insertion path, then the entry originally there is strictly smaller than that below it, so the smallest card on which entry r,s can be played is either in position $r+1,s$ or to the left of it.

Now the proof that RSK indeed produces a pair of semi-standard Young tableaux: the patience-sorting rules mean that all rows are weakly increasing. The columns have a segment replaced (where it intersects the insertion path, this is a single segment because the insertion path moves to the left). Because a lower card replaces a higher card, the segment is strictly increasing, so we only need to check the top (r,s) and bottom (r',s) entries of this replaced segment. The new entry in (r,s) came from the right of $(r-1,s)$ (as insertion paths move to the left), and since we always play on the leftmost pile possible, this entry is strictly bigger than what's in (r,s). The new (r',s) entry is smaller than what used to be there, so it's still strictly smaller than entry $(r'+1,s)$.

As for the bookkeeping tableaux, we always add numbers to the right or bottom, and since the sequence is weakly increasing, the rows and columns are both weakly increasing. To show that the columns are strictly increasing, we show that the columns can't grow by more than 1 box when we turn up the cards in the positions labelled by the same number in the bookkeeping sequence. The key is that these cards are weakly increasing, so, by the greedy algorithm, the sequences of replacements that they generate are also entrywise increasing, so each insertion path is strictly to the right of the previous. Since there is no box immediately to the right of a newly-added box, an insertion path cannot end below a path that it is to the right of.

Next, we show that RSK is indeed bijective to pairs of semi-standard Young tableaux of the same shape. In essence, the bookkeeping tableaux allows us to reconstruct the entire gameplay from the end result. From the last paragraph, we know that, for the plays indexed by the same number in the bookkeeping sequence, the insertion paths are on the right of the previous path. Hence, out of the boxes containing the largest number in the bookkeeping tableaux, the rightmost one was added last. Look at the card in this position (ie the corresponding entry in the other tableaux). If this is on the first level, this must be the card we turned up. Otherwise, the card that replaced it in the level above is the rightmost card that is lower than this card, and we can retrace the insertion path like this to find the card that was turned up.

So the coefficient of $x^\alpha y^\beta$ in $\sum_\lambda s_\lambda(x) s_\lambda(y)$ is the number of pairs of semi-standard Young tableaux of same shape and content α, β respectively. Via RSK, this is the number of pairs of sequences that are permutations of α, β respectively, with the second sequence weakly increasing and strictly increasing wherever the first sequence has a descent. We can bijectively send such pairs of sequences to $\mathbb{N}$-matrices with row sum α and column sum β: let a_{ij} be the number of times i occurs in the second sequence and j in the first sequence in the same positions (inverse: starting from the top left and working along each row in turn, put i a_{ij} times in the second sequence, and j a_{ij} times in the first sequence. So our example above corresponds to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$). This number is also $m^h_{\alpha\beta}$, the coefficient of $x^\alpha y^\beta$ in $\prod_{i,j}(1-x_iy_j)^{-1}$. Hence $\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda s_\lambda(x)s_\lambda(y)$, which means the Schur functions are an orthonormal basis.

Observe that the first row of the first tableaux is the result if we just played a single game of patience-sorting - so the length of the first row is the length of the longest weakly increasing subsequence (1222 in our example above). The lengths of the other rows are also related to increasing subsequences. In ordinary RSK, reversing the deck transposes the first tableaux, so the length of the first column must be the length of the longest decreasing subsequence. The un-standardisation (see below) of a decreasing subsequence is strictly decreasing, so, in the extended case, the length of the first column of the first tableaux is the length of the longest strictly decreasing subsequence.

As $\sum_\lambda m_\lambda(x)h_\lambda(y) = \prod_{i,j}(1-x_iy_j)^{-1} = \sum_\mu s_\mu(x)s_\mu(y) = \sum_{\mu\lambda} K_{\mu\lambda}m_\lambda(x)s_\mu(y)$, equating coefficients of $m_\lambda(x)$ gives $h_\lambda = \sum_\mu K_{\mu\lambda}s_\mu$
Recall that the number of $\mathbb{N}$-matrices with row sum λ and column sum μ is $\langle h_\lambda, h_\mu\rangle = \langle \sum_\nu K_{\nu\lambda}s_\nu, \sum_\tau K_{\tau\mu}s_\tau\rangle = \sum_\nu K_{\nu\lambda}K_{\nu\mu}$. Since computing $\langle h_\lambda, h_\mu\rangle$ is provably hard, this suggests computing $K_{\lambda\mu}$ is also quite hard.

Take $\lambda = 1^n$. Then $\sum_\mu K_{\mu,1^n}s_\mu = h_n = (x_1+x_2+\cdots)^n$, and $\sum_\mu K_{\mu,1^n}s_\mu = \sum_{\mu,\nu} K_{\mu,1^n}K_{\mu\nu}m_\nu$. Equating coefficients of $x_1x_2\cdots x_n$: $n! = \sum_\mu K_{\mu,1^n}K_{\mu,1^n}$, which we also know from applying original RSK.

Extended RSK can be obtained from ordinary RSK via standardisation (which is a useful technique for extending standard Young tableaux results to semi-standard Young tableaux). Given a deck with repeated cards, replace all 1s with 1, 2, ...

# of 1s (in order), then replace 2s with (# of 1s)+1, (# of 1s)+2 ... and bookkeep with 123⋯n. Playing ordinary RSK with this substitution is the same as playing extended RSK with the original sequences, since one cannot play a card on another of the same value. Reversing the substitution at the end recovers the tableaux produced by extended RSK.

A geometric interpretation of RSK reveals many symmetry properties - for example, if the matrix A corresponds to the tableaux (P,Q), then A^T corresponds to (Q,P). Transposing a permutation matrix produces the inverse permutation, so for the non-extended case, this says, if w corresponds to (P,Q), w^{-1} corresponds to (Q,P). So symmetric matrices correspond to semi-standard Young tableaux and involution permutations correspond to standard Young tableaux. The number of ways to fill a symmetric matrix with row sum α is precisely the coefficient of x^α in $\prod_k (1-x_k)^{-1} \prod_{i<j} (1-x_i x_j)^{-1}$ (the first product chooses the diagonal entries, the second chooses the upper triangular entries, which counts towards both row and column because of symmetry). Hence $\prod_k (1-x_k)^{-1} \prod_{i<j} (1-x_i x_j)^{-1} = \sum_\lambda s_\lambda$

Dual RSK differs from RSK in that a card is allowed to displace one of equal value (or of higher value, as before). To accommodate this, the bookkeeping sequence must increase at positions where the main sequence repeats, as well as at descents.

e.g. 13213, bookkeep with 11233

1   1

13   11

12   11
3   2

12   11
1   2
3   3

123   113
1   2
3   3

The first tableaux is now not semi-standard, but its transpose is, because the new rule prevents repetition in rows. Each insertion path still moves to the left as it goes down, and is to the right of the previous if they are labelled by the same number in the bookkeeping sequence. So the second tableaux is semi-standard.

As for RSK, we can reconstruct the whole game from the last two tableaux, and we recover sequences of the required form, so dual RSK is a bijection from 0-1 matrices to pairs (P,Q) of tableaux of the same shape, with P^T and Q semi-standard. Recalling our generating function for 0-1 matrices,

$\prod_{i,j}(1+x_iy_j)=\sum_\lambda s_\lambda(x)s_{\lambda^T}(y)$

As the involution ω on the y variables send $\prod_j(1-x_iy_j)$ to $\prod_j(1+x_iy_j)$, equating coefficients of $s_\lambda(x)$ shows $\omega(s_\lambda)=s_{\lambda^T}$.

Classically, Schur functions are defined via the alternant: for $\alpha\in\mathbb{N}^n$,

$a_\alpha(x)=\sum_{w\in S_n}\mathrm{sgn}(w)\,w(x^\alpha)=\det(x_i^{\alpha_j})$

If $\alpha_i=\alpha_j$, then the matrix $x_i^{\alpha_j}$ has two columns equal, so $a_\alpha(x)=0$. Swapping α_i and α_j exchanges two columns of the matrix $x_i^{\alpha_j}$, which changes the sign of a_α. So it suffices to study a_α where $\alpha_1>\alpha_2>\dots>\alpha_n\geq 0$ ie where $\alpha=\lambda+\delta$, where λ is a partition and $\delta=(n-1,n-2,\dots,1,0)$.

Exchanging x_i and x_j swaps two rows of $x_i^{\alpha_j}$, so changes the sign of a_α. Hence a_α is an alternating polynomial, and is a multiple of $\prod_{i<j}(x_i-x_j)=a_\delta(x)$. Hence a_α/a_δ is a symmetric polynomial. Since multiplication by a_δ turns a symmetric polynomial into an antisymmetric one, $a_\alpha\to a_\alpha/a_\delta$ is an isomorphism from antisymmetric polynomials to symmetric ones. $\{a_\alpha \mid \alpha-\delta \text{ a partition}\}$ gives a basis of antisymmetric polynomials, so a_α/a_δ gives a basis of symmetric polynomials: we will see that, under the inclusion of such polynomials to Λ, a_α/a_δ maps precisely to $s_{\alpha-\delta}$.

Jacobi-Trudi identity: $a_{\lambda+\delta}/a_\delta=\det(h_{\lambda_i-i+j})$ where $h_k=0$ if $k<0$, and otherwise $h_k=\sum_{1\leq i_1\leq i_2\leq\dots\leq i_k\leq n}x_{i_1}\cdots x_{i_k}$ (ie the symmetric polynomial in n variables whose image in Λ is h_k). Once we know that $a_{\lambda+\delta}/a_\delta$ maps to s_λ, applying the involution ω shows that $s_\lambda=\det(e_{\lambda^T_i-i+j})$.

In practice we find these matrices by filling the diagonal with h_{λ_i}, and then filling each row so that the indices increase by 1 along the row e.g. $s_{5,3,3} = \det\begin{pmatrix} h_5 & h_6 & h_7 \\ h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \end{pmatrix}$

Proof: Write $e_j^{(k)}$ for the coefficient of t^j in $\prod_{i \neq k}(1+tx_i)$

Recall that $(1+h_1t+h_2t^2+\dots)\prod(1-tx_i)=1$

So $(1+h_1t+h_2t^2+\dots)(1-e_1^{(k)}t+e_2^{(k)}t^2-\dots) = (1-tx_k)^{-1}$

Take coefficients of t^{α_i} of both sides: $\sum_{1,n-j} h_{\alpha_i-n+j}(-1)^{n-j}e_{n-j}^{(k)} = x_k^{\alpha_i}$

So $a_\alpha = \det(x_j^{\alpha_i}) = \det(h_{\alpha_i-n+j})\det((-1)^{n-i}e_{n-i}^{(j)})$ (with $1 \le i,j \le n$)

$a_{\lambda+\delta} = \det(h_{\lambda_i-i+j})\det((-1)^{n-i}e_{n-i}^{(j)})$

Take λ to be the empty partition: $a_\delta = \det(h_{-i+j})\det((-1)^{n-i}e_{n-i}^{(j)})$

h_{-i+j} is upper triangular with $h_0=1$ on the diagonal $\Rightarrow \det(h_{-i+j})=1$

so $a_\delta = \det((-1)^{n-i}e_{n-i}^{(j)}) \Rightarrow a_{\lambda+\delta}/a_\delta = \det(h_{\lambda_i-i+j})$

We can show using an involution principle that $\det(h_{\lambda_{k+1-j}+j-i})$ is the sum of the height-weights of all sets of k non-intersecting lattice paths from $(i,1)$ to $(\lambda_{k+1-i}+i, n)$. Each such set of paths correspond to a semi-standard Young tableaux with k rows whose entries lie in $\{1,2,\dots n\}$, and $x^{\text{its content}}$ is the weight of the set.

By reversing the k rows and columns of $h_{\lambda_{k+1-j}+j-i}$, we see that $\det(h_{\lambda_{k+1-j}+j-i}) = \det(h_{\lambda_j-j+i}) = s_\lambda$. This proves $a_{\lambda+\delta}/a_\delta = s_\lambda$ (when restricted to n variables).

More detail in my enumerative combinatorics notes, Constructive Combinatorics by Stanton and White, or Representation Theory of the Symmetric Group by Sagan. Total Positivity by Fomin and Zelevinsky discusses connections from lattice paths to double Bruhat cells and cluster algebras, and Vicious Walkers contains some applications to physics.

The analogue of Jacobi-Trudi for skew-Schur functions says $s_{\lambda\setminus\mu} = \det(h_{\lambda_i-\mu_j-i+j})$.

It can be shown, $\forall \nu$, $\langle s_{\lambda\setminus\mu}, s_\nu\rangle = \langle s_\lambda, s_\mu s_\nu\rangle$, and applying ω to all terms (as ω is an isometry) proves that $\omega(s_{\lambda\setminus\mu}) = s_{\lambda^T\setminus\mu^T}$

A rim hook (also called hooks, ribbons or border strips) is a skew-shape that is connected with no 2×2 blocks.

Rim hooks are specified by weak compositions (the number of boxes in each row), since the conditions force each row to "overlap" with the previous at precisely one column. Hence there are

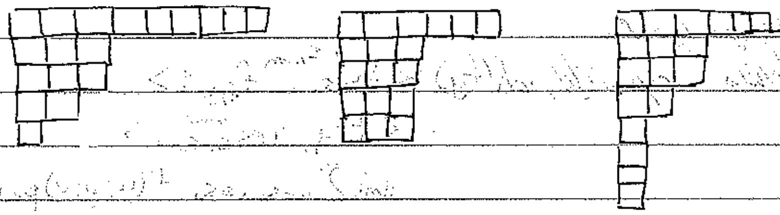
2^{n-1} rim hooks with n boxes
The height of a rim hook is the number of parts -1

Theorem: $s_\mu p_m = \sum_\lambda (-1)^{\text{height}(\lambda\setminus\mu)} s_\lambda$ summing over λ for which $\lambda\setminus\mu$ is a hook with m boxes.

Proof: Work in n variables, with $n > m + \#$ boxes in μ.
Recall that $a_{\mu+\delta} = \sum_{w\in S_n} \operatorname{sgn}(w) w(x^{\mu+\delta})$ (add 0s to μ if it has fewer than n parts) Write e_i for the vector with 1 in entry i and 0 otherwise
so $a_{\mu+\delta}\, p_m = \sum_{w\in S_n} \operatorname{sgn}(w) w(x^{\mu+\delta}) \left(\sum_{i=1}^n x^{me_i}\right)$
$= \sum_{w\in S_n} \operatorname{sgn}(w) w(x^{\mu+\delta}) \left(\sum_{i=1}^n w(x^{me_i})\right) = \sum_{i=1}^n a_{\mu+\delta+me_i}$
Rearrange the components of $\mu+\delta+me_i$ in descending order — ie if $(\mu+\delta+me_i)_q = \mu_q + (n-q) + m$ is bigger than the earlier coordinates, move it up, until we get the partition $\lambda+\delta$ "representing" $\mu+\delta+me_i$
(if $\mu+\delta+me_i$ has two components, just forget it, because then $a_{\mu+\delta+me_i} = 0$)
Suppose the rearranging places $(\mu+\delta+me_i)_q$ between the $(p-1)$th and pth components — ie $\mu_{p-1} + (n-p+1) > \mu_q + (n-q+r) \geq \mu_p + n - p$. Then
$\lambda = (\mu_1, \ldots, \mu_{p-1}, \mu_q + p - q + r, \mu_p + 1, \ldots, \mu_{q-1}+1, \mu_{q+1}, \ldots, \mu_n)$ and
$a_{\lambda+\delta} = (-1)^{q-p} a_{\mu+\delta+me_i}$ since each adjacent transposition incurs a change in sign.
Now $\lambda\setminus\mu$ has no squares in the first $p-1$ rows or rows $q+1, \ldots n$.
On rows $p+1$ to q, $\lambda\setminus\mu$ "extends" μ so it is 1 longer than the row above (of μ) — such skew shapes are precisely hooks, and the sign change is the height of the hook.
So $a_{\mu+\delta}\, p_m = \sum_\lambda (-1)^{\text{height}(\lambda\setminus\mu)} a_{\lambda+\delta}$ over the required set
Divide both sides by a_δ and note that, since $n \geq m + \#$ boxes in μ, $s_\mu p_m$ is uniquely determined by its preimage in symmetric functions in m variables.

Example: $\mu = 63321$, $m=3$. Take $n=9$
$\therefore \mu+\delta = 14, 10, 9, 7, 5, 3, 2, 1, 0$
$\mu+\delta+me_1 = 17, 10, 9, 7, 5, 3, 2, 1, 0$ which is already in order
so corresponding $\lambda = 9, 3, 3, 2, 1$
$\mu+\delta+me_4 = 14, 10, 9, 10, 5, 3, 2, 1, 0$ which has repeated entries, so we ignore it.

$\mu+\delta+me_5 = 14,10,9,7,8,3,2,1,0$ swap 7 and 8
$\therefore$ corresponding $\lambda = 6,3,3,3,3,3$, appears with sign (-1).
$\mu+\delta+me_8 = 14,10,9,7,5,3,2,4,0$ move 4 up two places
$\therefore$ corresponding $\lambda = 6,3,3,2,1,1,1,1$ appears with sign $(+1)$.
The tableaux for the λs above are:



We wish to iterate this construction to calculate $s_\mu p_r$ for general r.
Define a skew tableaux of shape $\lambda\setminus\mu$ and content α to be a border strip tableaux if its rows and columns are weakly increasing and the boxes containing each i form a hook. Its height is the sum of the heights of these hooks.

e.g. $\begin{array}{|c|c|c|c|} \hline & 1 & 2 & 2 \\ \hline 1 & 1 & 3 & 3 \\ \hline 3 & 3 & 3 & \\ \hline 3 & 4 & & \\ \hline \end{array}$ is a border strip tableaux of shape $(4,4,3,2)\setminus 1$ and content $(3,2,6,1)$. It has height $1+0+2+0=3$.

Then $s_\mu p_r = \sum_\lambda \chi_{\lambda\setminus\mu}(r) s_\lambda$ where $\chi_{\lambda\setminus\mu}(r) = \sum_T (-1)^{\text{height } T}$ summing over all border strip tableaux of shape $\lambda\setminus\mu$ and content r.
Taking $\mu = \phi$, we see $p_r = \sum_\lambda \chi_\lambda(r) s_\lambda$ and $s_\mu = \sum_r \langle s_\mu, p_r z_r^{-1}\rangle p_r$

$$= \sum_r \langle s_\mu, \sum_\lambda \chi_\lambda(r) s_\lambda z_r^{-1}\rangle p_r$$

this is the Murnaghan-Nakayama rule $= \sum_r \chi_\mu(r) z_r^{-1} p_r$

So $\sum_\lambda \chi_\lambda(r)\chi_\lambda(\mu) z_\mu^{-1} = \delta_{\mu r}$, $\sum_r \chi_\mu(r)\chi_\lambda(r) z_r^{-1} = \delta_{\lambda\mu}$ — you might guess from this and my suggestive notation that $\chi_\lambda(\mu)$ is the λ-character of S_n evaluated on the μ-conjugacy class (and $p_r = \sum_\lambda \chi_\lambda(r) s_\lambda$ is then a statement of Schur-Weyl duality)

To explain this character theory connection, look at the vector space $\bigoplus_n$ class functions on S_n.
Define the characteristic map from this space to symmetric functions:
$ch(f) = \frac{1}{n!}\sum_{w\in S_n} f(w)\, p_{\text{cycle type of } w} = \sum_\lambda z_\lambda^{-1} f(\text{permutation of type } \lambda) p_\lambda$ and extend linearly.
With the usual inner product on the class functions (on each piece, $\langle f,g\rangle = 0$ if f,g from different pieces), ch is an isometry: $\langle ch(f), ch(g)\rangle = \langle \sum_\lambda z_\lambda^{-1} f(\lambda) p_\lambda, \sum_\mu z_\mu^{-1} g(\mu) p_\mu\rangle = \sum_\lambda z_\lambda^{-1} f(\lambda) g(\lambda)$
$= \frac{1}{n!}\sum_{w\in S_n} f(w) g(w) = \langle f,g\rangle$ (it suffices to work over $\mathbb{R}$, since all characters of S_n are real).

Ch is also an algebra isomorphism, where multiplication on the class functions is determined by induction $S_n \times S_m \hookrightarrow S_{n+m}$:

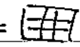
1. Ch is a ring homomorphism:
Extend the coefficient ring on the class functions of S_n to Λ - then $Ch(f) = \langle f, \psi \rangle$
where $\psi(w) = p_{\text{cycle type of } w}$.
Then, by Frobenius reciprocity, $Ch(fg) = \langle \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} f \times g, \psi \rangle$
$= \langle f \times g, \mathrm{Res}_{S_n \times S_m}^{S_{n+m}} \psi \rangle$ look at cycle lengths
$= \frac{1}{n!m!} \sum_{u \in S_n, v \in S_m} f(u) g(v) p_u p_v$
$= Ch f \, Ch g$

2. $Ch(\chi_{M^\lambda}) = h_\lambda$ where M^λ is the permutation module on tabloids, which can also be thought of as ordered set partitions
The stabiliser of an element in M^λ is $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$, so
$\chi_{M^\lambda} = \mathbb{1}_{S_{\lambda_1}} \mathbb{1}_{S_{\lambda_2}} \cdots \mathbb{1}_{S_{\lambda_k}}$, where $\mathbb{1}_H$ denotes the trivial representation on H.
By 1, it suffices to show that $Ch(\mathbb{1}_{S_n}) = h_n$
$Ch(\mathbb{1}_{S_n}) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda = h_n$ since $\sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda t^n = \sum h_n t^n$ by cycle-index theorem

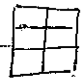
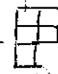
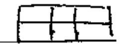
3. Every s_λ has a preimage under Ch (we already know Ch is injective, since it's an isometry)
$s_\lambda = \det(h_{\lambda_i + j - i}) = Ch[\det(\mathbb{1}_{\lambda_i + j - i})]$ (this can be thought of as "inversion" of 2)

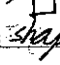
Orthonormality of s_λ means that $\langle \det(\mathbb{1}_{\lambda_i + j - i}), \det(\mathbb{1}_{\mu_i + j - i}) \rangle = \delta_{\lambda\mu}$
$\det(\mathbb{1}_{\lambda_i + j - i})$ is an integral linear combination of induced characters with "length" 1 $\therefore$ it is $\pm$ an irreducible character
$\det(\mathbb{1}_{\lambda_i + j - i})$ evaluated at 1 is $\sum_{w \in S_n} \frac{n!}{\prod_i (\lambda_i + w(i) - i)!} > 0$
so $\det(\mathbb{1}_{\lambda_i + j - i})$ is an irreducible character.

But we showed previously that $s_\mu = \sum_\nu \chi_\mu(\nu) z_\nu^{-1} p_\nu = Ch(\chi_\mu)$
$\therefore \sum_T (-1)^{\text{height } T}$ over all border strip tableaux of shape λ and content μ
$= \chi_\lambda(\mu) = \det(\mathbb{1}_{\lambda_i + j - i})(\mu)$ is an irreducible character.
So the Murnaghan-Nakayama rule allows us to compute the character values (although this isn't efficient unless the conjugacy class and character correspond to "simple partitions", but is still useful for obtaining bounds e.g. to calculate convergence rates of random walks on S_n, which are related to black holes).

e.g. $\chi_{(3,3,3)}((1234)(56)(789))$ is $\sum_T (-1)^{\text{height } T}$, summing over all $T =$  which decompose as a 1-hook with 4 squares, a 2-hook with 2 squares and a 3-hook with 3 squares.

To find all such, we consider the 3-hooks first. There are 3 possible locations for an outwardmost hook of 3 squares, and after removing this hook we have

,  or 

Next, we remove the 2-hook of 2 squares. The second shape above has no hook of 2 squares; the other shapes each have 2 possible locations for a hook of 2 squares, and removing those give respectively , , , .

These must be the 1-hook ∴ first and last shapes are illegal.

∴ the border strip tableaux we require are
$$\begin{array}{ccc} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \quad \text{and} \quad \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & 3 \end{array}$$

which have height $2+1+2=5$ and $1+0+0=1$ respectively.

so $\chi_{(3,3,3)}((1234)(56)(789)) = -2$

e.g. $\chi_\lambda(n\text{-cycle})$ is $\sum_T (-1)^{\text{height } T}$, summing over all λ-tableaux which is a single 1-hook ∴ $\chi_\lambda(n\text{-cycle}) = 0$ unless $\lambda = (k,1,1,\dots,1)$, in which case $\chi_\lambda(n\text{-cycle}) = (-1)^{\text{height of } \lambda} = (-1)^{n-k}$.

The ch map categorifies to $V \to \mathrm{Hom}_{S_n}(V, (\mathbb{C}^k)^{\otimes n})$, regarding the image as a GL_k representation.

The main references for what follows are Generalised Riffle Shuffles and Quasisymmetric Functions by Stanley, and Combinatorial Hopf Algebras and Generalised Dehn-Sommerville Relations by

These are current hot topics of research, and attempt to unify much of combinatorics.

A formal power series is quasi-symmetric if the coefficient of $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ is equal to the coefficient of $x_{j_1}^{\alpha_1} \cdots x_{j_k}^{\alpha_k}$ whenever $i_1 < \dots < i_k$, $j_1 < \dots < j_k$. Clearly all symmetric functions are quasi-symmetric. $\sum_{i<j} x_i^2 x_j$ is quasi-symmetric but not symmetric.

Given a composition $\alpha = (\alpha_1, \alpha_2, \dots \alpha_k)$ of n, set $S_\alpha = \{\alpha_1, \alpha_1+\alpha_2, \dots \alpha_1 + \dots + \alpha_{k-1}\} \subseteq [n-1]$.
The inverse map is $S = \{s_1, s_2, \dots, s_{k-1}\} \mapsto \alpha_S = (s_1, s_2 - s_1, \dots s_{k-1} - s_{k-2}, n - s_{k-1})$
e.g. $\alpha = (1,5,3)$, $S_\alpha = \{1, 6\}$
For $w \in S_n$, we will α_w for the composition corresponding to the descent set of w
(Recall that $i \in$ descent set of w if $w_i > w_{i+1}$)

An obvious basis for Q, the set of quasi-symmetric functions, is
$m_\alpha = \sum_{i_1 < i_2 < \dots < i_k} x^{\alpha_1} \dots x^{\alpha_k}$ across all compositions / $\mathbb{N}$-sequences α.
Another basis is given by $L_\alpha = \sum_{i_1 \le \dots \le i_k,\ i_j < i_{j+1} \text{ if } j \in S_\alpha} x_{i_1} x_{i_2} \dots x_{i_k}$
e.g. $L_3 = \sum_{i \le j \le k} x_i x_j x_k = h_3 = m_{1,1,1} + m_{2,1} + m_{1,2} + m_3$
$L_{2,1} = \sum_{i \le j < k} x_i x_j x_k = m_{2,1} + m_{1,1,1}$
$L_{1,2} = \sum_{i < j \le k} x_i x_j x_k = m_{1,2} + m_{1,1,1}$
$L_{1,1,1} = \sum_{i < j < k} x_i x_j x_k = m_{1,1,1}$
The L_α's are also denoted F_α, and are known as fundamental basis or Gessel's basis.
By considering where the strict inequalities in m_α can be relaxed, we see that
$L_\alpha = \sum_{S_\alpha \subseteq T \subseteq [n-1]} m_{\alpha_T}$
so, by inclusion-exclusion, $m_\alpha = \sum_{S_\alpha \subseteq T \subseteq [n-1]} (-1)^{|T \setminus S_\alpha|} L_{\alpha_T}$. Hence L_α do indeed form a basis.

Quasi-symmetric functions are related to card-shuffling in the following way:
we generalise an inverse shuffle to an inverse X shuffle: let X be a probability distribution on $\mathbb{N}$, and give each card a number independently according to X. Then reorder the cards according to these numbers, keeping the cards with the same number in the same relative order as before.
The description of the forward process is slightly more confusing: again, label the cards according to X, then perform the permutation corresponding to the standardisation of this labelling.
Equivalently, split the deck into piles so that the size of the i^{th} pile has distribution (# cards marked i when the deck is labelled with distribution X), then shuffle these piles together.

For fixed pile sizes $a_1, a_2, \dots a_k$, the forward shuffles are precisely the shortest length coset representatives for $S_{a_1} \times S_{a_2} \times \dots \times S_{a_k} \le S_n$, while inverse shuffles describe all permutations whose descent set $\subseteq \{a_1, a_1+a_2, \dots, a_1+a_2+\dots+a_{n-1}\}$.

Write x_i for the probability of i under X. Then the probability of obtaining ω after a single X-shuffle is $L_{\alpha_{\omega^{-1}}}$: ω^{-1} has descent set $\{i : \omega^{-1}(i) > \omega^{-1}(i+1)\} = \{i : i+1$ comes before i in $\omega\}$. We want the probability that $i_1 \dots i_n$ (IID from X) standardises to ω. Relabel the i's as $a_j = i_{\omega^{-1}(j)}$, so a_j is the number that, when standardised, becomes j. So we must have $a_1 \le a_2 \le \dots \le a_{n-1}$ with $a_i < a_{i+1}$ if $i+1$ comes before i ie if $i \in$ descent set of ω^{-1}. The probability this occurs is $x_{a_1} x_{a_2} \dots x_{a_n}$ for each sequence a_i satisfying the above condition, and the sum of these probabilities is precisely $L_{\alpha_{\omega^{-1}}}$.

Specialising to the usual k-shuffle (X is uniform on $\{1,2,\dots,k\}$) shows

$\mathbb{P}(\omega) = \frac{1}{k^n}\binom{k - \#\text{descents in } \omega^{-1} + n - 1}{n}$

Since repeating a 2-shuffle k times is equivalent to a 2^k-shuffle (in general, an X-shuffle followed by a Y-shuffle is the shuffle with distribution $\mathbb{P}(ij) = x_i y_j$), this can be used to calculate the convergence rate to a random deck. But the analogous problem for non-uniform X (e.g. $x_1 = p$, $x_2 = 1-p$) is more complicated, a sharp upper bound for the convergence rate is not known.

Let k be a field with characteristic $\neq 2$. Let R be the k-vector space with basis indexed by ranked posets (a finite poset with minimal element 0, maximal element 1, and all chains having equal length). The rank of the posets induce a grading on R.

Define a product on R by taking the cartesian product of posets - note that this respects the grading on R.

Define a coproduct on R: $\Delta P = \sum_z [0,z] \otimes [z,1]$ which also respects grading (the rank of the factors sum to the rank of P)

The character on R is $\zeta(P) = 1$ for all posets P

This is Rota's Hopf algebra

The vector space on all posets also form a combinatorial Hopf algebra: the product is disjoint union, the coproduct is $\Delta(P) = \sum_{I \subseteq P} I \otimes P \setminus I$ where I is an order ideal in P (ie if $y \in I$, $x \le y$, then $x \in I$). This is graded by the number of elements in each poset. The character is again the usual zeta function.

There is a combinatorial Hopf algebra morphism from the above to R, by taking the poset of order ideals under inclusion.

Montgomery's "Hopf Algebras and their Actions on Rings" is a good introduction to Hopf algebras.

A graded Hopf algebra is a graded vector space $A = A_0 \oplus A_1 \oplus \cdots$ with an associative, grading-preserving product and a linear coproduct $\Delta : A \to A \otimes A$ which is coassociative: if $\Delta(a) = \sum_i a_{i_1} \otimes a_{i_2}$, then $\sum_i \Delta(a_{i_1}) \otimes a_{i_2} = \sum_i a_{i_1} \otimes \Delta(a_{i_2})$, or, in operator form, $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. This needs to respect grading in the way explained above. The coalgebra structure also involves a counit map $\varepsilon : A \to k$ which satisfies $(\varepsilon \otimes \iota)\Delta = (\iota \otimes \varepsilon)\Delta$. For a connected Hopf algebra, ie $A_0 = k$, $\varepsilon = id$ on A_0 and $\varepsilon = 0$ on all other A_i (Hence $\Delta(a)$ must contain the terms $1 \otimes a$ and $a \otimes 1$). Finally, we need the algebra and coalgebra structure to be consistent: $\Delta(ab) = \Delta(a)\Delta(b)$.

A combinatorial Hopf algebra is a graded Hopf algebra with a character $\xi : A \to k$ which is an algebra homomorphism.

The characters come with a multiplication operation $fg(a) = \sum_i f(a_{i_1}) g(a_{i_2})$ ie fg is the composition (multiplication in k)$(f \otimes g)\Delta$. (This is associative by coassociativity of Δ, and is commutative if A is cocommutative.) Similarly, the linear functionals on an algebra have an induced coalgebra structure (though we have to be careful if the starting algebra is infinite dimensional.)

The quasi-symmetric functions are an algebra under usual multiplication of power series. The coalgebra structure is given by: $\Delta(M_\alpha) = \sum_{\beta\gamma = \alpha} M_\beta \otimes M_\gamma$,
e.g. $\Delta(M_{2,1}) = 1 \otimes M_{2,1} + M_2 \otimes M_1 + M_{2,1} \otimes 1$

The symmetric functions are also a Hopf algebra under the same operations. Since $M_{(n)} = \sum_i x_i^n = p_n$, we see $\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1$. It can be shown that $\Delta(e_n) = \sum_{i=0}^n e_i \otimes e_{n-i}$.

Consider the free associative algebra generated by non-commuting variables H_i, where $\deg(H_i) = i$. So a basis of the degree n subspace is $H_\alpha = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_l}$ across all weak compositions α of n. Define a product by concatenation, and set $\Delta(H_n) = \sum_{i=0}^n H_i \otimes H_{n-i}$. This is the Hopf algebra NCSym, which is dual to Q in that $\langle M_\alpha, H_\beta^* \rangle = \delta_{\alpha\beta}$ (to be precise, we should complete NCSym, ie take power series in H_i's). It can be shown that the abelianization of NCSym (ie the quotient by the commutator subgroup) is Λ, and H_β has image h_λ for any partition λ.

Theorem: Q, with counit character, is the terminal object in the category of combinatorial Hopf algebras. In other words, given any collection of combinatorial objects which can be combined and split up, and any character from such objects to the base field, there is a unique way to assign a generating function to each object (that is compatible with the character). In fact, this map is $h \mapsto \sum_\alpha \zeta^{\otimes k}(\text{deg-}\alpha\text{-projection of } \Delta^{k-1} h) M_\alpha$ ie take only terms in $\Delta^{k-1} h$ whose i^{th} factor has degree α_i.

And Λ is the terminal object for cocommutative combinatorial Hopf algebras.

e.g. Let G be the vector space of graphs, with product given by disjoint union, and grading by the number of vertices. Set $\Delta(G) = \sum_{S \subseteq G} S \otimes G \setminus S$ where S has the induced subgraph structure. Define the character as $\xi(G) = \mathbb{1}_{G \text{ is discrete}}$.
The generating function in this case is Stanley's chromatic polynomial:
$\chi_G(x_1, \ldots)$ = the probability that no adjacent vertices have the same colour when each vertex is assigned colour i with probability x_i. (this is clearly symmetric)
This is the generalised birthday problem, as χ_G computes the probability within a network that two people who know each other share a characteristic.

e.g. given a convex polytope, consider $f = (f_0, f_1, \ldots)$ where f_0 is the number of vertices, f_1 the number of edges. The relations between these numbers are important for linear programming. For three dimensional polytopes, the possible values of f_0, f_1, f_2 are completely known (they are constructable from one starting set of values by repeatedly coning faces or flattening vertices). The other solved case is for simplicial polytopes, where $h(x) = \sum_{i=0}^{d} f_{i-1}(x-1)^{d}$ where d is the dimension of the polytope. The coefficients h_i of this polynomial satisfies the Dehn-Somerville relation $h_i = h_{d-i}$, and together with two inequalities, give a necessary and sufficient condition for a polynomial to be $h(x)$ of some simplicial polytope (see Ziegler's work).
The faces of a convex polytope form a graded poset under inclusion, so we can enumerate flags instead of just faces. The flag enumerators admit a convolution, and convolution "preserves inequalities", so new conditions can be deduced from previous ones. There is also an analogue of $h(x)$ for flag enumerators. These data are the coefficients

of M_α and L_α in the generating function (image in Q) see Billera, Flag enumeration in polytopes, Eulerian posets and Coxeter groups.

The polynomial ring $k[x]$ is a Hopf algebra under the usual product and $\Delta(p(x)) = p(1\otimes x + x\otimes 1)$, so, by binomial expansion, $\Delta(x^k) = \sum_{i=0}^{k} \binom{k}{i} x \otimes x^{k-i}$. This structure is the divided powers algebra, often used in homological algebra. Define a Hopf map $Q \to k[x]$ by evaluation at $(1,1,1,\cdots 1,0,0,\cdots)$ where there are x 1's. For example, if α has k parts, then M_α maps to $\binom{x}{k}$. So postcomposing the map from a combinatorial Hopf algebra to Q with this produces a single-variable generating functions. Starting with the incidence algebra on posets, this gives the zeta function (the generating function on lengths of multichains).

The 2-dimensional analogue of partition is a plane partition, a 2d array of numbers such that each row and column is a partition.

e.g.

7 5 5 3 2 1 1 1 1		is a plane partition of 58, with 18 parts,
6 5 5 2 1 1		of shape (8,6,4), and trace 14.
6 3 2 2		(the trace is the sum along the main diagonal)

Because of the connection to physics, these are often thought of as tilings of a hexagon (which can be viewed as a projection of 3-dimensional towers of boxes), see "The Shape of a typical Boxed Partition" by Cohn, Larson and Propp. Alternatively, plane partitions are equivalent to order ideals in $\mathbb{N}^3$ (that is, $I \subseteq \mathbb{N}^3$ such that, if $(i,j,k) \in I$ and $i' \leq i, j' \leq j, k' \leq k$, then $(i',j',k') \in I$): in box i,j, put the largest k such that $(i,j,k) \in I$. This is a bijection, since the order ideal relation is equivalent to the rows and columns decreasing.

e.g. the plane partitions of 3 are

$$111 \qquad \begin{matrix}11\\1\end{matrix} \qquad \begin{matrix}1\\1\\1\end{matrix} \qquad 21 \qquad \begin{matrix}2\\1\end{matrix} \qquad \text{and } 3$$

and the corresponding order ideals are

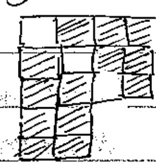
$$\left\{\begin{matrix}(1,1,1),(1,2,1),\\(1,3,1)\end{matrix}\right\} \quad \left\{\begin{matrix}(1,1,1),(1,2,1),\\(2,1,1)\end{matrix}\right\} \quad \left\{\begin{matrix}(1,1,1),(2,1,1)\\(3,1,1)\end{matrix}\right\} \quad \left\{\begin{matrix}(1,1,1),\\(1,1,2),\\(1,2,1)\end{matrix}\right\} \quad \left\{\begin{matrix}(1,1,1),\\(1,1,2),\\(2,1,1)\end{matrix}\right\}, \quad \left\{\begin{matrix}(1,1,1)\\(1,1,2)\\(1,1,3)\end{matrix}\right\}$$

let $P(r,c)$ be the plane partitions with at most r rows and c columns.
So $P(1,c)$ are ordinary partitions with at most c parts.
By truncating the partition generating function, we see

$$\prod_{j=1}^{c}(1-qx^j)^{-1} = \sum q^{\#\text{parts}} x^{\#\text{boxes}} \text{ over all partitions whose parts are } \leq c$$
$$= \sum_{\pi\in P(1,c)} q^{tr(\pi)} x^{\#\text{boxes in } \pi} \text{ by looking at the transpose.}$$

The generalisation to arbitrary r reads $\sum_{\pi\in P(r,c)} q^{tr(\pi)} x^{\#\text{boxes in }\pi} = \prod_{i=1}^{r}\prod_{j=1}^{c}(1-qx^{i+j-1})^{-1}$
In particular, taking $q=1$ and $r,c\to\infty$ shows $\sum \#\text{plane partitions of } n\ x^n = \prod_{i=1}^{\infty}(1-x^i)^{-i}$

The proof goes as follows. First, given λ,μ partitions into distinct parts with the same number of parts, build (the diagram of) a partition by shifting the i^{th} rows of the λ and μ diagram to the right by $i-1$ spaces, then transposing μ and pasting them along the common main diagonal. For example, $(4,3,2)$ and $(5,4,1)$ glue together to



$=(4,4,4,2,2)$

Because λ,μ had distinct parts, the gluing indeed results in a partition, and any partition can be broken along the main diagonal, so this construction is bijective:

$$\{\text{partitions of } n \text{ into } s \text{ distinct parts}\} \times \{\text{partitions of } m \text{ into } s \text{ distinct parts}\} \xrightarrow{1:1} \{\text{partitions of } m+n-s\}$$

Next, take two reverse semi-standard Young tableaux of the same shape (ie fill them so the rows are weakly decreasing and the columns are strictly decreasing), do the above construction with each corresponding pair of columns, and put these as columns of a plane partition. Then view each row of this plane partition as a partition and transpose it.

e.g.	4 4 2 1	5 3 2 2	give	4 4 2 1	which	4 3 2 2
	3 1 1	4 2 1		4 2 2 1	transposes	4 3 3 1
	2	1		4 2		2 2 1 1
				2	to	1 1
				2		1 1

(above we calculated the first column of the intermediate plane partition. The second column is obtained from gluing $(4,1)$ to $(3,2)$, etc. View this as doing the above gluing in 3d, with the content being the height. In the final step, $(4,4,2,1)^T=(4,3,2,2)$, $(4,2,2,1)^T=(4,3,3,1)$ etc.)
The rows in the intermediate plane partition are weakly decreasing since the rows of the two tableaux are. And the final step produces decreasing columns as the intermediate plane partition has decreasing columns, which means the rows (as partitions) are decreasing in $\leq$ ordering, which is preserved under transposition.

These steps are also reversable, so we have a bijection:

pairs of reverse semi-standard Young tableaux $\longrightarrow$ plane partitions of
of shape λ, of total content n and m respectively $\quad n+m-\#$ boxes in λ

The number of rows in the resulting plane partition is the number of parts in the first glued partition, which is the top left entry of the second tableaux.

The number of columns $=$ length of first column of partition in first row of intermediate
$=$ top left entry of the first tableaux.

Now to interpret the trace. i,i^{th} entry $= \#$ entries in i^{th} row of intermediate that are $\geq i$
$= \#$ glued partitions with i^{th} part $\geq i$
$= \#$ columns in starting tableaux of length $\geq i$
$=$ length of row i

$\therefore$ the trace is the number of boxes in the starting tableaux.

Finally, we match the pairs of reverse semi-standard Young tableaux to $\mathbb{N}$-matrices via reverse RSK, where one plays a high card on a lower card; alternatively, put minus signs in front of the tableaux entries, and change the signs back once you obtain the two sequences. Now the bookkeeping sequence is descending, and the main sequence weakly descends over where the bookkeeping sequence keeps the same value. Reading these sequences backwards gives sequences with the usual condition, so these pairs of tableaux correspond bijectively to $\mathbb{N}$-matrices also. Since the i,j entry contributes a_{ij} numbers to the sequences, the number of boxes in the tableaux is $\sum a_{ij}$. The total content of the first tableaux is $\sum i a_{ij}$, and that of the second tableaux is $\sum j a_{ij}$.

So $\mathbb{N}$-matrices A correspond bijectively to plane partitions of $\sum (i+j-1)a_{ij}$ with trace $\sum a_{ij}$. The number of rows of the plane partition is the last entry of the bookkeeping sequence, ie the last nonzero row of A. The number of columns is the last entry of the main sequence, which is the last nonzero row of A.

So $\sum_{\pi \in P(r,c)} q^{tr(\pi)} x^{|\pi|} = \sum_{A \text{ an } r\times c\ \mathbb{N}\text{-matrix}} q^{\sum a_{ij}} x^{(i+j-1)\sum a_{ij}}$
$= \prod_{i=1}^{r} \prod_{j=1}^{c} \sum_{a_{ij}\geq 0} (q x^{i+j-1})^{a_{ij}} = \prod_{i=1}^{r}\prod_{j=1}^{c} (1-qx^{i+j-1})^{-1}$

If we further limit the number in each box to be at most t, then the number of plane partitions remaining in $P(r,c)$ is $\prod_{i=1}^{r}\prod_{j=1}^{c}\prod_{k=1}^{t} \frac{(i+j+k-1)}{(i+j+k-2)}$ (if $r\leq c$)

S_3 acts on the order ideals of $\mathbb{N}^3$ by permuting the factors, so one can enumerate the plane partitions invariant under the induced S_3-action, or A_3-action. In total 10 groups act naturally on the set of plane partitions, and their invariants are counted in "Enumeration of totally symmetric plane partitions" by Stembridge.

Polya theory is the enumeration of orbits under group action. More formally, let G act on D, H act on R. Then $G\times H$ acts on $\mathrm{Maps}(D,R)$ by $[(g,h)f](d) = h(f(gd))$.
eg. let D be the four vertices of a square, R be two colours,
let G be $\mathbb{Z}/4\mathbb{Z}$ acting by rotation on the square, H be $\mathbb{Z}/2\mathbb{Z}$
then the $G\times H$ orbits on $\mathrm{Maps}(D,R)$ have representatives

Assign a weight to each element of R - ie. define a weight function $: R\to A$ where A is some algebra (e.g. a polynomial ring).
Then the weight of a function is given by $w(f) = \prod_{d\in D} w(f(d))$
We shall work with invariant weights: $w(f_1)=w(f_2)$ if f_1, f_2 are in the same $G\times H$ orbit (such orbits are also called patterns). Then, we have

de Bruijn's theorem: $\sum_{P \text{ pattern}} w(P) = \frac{1}{|G||H|}\sum_{g,h}\sum_{f \text{ fixed by } (g,h)} w(f)$, for invariant weight w
Proof: for each $a\in \mathrm{im}(w)\subseteq A$, look at the functions $: D\to R$ with weight a.
this is invariant under $G\times H$-action since the weight is invariant.
by Burnside, # patterns with weight a = # orbits of these functions
$= \frac{1}{|G||H|}\sum_{g,h} |\text{fix set of } (g,h)|$
Now $\sum_{P \text{ pattern}} w(P) = \sum_{a\in A} a$ # patterns with weight a
$= \frac{1}{|G||H|}\sum_{a\in A}\sum_{g,h} a$ # of functions of weight a fixed by (g,h)
$= \frac{1}{|G||H|}\sum_{g,h}\sum_{f \text{ fixed by } (g,h)} w(f)$.

By specialising to $H=\mathrm{id}$, we get Polya's theorem: $\sum_{P \text{ pattern}} w(P) = \frac{1}{|G|}\sum_g \sum_{f:\, g(f)=f} w(f)$
The right hand side is usually expressed in terms of the cycle index polynomial:
$P_G(x_1,\dots,x_d) = \frac{1}{|G|}\sum_{g\in G} x_1^{\#\text{1-cycles in } g}\cdots x_d^{\#d\text{-cycles in } g}$
where $d=|D|$, and the cycle representation of g comes from viewing $G\le S_d$.
Observe that f is fixed by g if and only if f is constant on each orbit of g, and $wf: D\to A$ can have any value in $\mathrm{im}\, w$ on such orbits. So we can rewrite Polya's theorem as

$\sum_{P \text{ pattern}} w(P) = P_G\left(\sum_{r\in R} w(r), \sum_r (w(r)^2), \dots, \sum_r (w(r)^d)\right)$

Setting $w \equiv 1$, we see that # of patterns $= P_G(|R|, |R|, \dots)$

e.g. recall the previous example of $\mathbb{Z}/4\mathbb{Z}$ acting on the square.

$P_G(x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1^4 + x_4 + x_2^2 + x_4)$

so $P_G(|R|,|R|,|R|,|R|) = \frac{1}{4}(2^4 + 2 + 2^2 + 2) = 6$

ie there are 6 patterns, the ones we drew before and :·, ·:

(because $H = id$ means we're now treating the colours as different)

To obtain the number of patterns with 2 points of each colour, we take the $y_1^2 y_2^2$ coefficient when the image of w is y_1 and y_2 ie the $y_1^2 y_2^2$ coefficient in $\frac{1}{4}\left((y_1+y_2)^4 + y_1^4 + y_2^4 + (y_1^2+y_2^2)^2 + y_1^4 + y_2^4\right) = y_1^4 + y_1^3 y_2 + 2y_1^2 y_2^2 + y_1 y_2^3 + y_2^4$

so there are two such patterns, as we saw previously.

When applying this theory to the classification of chemical molecules up to rotation, the groups involved are typically C_n and D_n, which have cycle index:

$P_{C_n} = \frac{1}{n}\sum_{d|n} \phi(d)\, x_d^{n/d}$ where $\phi(d) =$ # of integers between 0 and n coprime to d.

$D_{2n} = \frac{1}{2n}\left(n x_1 x_2^{\frac{n-1}{2}} + \sum_{d|n}\phi(d)\, x_d^{n/d}\right)$ if n odd

$\frac{1}{2n}\left(\frac{n}{2} x_2^{n/2} + \frac{n}{2} x_1^2 x_2^{\frac{n-2}{2}} + \sum_{d|n}\phi(d)\, x_d^{n/d}\right)$ if n even

If G_1 acts on D_1 and G_2 acts on D_2, then the cycle type of (g_1, g_2) on $D_1 \amalg D_2$ is the cycle type of g_1 concatenated with the cycle type of g_2

So $P_{G_1 \times G_2}(x, y) = P_{G_1}(x) P_{G_2}(y)$

Sometimes we wish to classify multiple objects up to rotation of each object and some symmetry amongst the objects: then we need the wreath product $G_1 \wr G_2$. Let I be the index set of the domains, each of which is D, and let G_1 act on I, G_2 act on D. Then $G_2^{|I|}$ acts on $D^{|I|}$ componentwise, and we can follow this with G_1-action permuting the components. So we have $G_1 \wr G_2 := G_1 \ltimes G_2^{|I|}$ acting on $D^{|I|}$

Then $P_{G_1 \wr G_2}(x_1, x_2, \dots) = P_{G_1}\left(P_{G_2}(x_1, x_2, \dots), P_{G_2}(x_2, x_4, \dots), P_{G_2}(x_3, x_6, \dots), \dots\right)$:

see "Representation theory of symmetric groups" by James and Kerber, "Groups acting on sets" by Kerber and Read's "Polya's theory of enumeration".

Unfortunately, cycle index polynomials are in general hard to compute, even for $(\mathbb{Z}/2\mathbb{Z})^n$ — this is the concern of computational Polya theory.

Now we return to the case of general H. The case where H is cyclic with generator h is particularly well understood:
$\sum_{P: \text{pattern}} \omega(P) = P_G(P_1(h), \cdots P_d(h))$ where $P_s(h) = \sum_{r: h^s(r)=r} \omega(r)\,\omega(h(r)) \cdots \omega(h^{s-1}(r))$
As before, if we are simply interested in the number of patterns with d_i elements of colour i, then we give colour i a weight of y_i, and then take the coefficient of $\prod y_i^{d_i}$ of the right hand side above.

This has been very successful in obtaining asymptotics for graph enumeration: fix a vertex set, let D be the pairs of vertices, $R = \{0,1\}$ (ie whether these vertices are adjacent or not) and take $G = S_n$, $H = id$ (or $\mathbb{Z}/2\mathbb{Z}$, if we want a graph to be equivalent to its complement) - see the work of Harary and Robinson.

Recall that $G \subseteq S_d$, so we can consider the permutation representation of S_d on the cosets of G. The character $\pi_{\text{cosets of } G}(g) = \mathrm{Ind}_G^{S_d} \mathbb{1}(g) = \frac{1}{|G|} \#\{x \in S_d : xgx^{-1} \in G\}$

$$= \frac{1}{|G|} \#\{\lambda \text{ cycles in } G\} \frac{1}{\text{stab of } g \text{ in } S_d}$$
$$= \frac{d!}{|G|} \frac{\#\{\lambda \text{ cycles in } G\}}{\#\{\lambda \text{ cycles in } S_d\}}$$

Then $ch(\pi_{\text{cosets of } G}) = \frac{1}{d!} \sum_{\omega \in S_d} \pi_{\text{cosets of } G}(\omega)\, p_{\text{cycle type of } \omega}$

$$= \frac{1}{d!} \sum_{\lambda \vdash d} \frac{d!}{|G|} \frac{\#\{\lambda \text{ cycles in } G\}}{\#\{\lambda \text{ cycles in } S_d\}} \#\{\lambda \text{ cycles in } S_d\}\, p_\lambda$$
$$= \frac{1}{|G|} \sum_{\lambda \vdash d} \#\{\lambda \text{ cycles in } G\}\, p_\lambda$$
$$= Z_G(p_1, \cdots p_d)$$

So, if $\{\omega(r) : r \in R\}$ is our variable set, $ch(\pi_{\text{cosets of } G}) = \sum_{P \text{ pattern}} \omega(P)$, hence the pattern enumerator is a GL_n-character.

Let $G = S_d$, and view $f: D \to R$ as an assignment of d balls to r boxes. Then the patterns are exactly the configurations of d unlabelled balls in r boxes. We know that the number of these is $\binom{d+r-1}{d}$, and Polya theory gives the same answer:
$\#\text{ patterns} = P_{S_d}(r, r, \cdots r) = \frac{1}{d!} \sum_{\omega \in S_d} r^{\#\text{cycles in } \omega} = \frac{1}{d!} r(r+1) \cdots (r+d-1) = \binom{d+r-1}{d}$

Suppose each such configuration is equally likely; then the moment generating function
$\mathbb{E}(\prod y_i^{\#\text{ balls in the } i^{th} \text{ box}}) = \frac{1}{\binom{d+r-1}{d}} \sum \omega(P) = \frac{1}{\binom{d+r-1}{d}} P_{S_d}(p_1, p_2, \cdots p_d)$.
It is much easier to work with $P_{S_d}(p_1, \cdots p_d)$ across all d's at once, using the cycle index theorem: $\sum_{d=0}^{\infty} t^d \binom{d+r-1}{d} \mathbb{E}_d(\prod y_i^{\#\text{ balls in } i^{th} \text{ box}}) = \sum_{d=0}^{\infty} P_{S_d}(p_1, \cdots p_d) t^d = \prod_{i=1}^{r} \frac{1}{1-y_i t}$
One application of this: $\mathbb{E}(\#\text{ balls in first box}) = \frac{d}{dy_1} \mathbb{E}(\prod y_i^{\#\text{ balls in } i^{th} \text{ box}})\big|_{y_i=1}$

$= \text{coefficient of } \binom{d+r-1}{d} t^d \text{ in } \frac{d}{dy_1} \prod_{i=1}^{r} \frac{1}{1-y_i t} \Big|_{y_i=1}$
$= \text{coefficient of } \binom{d+r-1}{d} t^d \text{ in } \left(\frac{-t}{1-y_1 t} \prod_{i=1}^{r} \frac{1}{1-y_i t}\right)\Big|_{y_i=1}$
$= \text{coefficient of } \binom{d+r-1}{d} t^d \text{ in } \frac{t}{(1-t)^{r+1}}$

To analyse the Bose-Einstein distribution (where all configurations of d unlabelled balls in r boxes are equally likely) we can work instead with independent geometric variables: if $\mathbb{P}(X_i = j) = (1-t)t^j$, then
$\mathbb{E}(\prod y_i^{X_i}) = \prod \mathbb{E}(y_i^{X_i}) = \prod_{i=1}^{r} \sum_j y_i^j \,\mathbb{P}(X_i=j) = \prod_{i=1}^{r} \frac{1-t}{1-y_i t}$
$= \sum_d (1-t)^r t^d \binom{d+r-1}{d} \,\mathbb{E}(\prod_i y_i^{\#\text{balls in box } i}$ under Bose-Einstein$)$
$= \sum_d \mathbb{P}(\sum X_i = d)\,\mathbb{E}(\prod_i y_i^{\#\text{balls in box } i}$ under Bose-Einstein$)$

We can carry out similar analysis with subgroups of S_d, to express the distributions where those patterns are equally likely as the result of filling the boxes independently according to some other distribution. Very few results have been obtained in this area.
e.g. $G = id$ describes the dropping of labelled balls into labelled boxes.
If X_i have Poisson distribution with parameter t, then
$\mathbb{E}(\prod y_i^{X_i}) = \prod \mathbb{E}(y_i^{X_i}) = \prod_{i=1}^{r} \sum_j y_i^j \,\mathbb{P}(X_i=j) = \prod_{i=1}^{r} \sum_j y_i^j e^{-t} \frac{t^j}{j!} = e^{-rt} \prod_{i=1}^{r} e^{y_i t} = e^{-rt} e^{\sum y_i t}$
$= e^{-rt} \sum_d \frac{t^d}{d!} p_d(y) = e^{-rt} \sum_d \frac{t^d}{d!} \,\mathbb{E}(\prod_i y_i^{\#\text{balls in box } i}$ under multinomial distribution$)$

Now we return to tableaux combinatorics.
Given a permutation (ie a word in distinct letters), we can perform a Knuth move as follows: find $a<b<c$ occurring in the string in order acb, cab, bac, and bca (ie a, c are adjacent), and switch a and c.
Two permutations are Knuth equivalent if one can be transformed into the other via Knuth moves.
Observe that Knuth moves "commute" with reversing the string, so if u, v are Knuth equivalent, so are u, v both read backwards.

let I_k denote the maximal number of elements in the union of k increasing subsequences
let D_k denote the maximal number of elements in the union of k decreasing subsequences
for example, for $w = 236145$, $I_1 = 4$ (attained by 2345), $I_2 = 6$ (attained by $236, 145$).

observe that none of the subsequences attaining I_2 have the maximal length.
$D_1=2$ (attained by e.g. 21), $D_2=4$ (attained by e.g. 64, 31)

Lemma 1: I_k and D_k are unchanged under Knuth moves

Proof: Since D_k of a string is I_k of its reverse, and Knuth moves "commute" with reversing, it suffices to prove this for I_k.

Suppose this is false - so, $\exists w \in S_n$ with $I_k(w)=m$, $I_k \neq m$ after a Knuth move.

Suppose this is the move $acb \to cab$. Then ac must be part of an increasing sequence attaining I_k (or their relative order would not affect I_k). b must be in another sequence attaining I_k, as otherwise we can replace $ac\cdots$ with $ab\cdots$ as $b<c$, and get a sequence of the same length that isn't affected by the Knuth move.

So we have $\cdots ac \cdots$ and $\cdots b \cdots$ in the sequences attaining I_k. When we switch a and c, switch the tails of the sequences from b or c onwards to obtain increasing sequences whose union have the same number of elements, a contradiction (because I_k cannot increase if we put a larger element in front of a smaller).

The case $bac \to bca$ is completely analogous, and the remaining two cases are inverses of the above.

Next, define the reading word of a tableaux to be the last row (read left to right) followed by the penultimate row, then the row above... e.g. if $T=$ 1356,
24
78

then read(T) = 7824 1356.

If T is standard or semi-standard, we can reconstruct T from its reading word by noting the positions of descents.

Lemma 2: any $w \in S_n$ is Knuth equivalent to read($P(w)$), where $P(w)$ is the first tableaux obtained from w in RSK.

Proof: it suffices to prove that read(P) with k appended is Knuth equivalent to read(P with k inserted the RSK way) - ie we consider one insertion at a time.

we also look at each row at a time, to show that read(P)$\cdot k \equiv$ read($P \setminus$ first row).

read(first row of P)$\cdot k \equiv$ read($P\setminus$ first row)$\cdot$ what k displaced from first row $\cdot$ read(new first row) $\equiv$ read($P\setminus$ first 2 rows) $\cdot$ what k displaced from second row $\cdot$ read(new top two rows) $\equiv \cdots$ ie. we show read(a row)$\cdot k \equiv$ what k displaced $\cdot$ read(new row)

this reduces to $a_1 a_2 \cdots a_n k \equiv a_i a_1 a_2 \cdots a_{i-1} k a_{i+1} \cdots a_n$ where $a_1 < a_2 \cdots < a_{i-1} < k < a_i \cdots$

and the required Knuth moves are: $a_{n-1} a_n k \to a_{n-1} k a_n$; $a_{n-2} a_{n-1} k \to a_{n-2} k a_{n-1} \cdots$

$a_i a_{i+1} k \to a_i k a_{i+1}$, $a_{i-1} a_i k \mapsto a_i a_{i-1} k$, $a_{i-2} a_i a_{i-1} \mapsto a_i a_{i-2} a_{i-1}$, ...
$a_1 a_i a_2 \to a_i a_1 a_2$. (ie move the k to behind a_i, then move a_i to the front)

lemma 3 : $P(\text{read}(T)) = T$
Proof: carrying out RSK to a reading word essentially fills T from the bottom row upwards. Each card is inserted at the top of the column it came from, and the column slides down one spot (the insertion path cannot go to the left, if you look at the tableaux two moves ago)

Combining these give Greene's theorem: I_k = sum of the lengths of the first k rows
D_k = sum of the lengths of the first k columns
of the tableaux obtained under RSK.
Proof: By lemmas 1 and 2, $I_k(\omega) = I_k\, \text{read}(P(\omega))$.
By lemma 3, $P(\text{read}\, P(\omega)) = P(\omega)$
so it suffices to consider reading words — ie to show that $I_k(\text{read}\, T)$ = sum of the lengths of the first k rows of T, and similarly for D_k.
Each row of T is an increasing sequence in read(T), and each column of T is a decreasing sequence in read(T)
$\therefore I_k \geq$ lengths of first k rows, $D_k \geq$ lengths of first k columns
Consider position k,l on the rim of T. (ie there is nothing diagonally to the right and below k,l). So every box is either in the first k rows, or in the first l columns, and this overcounts precisely kl boxes. So $I_k + D_l \geq n + kl$.
On the other hand, each increasing subsequence can intersect a decreasing subsequence at most once, so there are at most kl intersections between the subsequences attaining I_k and D_k.
Hence $I_k + D_l \leq n + kl$.
So we must have I_k = length of first k rows, D_l = length of first l columns for each pair k,l such that box k,l is on the rim.
We can find such rim boxes for every k (end of a row) and every l (end of a column), so these hold $\forall k,l$

So the numbers I_k, D_k completely describe the shape of the tableaux obtained under RSK. Using lemma 1, we can deduce that the shape of the RSK tableaux depends only on its Knuth class. In fact, more is true:

Theorem: The first RSK tableaux of v and w are equal if and only if they are in the same Knuth equivalence class.

Proof: let w_k be w with the numbers larger than k removed (so $w_k \in S_k$)
then $P(w_k)$ is the subtableaux of $P(w)$ involving $1, 2, \cdots k$, since playing higher cards does not affect the position of lower ones.
suppose we do a Knuth move on w. w_k is unchanged if $c > k$ (as then c doesn't appear in w_k); otherwise, a, b, c all appear in w_k, so we do a Knuth move on w_k.
Hence w_k is Knuth equivalent to v_k for each k. So $P(w_k)$ and $P(v_k)$ have the same shape, for each k. But the difference between $P(w_k)$ and $P(w_{k-1})$ is the box containing k, hence k is in the same spot in $P(w)$ as in $P(v)$, and this holds for all k.

Here's how to play Jeu de Taquin on a semi-standard Young tableaux with distinct entries (ie one that can be relabeled to give a standard tableaux):
- remove the lowest entry
- slide the smallest adjacent entry into the hole
- continue until what remains is the Young diagram of a partition

For example: $\begin{matrix} 1 & 2 & 5 \\ 3 & 4 \end{matrix} \mapsto \begin{matrix} & 2 & 5 \\ 3 & 4 \end{matrix} \mapsto \begin{matrix} 2 & & 5 \\ 3 & 4 \end{matrix} \mapsto \begin{matrix} 2 & 4 & 5 \\ 3 \end{matrix}$

We construct the evacuation $\mathrm{evac}(Q)$ of a standard Young tableaux Q as follows: play Jeu de Tacquin successively, starting with Q, until nothing remains. This gives a decreasing (under $\subseteq$) sequence of shapes (ignore the filling now) – $\mathrm{evac}(Q)$ is the bookkeeping tableaux of this sequence read backwards

e.g. successive games of Jeu de Tacquin on $\begin{matrix} 1 & 2 & 5 \\ 3 & 4 \end{matrix}$ give the sequence of tableaux

$\begin{matrix} 1 & 2 & 5 \\ 3 & 4 \end{matrix}$, $\begin{matrix} 2 & 4 & 5 \\ 3 \end{matrix}$, 345, 45, 5 so $\mathrm{evac}\begin{pmatrix} 125 \\ 34 \end{pmatrix} = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 \end{matrix}$.

It turns out that applying RSK to w reversed (ie $w_n w_{n-1} \cdots w_1$) gives (tableaux of w)^T (as mentioned before) and evac(bookkeeping tableaux of w)^T. Applying RSK to $n+1-w_n, n+1-w_{n-1}, \cdots n+1-w_1$ (ie subtract the reverse of w termwise from $n+1$) gives the evacuation of both tableaux of w.

The subject of Hall polynomials began with the enumeration of abelian groups. Given G an abelian p-group, by the structure theorem, $G = \oplus \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$, and λ is then called the type of G.

We are interested in $g^{\lambda}_{\mu\nu}(p)$, the number of subgroups of $\oplus \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$ that are isomorphic to $\oplus \mathbb{Z}/p^{\mu_i}\mathbb{Z}$ whose quotient is $\oplus \mathbb{Z}/p^{\nu_i}\mathbb{Z}$. Hall proved that $g^{\lambda}_{\mu\nu}$ are polynomial in p — these are the Hall polynomials.

(We can ask the same question over any discrete valuation ring, where we replace $p\mathbb{Z}$ with a prime ideal).

If ν has a single part, then $g^{\lambda}_{\mu\nu}(t) = (1-\frac{1}{t})^{-1} t^{\sum(\lambda_i - \mu_i)(i-1)} \prod_j (1 - t^{-\#\text{parts of size } j \text{ in } \lambda})$ if $\lambda \setminus \mu$ has at most one box in each column, and we product over j such that the j^{th} column of $\lambda\setminus\mu$ has one box, and the $j+1^{th}$ column of $\lambda\setminus\mu$ has no boxes. (This counts cyclic quotients)

e.g. if $\lambda = (2,0,\cdots)$, $\mu = (1,0,\cdots)$, then $g^{\lambda}_{\mu\nu} = 0$ unless $\nu = (1,0,\cdots)$ (as the number of boxes in μ and in ν must total the number of boxes in λ)

Then, $\lambda\setminus\mu$ has shape , so $g^{\lambda}_{\mu\nu}(t) = (1-\frac{1}{t})^{-1} t^{1\cdot 0} (1 - t^{-\#\text{parts of size 2 in } \lambda})$

$$= (1-\tfrac{1}{t})^{-1}(1-t^{-1}) = 1$$

(and indeed, there is a unique copy of $\mathbb{Z}/p\mathbb{Z}$ in $\mathbb{Z}/p^2\mathbb{Z}$)

e.g. if $\lambda = (1,1,0,\cdots)$, $\mu = (1,0,\cdots)$, then $g^{\lambda}_{\mu\nu} = 0$ unless $\nu = (1,0,\dots)$

Then, $\lambda\setminus\mu$ has shape so $g^{\lambda}_{\mu\nu}(t) = (1-\frac{1}{t})^{-1} t^{0\cdot 0 + 1\cdot 1}(1 - t^{\#\text{parts of size 1 in } \lambda})$

$$= (1-\tfrac{1}{t})^{-1} t (1-t^{2}) = t+1$$

this counts the number of lines in a plane over $\mathbb{F}_p$.

Fix a p, and let G_λ denote the p-group of type λ. Then $\sum_{\lambda \vdash n} \frac{1}{|\mathrm{Aut}\, G_\lambda|} = \sum_{\lambda\vdash n} \frac{1}{|G_\lambda|}$

— see Lynn Butler, Subgroup lattices and symmetric functions.

Hall's great idea is to use the $g^{\lambda}_{\mu\nu}$ as connection coefficients in what became known as the Hall algebra $H(t)$: it is a $\mathbb{Q}(t)$-vector space with basis u_λ indexed by partitions, under the product $u_\mu u_\nu = \sum_\lambda g^{\lambda}_{\mu\nu}(t) u_\lambda$.

$H(t)$ turns out to be freely generated by $u_{(1,1,\dots,0,\dots)}$, so we can define a map $u_{(1,1,\dots,1,0,\dots)} \to t^{-r(r-1)} e_r$.
The images of u_λ are then the Hall-Littlewood polynomials $P_\lambda(x,t)$ (in $\mathbb{Q}(t)\Lambda(x)$).
They also have an explicit formula $P_\lambda(x,t) = \sum_w w\left(x^\lambda \prod_{\lambda_i > \lambda_j \geq 0} \frac{x_i - t x_j}{x_i - x_j}\right)$, where λ is a partition of n, and we sum over w coset representatives of $S_{\lambda_1} \times S_{\lambda_2} \times \cdots$ in S_n.
In particular, $P_\lambda(x,0) = s_\lambda$, $P_\lambda(x,1) = m_\lambda$, $P_{(1,1,\dots,1,0,\dots)}(x,t) = e_r$ independent of t.

The "correct" inner product on $\mathbb{Q}(t)\Lambda$ is $\langle p_\lambda, p_\mu \rangle = z_\lambda \prod (1-t^{\lambda_i})^{-1} \delta_{\lambda\mu}$.
Then, P_λ can be characterised as the orthogonal basis of $\mathbb{Q}(t)\Lambda$ such that the change of basis matrix from m_λ to P_λ is upper triangular with diagonal entries 1. Equivalently, P_λ is what we obtain from applying Gram-Schmidt without scaling to m_λ, so each P_λ is m_λ + lower order terms.

$P_\lambda(x,-1)$ are Schur's Q-functions, which are related to the projective representations of S_n, the representation theory of Lie superalgebras, the cohomology of Grassmanians, etc. They also help us evaluate the characters on $GL_n(\mathbb{F}_q)$, for analysis of the transvections random walk on GL_n, or to compute the probability of a random element of GL_n being semisimple.

A further generalisation are the MacDonald polynomials, the unique orthogonal basis $P_\lambda(x,q,t)$ of $\mathbb{Q}(q,t)\Lambda$ which has an upper triangular, diagonal entries 1 change of basis matrix from the m_λ, under the inner product $\langle p_\lambda, p_\mu \rangle = z_\lambda \prod \frac{(q^{\lambda_i}-1)}{(t^{\lambda_i}-1)}$.
If $q=t$, then the inner product loses the extra scaling factor, so $P_\lambda(x,q,q) = s_\lambda$.
If $q=0$, then the inner product reduces to the one in $\mathbb{Q}(t)\Lambda$ $\therefore$ $P_\lambda(x,0,t) = P_\lambda(x,t)$
If we set $q=t^\alpha$, and take the limit $q \to 1$, we get the Jack polynomials $J_\lambda^\alpha(x)$. In particular, $J_\lambda^2(x)$ are the Zonal polynomials

It is known that $P_2(x,q,t) = \frac{1}{2(1-qt)}\{-(1+t)p_2 + (1-t)(1+q)p_{1,1}\}$
and $P_{1,1}(x,q,t) = \frac{1}{2(1-qt)}\{-(1-t^2)p_2 + (1+t^2)p_{1,1}\}$.

The Schur functions and Jack polynomials can be viewed as eigenfunctions of certain difference operators, and MacDonald's original definition of his polynomials was as the eigenfunction of an operator that generalised these: $f \to \sum_i t^{\binom{n}{2}} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} f(x_1, x_2, \dots, x_{i-1}, q x_i, x_{i+1}, \dots)$ with eigenvalue $\sum q^{\lambda_i} t^{\#\text{parts in }\lambda - i}$.

Expand $P_\lambda(x,q,t)$ as $\sum_\mu K_{\mu\lambda}(q,t)\, s_\mu(x)$, and set $\chi^\lambda_\rho(q,t) = \sum_\mu \chi_{s^\lambda}(\rho) K_{\mu\lambda}(q,t)$.
Then $P_\lambda(x,q,t) = \sum_\rho [z_\rho^{-1} \prod_i (1-t^{\rho_i}) \chi^\lambda_\rho(q,t)]\, p_\rho(x)$.
So there is a lot of interest in $K_{\mu\lambda}(q,t)$, especially as Macdonald showed that it is polynomial in q,t, so it is likely to enumerate something.
The work of Garsia and Haiman has shown the following: fix μ a partition of n.
Define $\Delta_\mu(x,y) = \det(x_i^{a_j} y_i^{b_j})$ where (a_j, b_j) are the coordinates of the squares of μ. For example, $\mu = (2,2,1)$ has coordinates $(0,0), (0,1), (1,0)$, $(1,1)$ and $(2,0)$, so

$$\Delta_{(2,2,1)}(x,y) = \det \begin{pmatrix} 1 & y_1 & x_1 & x_1y_1 & x_1^2 \\ 1 & y_2 & x_2 & x_2y_2 & x_2^2 \\ 1 & y_3 & x_3 & x_3y_3 & x_3^2 \\ 1 & y_4 & x_4 & x_4y_4 & x_4^2 \\ 1 & y_5 & x_5 & x_5y_5 & x_5^2 \end{pmatrix}$$

($x_i^{a_j} y_i^{b_j}$ is an $n \times n$ matrix)
There are μ_i columns with entries that have degree $i-1$ in x, so $\Delta_\mu(x,y)$ has degree $\sum \mu_i (i-1)$ in x. Similarly the degree of $\Delta_\mu(x,y)$ in y is $\sum \mu_i^T (i-1)$.
Let D_μ be the space of derivatives of Δ_μ, and $(D_\mu)_{r,s}$ be those elements of D_μ that have degree r in x and degree s in y.
Then, using Hilbert schemes (see J. Hoglund's monograph), we have the $n!$ conjecture (now a theorem): $\dim D_\mu = n!$ for all μ a partition of n. Furthermore S_n-action on D_μ (by permuting x's and y's identically) is the regular representation. Since S_n-action preserves the double-grading on D_μ, we can ask for the multiplicities of each irreducible in $(D_\mu)_{r,s}$. The generating function for these turn out to be $\sum t^r q^s \langle \chi_{s^\lambda}, \chi_{(D_\mu)_{r,s}} \rangle = K_{\lambda\mu}(q,t)$

The theory of Macdonald polynomials allows us to analyse the following Markov chain on the partitions of k: if I'm currently at λ, flip a $\frac{1}{q}$-coin for each square of λ, and throw out a part of λ if I don't get heads for all squares of that part. There is an analogous process to generate new parts out of the removed squares, using t instead of q. This is one example of an auxiliary variables chain, and another is block spin dynamics from physics, regarding random sampling from Ising models.
For the partition walk, formulae of the eigenvalues are known, and the eigenfunctions are $f_\lambda(\rho) = \chi^\lambda_\rho(q,t) \prod (1-q^{\rho_i})$, which is almost the coefficient in the expansion of P_λ in terms of p_λ.