Hopf-power Markov chains


Why: Can use Hopf-algebra structure theory (Eulerian idempotent, Poincare-Birkhoff-Witt) to diagonalise matrix of transition probabilities and get convergence rate.

How: A combinatorial Hopf algebra has basis \( \Pi B_n \) indexed by combinatorial objects, graded by “size”.

The \( n \)th Hopf-power map \( \Psi_n : m^n \Delta^n \) represents breaking into \( n \) parts and recombining.

For \( x, y \in B_n \), set

\[
\text{Prob}(x \rightarrow y) = \text{coefficient of } y \text{ in } a^{-m} \Psi_m(x).
\]

(In most cases, \( B \) can be reweighted so that these coefficients sum to 1.)

The descent set process of a deck of cards.

Theorem: The descent composition \( DC(w) \) is the lengths of the rising sequences in the word \( w \): \( DC(4261) = (1, 2, 1) \).

Proof: Apply the universal construction of Aguiar-Bergeron-Sottile to the character \( \zeta : S \rightarrow \mathbb{R} \).

The ζ(\( w \)) = 1 if \( w_1 < w_2 < \cdots < w_n \),

0 otherwise.

Theorem: The descent set process of a deck of \( n \) distinct cards under \( a \)-shuffle is the Hopf-power Markov chain on \( QSym \) with respect to \( \{ F_I \} \).

So we can study the descent set process using Hopf-algebraic techniques.

Theorem: Eigenvalues are: \( 1, a^{-1}, a^{-2}, \ldots, a^{-n+1} \); multiplicity of \( a^{-n+k} \) is coefficient of \( x^k y^n \) in \( \Pi_1 (1 - xy)^{-n} \), where \( d_n \) is number of Lyndon compositions \( I \) with \( |I| = n \).

Using the eigenfunction formulae below:

Corollary: \( \text{Prob}(I \rightarrow J \text{ in } m \text{ steps}) = \frac{1}{n!} \sum_{\sigma \in S_n} a^{a_{\sigma(I)}} \gamma(\sigma) \).

Left eigenfunctions \( g_\lambda \).

Theorem: \( g_\lambda(J) = \chi^J(\lambda) = \text{ribbon character with skew-shape } J \text{ evaluated at cycle type } \lambda \). Eigenvector \( a^{-\mu(\lambda)} \).

Example: Fillings of skew-shape of \( J = (3, 5, 2, 1) \) with \( \lambda = (4, 4, 3) \) is

\[
\begin{array}{cccc}
2 & 2 & 2 & 2 \\
3 & 3 & 1 & 1 \\
1 & 1 & 1 & 2 \\
\end{array}
\]

\( g_{(4,4,3)}((3, 5, 2, 1)) = (-1)^{0+2+0} = 1. \)

Corollary: Stationary distribution = \( g_{(1,1,\ldots,1)}(J) = \text{proportion of permutations with descent composition } J \).

Right eigenfunctions \( f_J \).

\[
f(J) := \frac{1}{|J|} \left( \frac{1}{|\lambda_\lambda|} \right)^{\mu(\lambda)} \sum_{\sigma \in S_n} \mu(J) \chi(\sigma),
\]

Theorem: \( f_J(J) = \text{coefficient of any permutation with descent composition } J \) in Garsia-Reutenauer idempotent \( E_J \). Eigenvector \( a^{-\mu(\lambda)} \).

Example: Compositions \( J \) with underlying set partition \( \lambda = (4, 4, 3) \) are \( (4, 4, 3), (4, 3, 4), (3, 4, 4) \).

Decompositions \( J' \) of \( J = (3, 5, 2, 1) \) with respect to \( J' \) are

\[
(\cdots, 5, \cdot, 2, 1) \quad (\cdots, 3, \cdot, 2, 1) \quad (\cdots, 3, 2, 1) \quad (\cdots, 3, 2, 1)
\]

\[
f_{(4,4,3)}((3, 5, 2, 1)) = \frac{1}{9} \left( \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \right) \left( \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \right) = \frac{7}{5184}.
\]

Corollary: Normalised number of descents = \( f_{(1,1,\ldots,1)}(J) \). So expected number of descents after shuffling \( t \) times

\[
= (1 - a^{-1}) \frac{t}{2} + a^{-1}(\# \text{ descents at start}).
\]