Representations of Quantum Groups

Introduction

The term *quantum group* usually refers to two types of objects, both associated with a Lie group G and dependent on some parameter q in the underlying field. The *quantised co-ordinate ring* $O_q(G)$ is a deformation of $O(G)$, the ring of polynomial functions on G, and the *quantised enveloping algebra* $U_q(\mathbf{g})$ is a deformation of $U(\mathbf{g})$, the universal enveloping algebra of the Lie algebra corresponding to $\mathbf{g}$. These are dual as Hopf algebras, and we recover the classical objects $O(G)$ and $U(\mathbf{g})$ when we take the limit $q \longrightarrow 1$.

This essay will almost entirely deal with the simplest case of $U_q(\mathbf{sl_2})$, our main result being a description of all its irreducible representations over $\mathbb{C}$. $U_q(\mathbf{sl_2})$ behaves very differently depending on whether or not q is a root of unity, so we need to treat the two cases separately. The intuitive moral is that, when q is not a root of unity, complex representations of $U_q(\mathbf{sl_2})$ exhibit behaviour strikingly similar to complex representations of $\mathbf{sl_2}$, so many techniques for analysing $\mathbf{sl_2}$-representations carry over almost verbatim. Everything works "as well as possible" in this non-root-of-unity setting - all finite-dimensional representations are completely reducible, and a neat combinatorial algorithm using very little data allows us to decompose any finite-dimensional representation into irreducibles. In contrast, when q is a root of unity, the complex representations of $U_q(\mathbf{sl_2})$ resemble modular representations of $\mathbf{sl_2}$ - for example, the dimension of irreducible representations is bounded - and our results are far less pleasant. We will encounter representations which are not completely reducible, and finding the irreducible composition factors becomes much more tricky. I will close by outlining how this viewpoint extends to $U(\mathbf{g})$ for arbitrary Lie algebras $\mathbf{g}$.

Contents

1 The definition of $U_q(\mathbf{sl_2})$

Recall that is the Lie algebra with basis elements where ,

$$[H,E] = 2E,\ [H,F] = -2F,\ [E,F] = H \tag{1.1}$$

Its *universal enveloping algebra* $U(\mathbf{sl_2})$ is an associative algebra containing $\mathbf{sl_2}$, with the Lie bracket given by the commutator operator. An explicit construction involves taking the tensor algebra of $\mathbf{sl_2}$ and quotienting out by the two-sided ideal generated by the following *Serre relations*,

$$HE - EH = 2E,\ HF - FH = -2F,\ EF - FE = H \tag{1.2}$$

By the Poincare-Birkoff-Witt theorem, the set of monomials $\{E^r H^s F^t : r, s, t \in \mathbb{N}\}$ form a basis for $U(\mathbf{sl_2})$. Another important corollary of this theorem is that the representations of $U(\mathbf{sl_2})$ are in bijection with those of $\mathbf{sl_2}$ - more precisely, any representation of $U(\mathbf{sl_2})$ is completely determined by the image on $\mathbf{sl_2}$, and any $\mathbf{sl_2}$-action on a vector space can be extended to a valid $U(\mathbf{sl_2})$-action.

How might one derive some conditions for a quantised version of by "deforming" the usual Serre relations (1.2)? In the spirit of quantum mechanics, set $q = e^{\hbar/2}$, and consider:

$$K = e^{\hbar H/2} = \sum_{i=0}^{\infty} \frac{1}{i!}\left(-\frac{\hbar H}{2}\right)^i \tag{1.3}$$

(Since we aim only for an intuitive argument, assume without proof that the usual exponential properties hold with this definition.) Differentiating expressions in K with respect to $\hbar$ and setting $\hbar = 0$ would give expressions in H; we will do this by naively taking first-order approximations in $\hbar$. For example, K will be approximated by $1 - \frac{\hbar}{2}H$.

First observe that as

$$\frac{K - K^{-1}}{q - q^{-1}} = \frac{\sinh(-\hbar H/2)}{\sinh(-\hbar/2)} \simeq \frac{(-\hbar H/2)}{(-\hbar/2)} = H \tag{1.4}$$

as $\hbar \to 0$ so a plausible change to the third Serre relation reads $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$.

We obtain a commutator when we differentiate a conjugate. Hence, using the first Serre relation, we see

$$KEK^{-1} \simeq \left(1 - \frac{\hbar}{2}H\right) E \left(1 + \frac{\hbar}{2}H\right) \simeq 1 - \frac{\hbar}{2}(HE - EH) = 1 - \frac{\hbar}{2}(2E) \simeq q^2 E \tag{1.5}$$

and we clearly can do the same with the second relation. So make the following

Definition. $U_q(\mathbf{sl_2})$ is the free algebra generated by the symbols E, F, K, K^{-1}, quotiented out by the two-sided ideal generated by $KK^{-1} = 1, K^{-1}K = 1$, together with the following *quantised Serre relations*:

$$KEK^{-1} = q^2E \tag{QSR1}$$

$$KFK^{-1} = q^{-2}F \tag{QSR2}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \tag{QSR3}$$

Without introducing Yang-Baxter equations, whose solution prompted the invention of quantum groups, it is difficult to justify setting $K = q^H$ rather than some other function of q and H. One possible answer is that as defined above retains two desirable properties of $U(\mathbf{sl_2})$: the monomials $\{E^rH^sF^t : r, s, t \in \mathbb{N}\}$ provide a PBW-type basis; and it admits the following Hopf algebra structure:

$$\Delta(E) = K \otimes E + E \otimes 1; \Delta(F) = 1 \otimes F + F \otimes 1K^{-1}; \Delta(K) = K \otimes K \tag{1.6}$$

$$\varepsilon(E) = 0; \varepsilon(F) = 0; \varepsilon(K) = 1$$
$$S(E) = -K^{-1}E; S(F) = -FK; S(K) = K^{-1}$$

(Other Hopf algebra structures exist on $U_q(\mathbf{sl_2})$, but the above is most common in the literature.) This enables us to define well-behaved tensor product and dual representations of $U_q(\mathbf{sl_2})$.

2 Finite-dimensional representations of $U_q(\mathbf{sl_2})$ when q is not a root of unity

2.1 A baby example

We defined $U_q(\mathbf{sl_2})$ as a free algebra quotiented out by an ideal of relations. So a representation of $U_q(\mathbf{sl_2})$ is precisely an algebra homomorphism $\rho : A \to GL(V)$ for some vector space V with the relation ideal contained in $\ker \rho$. Since ρ is a homomorphism, it is completely specified by the image of the generators. Hence we aim to find linear maps $\mathbf{E}, \mathbf{F}, \mathbf{K}, \mathbf{K^{-1}}$ which satisfy the quantised Serre relations. Because we also require $\mathbf{KK^{-1}} = \mathbf{I}$ and $\mathbf{K^{-1}K} = \mathbf{I}$, $\mathbf{K}$ must be an invertible linear map, and $\mathbf{K^{-1}}$ is the inverse map. Hence it suffices to define $\mathbf{E}, \mathbf{F}, \mathbf{K}$.

Let's create our first representation by modifying the familiar action on $\mathbf{sl_2}$ given by:

$$\mathbf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

Naively set $K = e^{\hbar H/2}$ again. Since $\mathbf{H}$ is diagonal, the non-zero components of powers of $\mathbf{H}$ are simply powers of each component, so $\mathbf{K}$ has the form shown in (2.2) below. Direct computation shows that the quantised Serre relations are satisfied; hence we have a representation defined by:

$$\mathbf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \tag{2.2}$$

The key feature here is that, analogous to the classical case, $\mathbf{K}$ is completely diagonalisable, and $\mathbf{E}, \mathbf{F}$ permute the $\mathbf{K}$-eigenspaces in opposite directions. This can be summarised pictorially, with each node representing a one-dimensional $\mathbf{K}$-eigenspace:

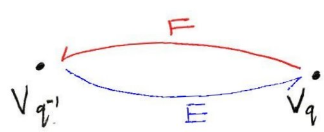

Figure 2.3 - $L(1,+)$ representation of $U_q(\mathbf{sl_2})$

It will follow from the upcoming theorem that every irreducible $U_q(\mathbf{sl_2})$-module has a similar visual description.

2.2 The structure of irreducible representations

Our goal is to prove:

Theorem. *The irreducible finite-dimensional representations of $U_q(\mathbf{sl_2})$, when q is not a root of unity, are $L(n,+)$ and $L(n,-)$, for all $n \geq 0$. $L(n,+)$ and $L(n,-)$ are both $n+1$-dimensional.*

- *For any $v \in L(n,+)$ with $\mathbf{E}v = 0$, $\left\{v, \mathbf{F}v, \mathbf{F}^2 v, \dots \mathbf{F}^n v\right\}$ is a basis of $L(n,+)$, and $U_q(\mathbf{sl_2})$-action on $L(n,+)$ is given by:*

$$\mathbf{E}\mathbf{F}^r v = \frac{q^r - q^{-r}}{\left(q-q^{-1}\right)^2}\left(q^{n-r+1} - q^{-n+r-1}\right)\mathbf{F}^{r-1} v \ \forall r > 0;\ \mathbf{E}v = 0$$

$$\mathbf{F}\mathbf{F}^r v = \mathbf{F}^{r+1} v \ \forall r < n;\ \mathbf{F}\mathbf{F}^n v = 0$$

$$\mathbf{K}\mathbf{F}^r v = q^{n-2r}\mathbf{F}^r v$$

In other words:

$$\mathbf{E} = \begin{pmatrix} & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \ddots & \\ & & & & \alpha_n \\ & & & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}; \mathbf{K} = \begin{pmatrix} q^n & & & & \\ & q^{n-2} & & & \\ & & \ddots & & \\ & & & q^{2-n} & \\ & & & & q^{-n} \end{pmatrix}$$

where $\alpha_r = \frac{q^r - q^{-r}}{(q-q^{-1})^2}\left(q^{n-r+1} - q^{-n+r-1}\right)$.

- *For any $v \in L(n,-)$ with $\mathbf{E}v = 0$, $\left\{v, \mathbf{F}v, \mathbf{F}^2 v, \dots \mathbf{F}^n v\right\}$ is a basis of $L(n,-)$, and $U_q(\mathbf{sl_2})$-action on $L(n,-)$ is given by:*

$$\mathbf{E}\mathbf{F}^r v = -\frac{q^r - q^{-r}}{\left(q-q^{-1}\right)^2}\left(q^{n-r+1} - q^{-n+r-1}\right)\mathbf{F}^{r-1} v \ \forall r > 0;\ \mathbf{E}v = 0$$

$$\mathbf{F}\mathbf{F}^r v = \mathbf{F}^{r+1} v \ \forall r < n;\ \mathbf{F}\mathbf{F}^n v = 0$$

$$\mathbf{K}\mathbf{F}^r v = -q^{n-2r}\mathbf{F}^r v$$

In other words:

$$\mathbf{E} = - \begin{pmatrix} & -\alpha & & & \\ & & \alpha_2 & & \\ & & & \ddots & \\ & & & & \alpha_n \\ & & & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}; \mathbf{K} = - \begin{pmatrix} q^n & & & & \\ & q^{n-2} & & & \\ & & \ddots & & \\ & & & q^{2-n} & \\ & & & & q^{-n} \end{pmatrix}$$

where $\alpha_r = \frac{q^r - q^{-r}}{(q-q^{-1})^2}\left(q^{n-r+1} - q^{-n+r-1}\right)$.

Under this notation, $L(1,+)$ is the example shown in Subsection 2.1, and $L(0,+)$ is the trivial representation given by the counit map. Figure 2.4 illustrates two more examples, $L(3,+)$ and $L(2,-)$; string diagrams for the other irreducible representations are similar except in the number of nodes (since this is the dimension). The diagrams for $L(n,+)$ and $L(n,-)$ are identical up to labelling, since in both cases the $\mathbf{K}$-eigenspaces are permuted by $\mathbf{E}$ and $\mathbf{F}$ in the same manner - indeed, $L(n,-)$ can be obtained $L(n,+)$ from by composing $\mathbf{E}$, $\mathbf{K}$ with $-\mathbf{I}$.

Figure 2.4 - $L(3,+)$ and $L(2,-)$ representations of $U_q(\mathbf{sl_2})$, when q is not a root of unity

It is relatively straightforward to check that the matrices presented above define valid representations - that is, they satisfy the quantised Serre relations. It is easiest to check this separately for each basis vector (we have $\mathbf{KEK}^{-1}(\mathbf{F}^r v) = q^2\mathbf{E}(\mathbf{F}^r v)\forall r$ and so on). Now we show that $L(n,+)$ and $L(n,-)$ are irreducible. Any $U_q(\mathbf{sl_2})$-submodule must contain a $\mathbf{K}$-eigenvector; in all the cases listed above, $\mathbf{K}$ has distinct eigenvalues, so this must be a multiple of one of the basis vectors. Given any pair of basis vectors, repeated application of $\mathbf{E}$ or $\mathbf{F}$ sends one to a non-zero multiple of the other, as $\alpha^r \neq 0$. Hence this submodule must contain all basis vectors, and is therefore all of $L(n,+)$ or $L(n,-)$.

So the meat of the proof will be to show that no other irreducible representations can exist. Before we start on this, we need some more terminology. Let V be an arbitrary finite-dimensional irreducible representation. Write V_μ for the subspace of $\mathbf{K}$-eigenvectors of eigenvalue μ; this is the *μ-weight space* of V, and any $v \in V_\mu$ is a *μ-weight vector*. If, in addition, $\mathbf{E}v = 0$ (so $v \in V_\mu \cap \ker \mathbf{E}$), then v is a *highest weight vector*, and (provided $v \neq 0$) μ is a *highest weight* of the representation V (any μ with $V_\mu \neq 0$ is a *weight* of V). Hence highest weight vectors correspond to the end nodes of a string diagram. In the statement of the theorem, v is a highest weight vector, and the form of the matrices show that, for each irreducible representation, the unique highest weight is precisely the weight involving the highest power of q, hence the name. Explicitly, $L(n,+)$ has highest weight q^n and $L(n,-)$ has highest weight q^{-n}; since q is not a root of unity, these are all distinct, so irreducible representations of $U_q(\mathbf{sl_2})$ can be recognised by their highest weights alone.

The proof of this classification theorem is only very slightly modified from the standard procedure for analysing finite-dimensional irreducible $\mathbf{sl_2}$ representations (found, for example, in Chapter 7 of [Humphreys]). In short, the argument runs as follows: finite-dimensionality guarantees the existence of a highest weight vector; its image under repeated $\mathbf{F}$-application is $U_q(\mathbf{sl_2})$-invariant, and using the finite-dimensional hypothesis a second time gives a restriction on possible values of the highest weight. Pay particular attention to the calculations (2.5), (2.6) and (2.7); we will use these over and over.

Step 1 *V contains a highest weight vector v.*

Since $\mathbb{C}$ is algebraically closed, some V_μ is non-zero. Take non-zero $w \in V_\mu$. Recall that $\mathbf{K}$ is invertible, so $\mu \neq 0$ and $\mathbf{K}^{-1}w = \mu^{-1}w$. Using (QSR1):

$$\mathbf{K}(\mathbf{E}w) = \mathbf{KEK}^{-1}(\mu w) = q^2\mathbf{E}(\mu w) = q^2\mu(\mathbf{E}w) \tag{2.5}$$

So $\mathbf{E}w$ is another eigenvector of $\mathbf{K}$, with eigenvalue $q^2\mu$. (2.5) holds for any $\mathbf{K}$-eigenvector, so by inductively applying it to $\mathbf{E}^r w$ we find that, $\forall r \geq 0$, $\mathbf{E}^r w \in V_{q^{2r}\mu}$. As q is not a root of unity, these eigenvalues are all distinct. Hence the non-zero elements of $\{w, \mathbf{E}w, \mathbf{E}^2 w, ...\}$ are linearly independent. V is finite-dimensional, so, for at least one i, we must have $\mathbf{E}^i w = 0$. By definition of w, $i \geq 1$, so we can define $v = \mathbf{E}^{i-1}w$, then $v \in V_{q^{2(i-1)}\mu} \cap \ker \mathbf{E}$ as required. Call this highest weight λ instead of $q^{2(i-1)}\mu$.

Step 2 *The non-zero vectors of the set $\{v, \mathbf{F}v, \mathbf{F}^2 v, ...\}$ form a basis for V.*

Perform the same calculation as (2.5) above, with $\mathbf{F}$ in place of $\mathbf{E}$:

$$\mathbf{K}(\mathbf{F}w) = \mathbf{KFK}^{-1}(\mu w) = q^{-2}\mathbf{F}(\mu w) = q^{-2}\mathbf{F}(\mu w) \tag{2.6}$$

for any w in any V_μ. Repeated application of (2.6) to $\mathbf{F}^r v$ then shows that $\mathbf{F}^r v \in V_{q^{-2r}\lambda}$, so, by the previous argument, non-zero elements of $\{v, \mathbf{F}v, \mathbf{F}^2 v, ...\}$ are linearly independent.

To show $\{v, \mathbf{F}v, \mathbf{F}^2 v, ...\}$ that span V, we show that $\mathbf{E}, \mathbf{F}, \mathbf{K}$ each send each $\mathbf{F}^r v$ into the subspace spanned by $\{v, \mathbf{F}v, \mathbf{F}^2 v, ...\}$. Since these three elements generate $U_q(\boldsymbol{sl_2})$, all of $U_q(\boldsymbol{sl_2})$ must then fix this subspace, which must be all of V as V is irreducible. Our assertion is clear for $\mathbf{F}$ and $\mathbf{K}$. As we saw in Step 1, $\mathbf{E}$ maps $V_{q^{-2r}\lambda}$ to $V_{q^{-2(r-1)}\lambda}$, so if $\{v, \mathbf{F}v, \mathbf{F}^2 v, ...\}$ indeed span V, $\mathbf{EF}^r v$ must be some multiple α_r of $\mathbf{F}^{r-1}v$. We can verify this inductively, with the help of (QSR3) (in the base case of $r = 1$, use the fact that $\mathbf{E}v = 0$):

$$\begin{aligned}
\mathbf{EF}^r v &= \frac{\mathbf{K} - \mathbf{K}^{-1}}{q - q^{-1}}\mathbf{F}^{r-1}v + \mathbf{FEF}^{r-1}v \\
&= \frac{q^{-2r+2}\lambda - q^{2r-2}\lambda^{-1}}{q - q^{-1}}\mathbf{F}^{r-1}v + \mathbf{F}\alpha_{r-1}\mathbf{F}^{r-2}v \\
&= \alpha_r \mathbf{F}^{r-1}v
\end{aligned}$$

(2.7) allows us to calculate α_r explicitly, since $\alpha_0 = 0$:

$$\begin{aligned}
\alpha_r &= \frac{q^{-2r+2}\lambda - q^{2r-2}\lambda^{-1}}{q - q^{-1}} + \alpha_{r-1} \\
&= \sum_{i=1}^{r} \frac{q^{-2i+2}\lambda - q^{2i-2}\lambda^{-1}}{q - q^{-1}} \\
&= \frac{q^r - q^{-r}}{(q - q^{-1})^2}(q^{1-r}\lambda - q^{r-1}\lambda^{-1})
\end{aligned}$$

Step 3 *For each integer $n \geq 0$, there are precisely two irreducible representations of dimension $n+1$, whose highest weights are q^n and q^{-n} respectively.*

As V is finite-dimensional, the infinite set $\{v, \mathbf{F}v, \mathbf{F}^2 v, ...\}$ must contain some elements equal to zero. Let n be minimal with $\mathbf{F}^{n+1}v = 0$; then for all $r \geq 0$, and by minimality of n, $\{v, \mathbf{F}v, ..., \mathbf{F}^n v\}$ are all non-zero, so V has dimension $n+1$. $\mathbf{EF}^{n+1}v = 0$ forces

$$\alpha_{n+1} = 0 \Rightarrow q^{-n}\lambda = q^n \lambda^{-1} \Rightarrow \lambda = \pm q^n \tag{2.9}$$

as claimed. Since q is not a root of unity, these two possibilities are distinct.

We now have enough information to write down $\mathbf{E}, \mathbf{F}, \mathbf{K}$ explicitly with respect to the basis $\{v, \mathbf{F}v, ..., \mathbf{F}^n v\}$. The form of $\mathbf{F}$ is obvious; (2.6) in Step 2 states that $\mathbf{F}^r v \in V_{q^{-2r}\lambda} = V_{\pm q^{n-2r}}$, so $\mathbf{K}$ is diagonal and has entries has shown in the theorem for $L(n,+)$; substituting $\lambda = \pm q^n$ into (2.8) gives the correct form for $\mathbf{E}$. QED

Finally, we isolate out Step 2 as a separate result, since this observation is useful when decomposing representations into irreducibles. It concerns the structure of $U_q(\mathbf{sl_2})$-modules which are generated by a single highest weight vector, called *highest weight modules*.

Proposition. *Let v be a highest weight vector, and suppose q is not a root of unity. Then the non-zero elements of $\{v, \mathbf{F}v, \mathbf{F}^2 v, ...\}$ is a basis of weight vectors for the $U_q(\mathbf{sl_2})$-submodule generated by v.*

2.3 Complete Reducibility

Recall that finite-dimensional representations $\boldsymbol{sl_2}$ of are completely reducible - in this subsection we prove the same for $U_q(\mathbf{sl_2})$, when q is not a root of unity. We will again adapt an $\boldsymbol{sl_2}$ argument; this differs from the standard proof in the literature (found, for example in I.4.5 of [Brown, Goodearl] and 2.9 of [Jantzen]). However, the core idea is the same - we locate some highest weight vectors, associate subspaces to them, then show that these are linearly independent, span V and are $U_q(\mathbf{sl_2})$-invariant.

First, we define the *Casimir element*

$$\Omega = FE + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2} \qquad ((2.10))$$

Step 1 Ω *is central in* $U_q(\mathbf{sl_2})$

It suffices to show that Ω commutes with the generators $\mathbf{E}, \mathbf{F}, \mathbf{K}$. Taking the product of (QSR1) and (QSR2):

$$KFEK^{-1} = (KEK^{-1})(KFK^{-1}) = q^{-2}Fq^2E = FE \tag{2.11}$$

and K clearly commutes with the second term of Ω . Observe also

$$\begin{aligned} E\Omega &= EFE + \frac{EKq + EK^{-1}q^{-1}}{(q-q^{-1})^2} \\ &= FE^2 + \frac{KE - K^{-1}E}{q-q^{-1}} + \frac{KEq^{-1} + K^{-1}Eq}{(q-q^{-1})^2} = \Omega E \end{aligned}$$

$$\begin{aligned} \Omega F &= FEF + \frac{KFq + EKFq^{-1}}{(q-q^{-1})^2} \\ &= F^2E + \frac{FK - FK^{-1}}{q-q^{-1}} + \frac{FKq^{-1} + FK^{-1}q}{(q-q^{-1})^2} = F\Omega \end{aligned}$$

Now take any finite-dimensional representation of V and let $\{0\} = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^d = V$ be a *composition series* for V - the V^i are $U_q(\mathbf{sl_2})$-submodules with each *composition factor* V^i/V^{i-1} irreducible.

Step 2 *The eigenvalues of* $\mathbf{\Omega}$ *on* V *are of the form* $\epsilon\frac{q^{n+1}+q^{-n-1}}{(q-q^{-1})^2}$, *where* $n \in \mathbb{N}$, $\epsilon \in \{+,-\}$

Suppose $v \in V$ satisfies $\mathbf{\Omega}v = cv$. Take i minimal with $v \in V^i$, so $\bar{v}$, the image of v under quotienting by V^{i-1}, is non-zero. Observe that $\mathbf{\Omega}\bar{v} = c\bar{v}$ also. V^i/V^{i-1} is an irreducible representation, so Step 1 and Schur's lemma says that Ω acts on V^i/V^{i-1} as scalar multiplication. Hence we must have $\mathbf{\Omega}\bar{w} = c\bar{w}$ $\forall v \in V^i/V^{i-1}$. By the classification Theorem 2.2, $V^i/V^{i-1} \simeq L(n,\epsilon)$ for some $n \in \mathbb{N}$, $\epsilon \in \{+,-\}$. Then, taking $\bar{w}$ to be the highest weight vector shows that $c = \epsilon\frac{q^{n+1}+q^{-n-1}}{(q-q^{-1})^2}$.

Step 3 *For each* $c \in \left\{\epsilon\frac{q^{n+1}+q^{-n-1}}{(q-q^{-1})^2} : n \in \mathbb{N}, \epsilon \in \{+,-\}\right\}$, *set* $V_{(c)} = \left\{v \in V : (\mathbf{\Omega} - c\mathbf{I})^{\dim V} = 0\right\}$. *Then* $V = \bigoplus_c V_{(c)}$ *where* c *ranges over the above set. For each* $c = \epsilon\frac{q^{n+1}+q^{-n-1}}{(q-q^{-1})^2}$, $V_{(c)}$ *is an* $U_q(\mathbf{sl_2})$*-submodule, and each of its composition factors is* $L(n,\epsilon)$.

The vector space decomposition $V = \bigoplus_c V_{(c)}$ follows from considering the Jordan normal form of $\mathbf{\Omega}$. Each $V_{(c)}$ contains a $\mathbf{\Omega}$-eigenvector of eigenvalue c, so by Step 2, only cs of the form $\epsilon\frac{q^{n+1}+q^{-n-1}}{(q-q^{-1})^2}$ occur in the sum. Since $\mathbf{\Omega}$,$c\mathbf{I}$ are both central in $U_q(\mathbf{sl_2})$, $(\mathbf{\Omega}-c\mathbf{I})^{\dim V}$ is central also, so $V_{(c)}$, being its kernel, is $U_q(\mathbf{sl_2})$-invariant as required.

From now on, fix $c=\epsilon\frac{q^{n+1}+q^{-n-1}}{(q-q^{-1})^2}$ and fix $\{0\}=V_{(c)}^0\subseteq V_{(c)}^1\subseteq\cdots\subseteq V_{(c)}^d=V_{(c)}$ a composition series of $V_{(c)}$. Suppose $V_{(c)}^i/V_{(c)}^{i-1}\simeq L(m_i,\delta_i)$. For each i, take $v_i\in V_{(c)}^i\backslash V_{(c)}^{i-1}$ and let $\bar{v}_i\in V_{(c)}^i/V_{(c)}^{i-1}$ be its image under natural projection. As in the proof of Step 2, $\mathbf{\Omega}\bar{v}_i=\delta_i\frac{q^{m_i+1}+q^{-m_i-1}}{(q-q^{-1})^2}\bar{v}_i$. But $v_i\in V_{(c)}$ means $(\mathbf{\Omega}-c\mathbf{I})^{\dim V}\bar{v}_i=0$, so we must have

$$\delta_i\frac{q^{m_i+1}+q^{-m_i-1}}{(q-q^{-1})^2}=c=\epsilon\frac{q^{n+1}+q^{-n-1}}{(q-q^{-1})^2}\forall i \tag{2.14}$$

$$\Rightarrow\left(\epsilon q^{n+1}-\delta_i q^{m_i+1}\right)\left(1-\epsilon\delta_i q^{-n-1-m_i-1}\right)=0$$

$$\Rightarrow \epsilon q^{n+1}=\delta_i q^{m_i+1}\quad\text{or}\quad \epsilon\delta_i=q^{-n-1-m_i-1}$$

Since q is not a root of unity, the second alternative cannot hold, and the first alternative is only true when $m_i=n,\delta_i=\epsilon$. Hence all the composition factors $L(m_i,\delta_i)\cong V_{(c)}^i/V_{(c)}^{i-1}$ are $L(n,\epsilon)$ as desired.

So it suffices to show that each $V_{(c)}$ is completely reducible - to do this, we recycle many concepts from the proof of the classification Theorem 2.2.

Step 4 *Any weight of $V_{(c)}$ must also be a weight of $L(n,\epsilon)$.*

Take $v\in V_{(c)}$with $\mathbf{K}v=\lambda v$ and choose i minimal with $v_i\in V_{(c)}^i$. As usual, write $\bar{v}_i\in V_{(c)}^i/V_{(c)}^{i-1}$ for its image under natural projection. $\mathbf{K}\bar{v}=\lambda\bar{v}$, so λ is a weight of $V_{(c)}^i/V_{(c)}^{i-1}\cong L(n,\epsilon)$.

Step 5 *For $x\in V_{(c)}$, $x\in\ker\mathbf{E}\iff(\mathbf{K}-\epsilon q^n\mathbf{I})^m x=0$ for some $m\in\mathbb{N}$*

Suppose $y\in\ker\mathbf{E}$ is a $\mathbf{K}$-eigenvector. Then, as shown in Step 4, $\bar{y}\in V_{(c)}^i/V_{(c)}^{i-1}$ is a $\mathbf{K}$-eigenvector with the same eigenvalue. The same logic says $\bar{y}\in\ker\mathbf{E}$, so from the structure of $L(n,\epsilon)$, we see that $\bar{y}$ must be a highest weight vector, with weight ϵq^n. Hence y has weight ϵq^n also.

From (QSR1), $\ker\mathbf{E}$ is invariant under $\mathbf{K}$. Consider the Jordan normal form of $\mathbf{K}$-action on $\ker\mathbf{E}$. This is an upper triangular matrix all of whose diagonal entries are eigenvalues of the $\mathbf{K}$-action on $\ker\mathbf{E}$ - by the previous paragraph, these eigenvalues are all ϵq^n. It follows that $(\mathbf{K}-\epsilon q^n\mathbf{I})^{\dim\ker\mathbf{E}}$ must be the zero map on $\ker\mathbf{E}$.

Conversely, if $(\mathbf{K}-\epsilon q^n\mathbf{I})^m x=0$, then $(\mathbf{K}-\epsilon q^n\mathbf{I})^m(\mathbf{E}x)=0$, since (QSR1) shows

$$(K-\epsilon q^{n+2})E=q^2EK-\epsilon q^{n+2}E=q^2E(K-\epsilon q^n) \tag{(2.15)}$$

repeated applications of which give

$$(K - \epsilon q^{n+2})^m E = q^{2m} E(K - \epsilon q^n) \tag{2.16}$$

From Step 4, we know ϵq^{n+2} is not an weight of $V_{(c)}$, so $\det(\mathbf{K} - \epsilon q^{n+2}\mathbf{I}) \neq 0$. Hence $(\mathbf{K} - \epsilon q^n \mathbf{I})^m$ is non-singular, so we must have $\mathbf{E}x = 0$.

Step 6 *Construct a basis* $\{x_1, x_2, .., x_d\}$ *of* $\ker \mathbf{E}$ *such that* $\{\mathbf{F}^r x_i : 1 \leq i \leq d, 0 \leq r \leq n\}$ *is a basis of* $V_{(c)}$.

First let $\{e_1, e_2, ..., e_{d(n+1)}\}$ be a basis of $V_{(c)}$ such that, for each i, $e_1, e_2, ..., e_{i(n+1)}$ span $V_{(c)}^i$ and $\bar{e}_{(i-1)(n+1)+1}, \bar{e}_{(i-1)(n+1)+2}, ..., \bar{e}_{i(n+1)}$ is a standard basis for $V_{(c)}^i / V_{(c)}^{i-1} \cong L(n, \epsilon)$. In other words, with respect to this basis, the matrices have this block form:

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}' & * & \cdots & * \\ & \mathbf{E}' & \ddots & \vdots \\ & & \ddots & * \\ & & & \mathbf{E}' \end{pmatrix}; \mathbf{F} = \begin{pmatrix} \mathbf{F}' & * & \cdots & * \\ & \mathbf{F}' & \ddots & \vdots \\ & & \ddots & * \\ & & & \mathbf{F}' \end{pmatrix}; \mathbf{K} = \begin{pmatrix} \mathbf{K}' & * & \cdots & * \\ & \mathbf{K}' & \ddots & \vdots \\ & & \ddots & * \\ & & & \mathbf{K}' \end{pmatrix} \tag{2.17}$$

where $\mathbf{E}', \mathbf{F}', \mathbf{K}'$ are as displayed in the classification Theorem 2.2 and describe $U_q(\mathbf{sl_2})$-action on (* denotes unknown entries).

Take $x_1 = e_1$; then $x_1 \in \ker \mathbf{E}$, and $\mathbf{F}^r x_1 = e_{r-1}$ for $0 \leq r \leq n$. By choice of e_is,

$$(\mathbf{K} - \epsilon q^n) e_{(n+1)+1} = \sum_{i-1}^{n+1} a_i e_i \tag{2.18}$$

for some $a_i \in \mathbb{C}$. Note that

$$(\mathbf{K} - \epsilon q^n) e_i = (\epsilon q^{n-2(i-1)} - \epsilon q^n) e_i \neq 0 \tag{2.19}$$

for $2 \leq i \leq n = 1$, so, by subtracting suitable multiples of these e_i from $e_{(n+1)+1}$, we find x_2 with $(\mathbf{K} - \epsilon q^n) x_2 = a_1 e_1$. $\mathbf{K} e_1 = \epsilon q^n e_1$, so , $(\mathbf{K} - \epsilon q^n)^2 x_2 = 0$ and $x_2 \in \ker \mathbf{E}$,, by Step 5.

Repeat this process with $e_{2(n+1)+1}$: for each i with $(n+1) + 2 \leq i \leq 2)n+1)$,

$$(\mathbf{K} - \epsilon q^n) e_i = \text{non-zero multiple of } e_i + \text{element of } V_{(c)}^1 \tag{2.20}$$

so subtracting multiples of these e_i from $e_{2(n+1)+1}$ can ensure that $(\mathbf{K} - \epsilon q^n)(\text{resulting vector}) \in \text{span of } \{e_1, e_2, ..., e_{(n+1)+2}\}$. Then we remove the e_i components, $2 \leq i \leq n+1$, on the right hand side, in the same way as before, and produce x_3 with $(\mathbf{K} - \epsilon q^n)^3 x_3 = 0$.

Continuing this process gives $\{x_1, x_2, .., x_d\} \in \ker \mathbf{E}$ with $\bar{x_i} = \bar{e}_{(i-1)(n+1)+1}$. More precisely, $x_i = e_{(i-1)(n+1)+1} + v_{i-1}$ with $v_{i-1} \in V_{(c)}^{i-1}$. From the form of

$\mathbf{F}$, we see that $\mathbf{F}^r x_i = e_{(i-1)(n+1)+r+1}$ + element of $V_{(c)}^{i-1}$. Hence the matrix expressing $\{x_1, \mathbf{F}x_1, \mathbf{F}^2 x_1, ..., \mathbf{F}^n x_1, x_2, \mathbf{F}x_2, ..., \mathbf{F}^n x_2, x_3, ..., \mathbf{F}^n x_d\}$ in terms of $\{e_1, e_2, ..., e_{d(n+1)}\}$ is upper triangular with all diagonal entries equal to 1, and therefore is invertible. So $\{\mathbf{F}^r x_i : 1 \leq i \leq d, 0 \leq r \leq n\}$ indeed form a basis of $V_{(c)}$.

To see that $\{x_1, x_2, .., x_d\}$ span $\ker \mathbf{E}$, take any $v \in \ker \mathbf{E}$ and set i minimal with $v_i \in V_{(c)}^i$. Then, since $\ker \mathbf{E} \subseteq V_{(c)}^i / V_{(c)}^{i-1}$ is one-dimensional, $\bar{v} \in V_{(c)}^i / V_{(c)}^{i-1}$ is a multiple of $\bar{e}_{(i-1)(n+1)+1}$. In other words, some linear combination of v and x_i lives in $V_{(c)}^{i-1} \cap \ker \mathbf{E}$. Applying induction on i allows v to be expressed in terms of the x_i, so indeed $\{x_1, x_2, .., x_d\}$ span $\ker \mathbf{E}$.

Define X^i = span of $\{\mathbf{F}^r x_i : 0 \leq r \leq n\}$. By Step 6, these span $V_{(c)}$ and are linearly independent. It remains to show that each X^i is $U_q(\mathbf{sl_2})$-invariant. $\mathbf{F}$-invariance will be clear once we figure out how $\mathbf{F}$ permutes the generalised weight spaces, and deduce that $\mathbf{F}^{n+1} x_i = 0$; afterwards we prove that the $\mathbf{F}^r x_i$ are all weight vectors. $\mathbf{E}$-invariance then follows from (QSR3) via (2.7), as in the proof of the classification theorem.

Since complete reducibility means that all generalised weight spaces are in fact weight spaces, the following should not come as a surprise:

Step 7 *$\forall x \in \ker \mathbf{E}$, and all $r \geq 0$, $(\mathbf{K} - \epsilon q^{n-2r}\mathbf{I})^m \mathbf{F}^r x = 0$ for some $m \in \mathbb{N}$*

From (QSR2):

$$(K - \epsilon q^{n-2r})F = q^{-2}FK - \epsilon q^{n-2r}F = q^{-2}F(K - \epsilon q^{n-2(r-1)}) \tag{2.21}$$

so repeated application shows

$$(K - \epsilon q^{n-2r})^m F = q^{-2m} F(K - \epsilon q^{n-2(r-1)})^m \tag{2.22}$$

Now use induction on r:

$$(\mathbf{K} - \epsilon q^{n-2r}\mathbf{I})^m \mathbf{F}^r x = q^{-2m}\mathbf{F}(\mathbf{K} - \epsilon q^{n-2(r-1)}\mathbf{I})^m \mathbf{F}^{r-1} x \tag{2.23}$$

the base case of $r = 0$ being the assertion of Step 5.

Step 8 *$\mathbf{K}$ acts on $\ker \mathbf{E}$ as a scalar map*

From Step 7, $\forall i$, $(\mathbf{K} - \epsilon q^{-n-2}\mathbf{I})^m \mathbf{F}^{n+1} x_i = 0$ for some m, but ϵq^{-n-2} is not a weight of $L(n, \epsilon)$, so $(\mathbf{K} - \epsilon q^{-n-2}\mathbf{I})^m$ cannot be the zero map, forcing $\mathbf{F}^{n+1} x_i = 0$ (this is the same argument as the last paragraph of Step 5). Hence $\mathbf{E}\mathbf{F}^{n+1} x_i = 0$. Later (see (3.13)) we will show that

$$EF^{n+1} = \frac{q^{n+1} - q^{-n-1}}{(q-q^{-1})^2} F^n (q^{-n}K + q^n K^{-1}) + F^{n+1}E \qquad (2.24)$$

Since q is not a root of unity, we must have $\mathbf{F}^n(q^{-n}\mathbf{K} + q^n\mathbf{K}^{-1})x_i = 0 \forall i$. $\ker \mathbf{E}$ is $\mathbf{K}$-invariant, so $(q^{-n}\mathbf{K} + q^n\mathbf{K}^{-1})x_i = \sum_{j=1}^{d} a_{ij}x_j$ for some $a_{ij} \in \mathbb{C}$. Then

$$\mathbf{F}^n(q^{-n}\mathbf{K} + q^n\mathbf{K}^{-1})x_i = \sum_{j=1}^{d} a_{ij}\mathbf{F}^n x_j \qquad (2.25)$$

$\mathbf{F}^n x_j$ form part of a basis set, so they are linearly independent. Consequently, we must have $a_{ij} = 0$, so $q^{-n}\mathbf{K} + q^n\mathbf{K}^{-1}$ is the zero map on $\ker \mathbf{E}$. Rearranging, we see $\mathbf{K}^2 = q^{2n}\mathbf{I}$ on $\ker \mathbf{E}$. Write $\mathbf{K}$ in Jordan normal form, and recall from Step 5 that all the diagonal entries are ϵq^n. Explicitly calculating the square of such a matrix shows that all off-diagonal entries must be 0, so $\mathbf{K}x = \epsilon q^n x \, \forall x \in \ker \mathbf{E}$.

So the x_i are weight vectors as claimed, and iteratively applying (QSR2) (as in (2.6)) shows that all $\mathbf{F}^r x_i$ must also be weight vectors. By the discussion before Step 7, this completes the proof. QED

2.4 Tensor representations, and decomposition into irreducibles

In this subsection, we address how to decompose any arbitrary finite-dimensional $U_q(\mathbf{sl_2})$-representation into a direct sum of irreducibles. The proof of complete reducibility suggests examining $\ker \mathbf{E}$ or Casimir action. However, if we are simply interested in the isomorphism types of these submodules rather than their particular location within V, then there is a far easier combinatorial method which only requires knowing the weights of V and their *multiplicities* (which is the dimension of the corresponding weight space). We illustrate such an algorithm on the example of tensor products, and deduce an analogue of the Clebsch-Gordan formula, which we then rephrase in terms of character polynomials.

First, consider the simple case of $L(1,+) \otimes L(1,+)$, which is four-dimensional. Denote by v and w the highest weight vectors of the two factors. So $\mathbf{E}v = \mathbf{E}w = 0$, $\mathbf{K}v = qv$, $\mathbf{K}w = qw$. (We abuse notation and write $\mathbf{K}$ for the K-action on both factors of $L(1,+) \otimes L(1,+)$, and similarly for $\mathbf{E}, \mathbf{F}$.) Let $\tilde{\mathbf{E}}, \tilde{\mathbf{F}}, \tilde{\mathbf{K}}$ be matrices describing $U_q(\mathbf{sl_2})$-action on the tensor product. From the Hopf algebra structure of (1.6):

$$\begin{aligned}
\tilde{\mathbf{K}}(v \otimes w) &= \mathbf{K}v \otimes \mathbf{K}w = qv \otimes qw = q^2 v \otimes w \\
\tilde{\mathbf{K}}(v \otimes \mathbf{F}w) &= \mathbf{K}v \otimes \mathbf{KF}w = qv \otimes q^{-1}\mathbf{F}w = q^2 v \otimes \mathbf{F}w \\
\tilde{\mathbf{K}}(\mathbf{F}v \otimes w) &= \mathbf{KF}v \otimes \mathbf{K}w = q^{-1}\mathbf{F}v \otimes qw = q^2 \mathbf{F}v \otimes w \\
\tilde{\mathbf{K}}\mathbf{F}(v \otimes \mathbf{F}w) &= \mathbf{KF}v \otimes \mathbf{KF}w = q^{-1}\mathbf{F}v \otimes q^{-1}\mathbf{F}w = q^{-2}\mathbf{F}v \otimes \mathbf{F}w
\end{aligned}$$

so $\{v \otimes w, v \otimes \mathbf{F}w, \mathbf{F}v \otimes w, \mathbf{F}v \otimes \mathbf{F}w\}$ is a basis of weight vectors (that this set is a basis follows from the definition of tensor product). Hence the weights of $L(1,+) \otimes L(1,+)$ are precisely q^2, 1 with multiplicity two (ie $\dim [L(1,+) \otimes L(1,+)]_1 = 2$), and q^{-2}. For $n > 2$, $q^n, -q^n$ are not weights of $L(1,+) \otimes L(1,+)$, so cannot contain $L(n,+)$ or $L(n,-)$. Having ruled these out, the only remaining irreducible representation containing q^2 as a weight is $L(2,+)$. Hence $L(1,+) \otimes L(1,+)$ must contain copies of $L(2,+)$ - in fact, exactly one copy, since q^2 occurs as a weight of $L(1,+) \otimes L(1,+)$ with multiplicity one. Indeed,

$$\tilde{\mathbf{E}}(v \otimes w) = \mathbf{K}v \otimes \mathbf{E}w + \mathbf{E}v \otimes w = 0 \tag{2.27}$$

so $v \otimes w$ is a highest weight vector of weight q^2. By Proposition 2.2 on the structure of highest weight modules, $v \otimes w$ generates a copy of $L(2,+)$.

The weights of $L(2,+)$ are q^2,1 and q^{-2}, each with multiplicity one, so the only weight unaccounted for in $L(1,+) \otimes L(1,+)$ is 1. The only irreducible $U_q(\mathbf{sl_2})$-module with 1 as its sole weight is $L(0,+)$. Hence we deduce $L(1,+) \otimes L(1,+) \cong L(2,+) \oplus L(0,+)$.

Now turn to the general case - take two irreducible representations $L(n,\epsilon)$ and $L(m,\delta)$ (where $\delta, \epsilon \in \{+,-\}$) with highest weight vectors v, w respectively. Recall that $\{\mathbf{F}^r v : 0 \leq r \leq n\}$ and $\{\mathbf{F}^s w : 0 \leq s \leq m\}$ are bases of weight vectors for the two factors. Now, generalising (2.26):

$$\begin{aligned}
\tilde{\mathbf{K}}(\mathbf{F}^r v \otimes \mathbf{F}^s w) &= \mathbf{K}\mathbf{F}^r v \otimes \mathbf{K}\mathbf{F}^s w \\
&= \epsilon q^{n-2r} \mathbf{F}^r v \otimes \delta q^{m-2s} \mathbf{F}^s w \\
&= \epsilon\delta q^{n+m-2(r+s)} \mathbf{F}^r v \otimes \mathbf{F}^s w
\end{aligned}$$

So $\{\mathbf{F}^r v \otimes \mathbf{F}^s w : 0 \leq r \leq n, 0 \leq s \leq m\}$ is a basis of weight vectors for $L(n,\epsilon) \otimes L(m,\delta)$, and the weight of any such basis vector is the product of the weights of its factors. We deduce that the weights of $L(n,\epsilon) \otimes L(m,\delta)$ are $\{\epsilon\delta q^{n+m}, \epsilon\delta q^{n+m-2}, ..., \epsilon\delta q^{-n-m}\}$ where the $\epsilon\delta q^{n+m-2t}$-weight space has dimension $|\{(r,s) : 0 \leq r \leq n, 0 \leq s \leq m, r+s=t\}|$. If $t \leq \min\{m,n\}$ (equivalently, $n+m-2t \geq |n-m|$), this expression simplifies to $|\{r : 0 \leq r \leq t\}| = t+1$.

The highest power of q, up to sign, which occurs here is $\epsilon\delta q^{n+m}$, with multiplicity one. By the same reasoning as in the simple example above, $L(n,\epsilon) \otimes L(m,\delta)$ must contain exactly one copy of $L(n+m,\epsilon\delta)$ and no copies of $L(i,\pm)$ for $i > n+m$.

We calculated above that the $\epsilon\delta q^{n+m-2}$-weight space has dimension two, and $L(n+m,\epsilon\delta)$ contains only a linear subspace of this. Hence a single copy of $L(n+m-2,\epsilon\delta)$ is responsible for the second dimension.

Continuing like this, we see the irreducible summands of $L(n,\epsilon) \otimes L(m,\delta)$ include $L(n+m,\epsilon\delta)$, $L(n+m-2,\epsilon\delta)$, $L(n+m-4,\epsilon\delta)$,..., $L(|n-m|,\epsilon\delta)$, and our simplified expression for the weight multiplicities does not allow us to go further. However, the total dimension of these summands is $(n+m+1)+(n+m-1)+\cdots+(|n-m|+1) = (n+1)(m+1) = \dim L(n,\epsilon) \otimes L(m,\delta)$, so this list is in fact complete. We have proved the quantised Clebsch-Gordan formula:

Proposition. $L(n,\epsilon)\otimes L(m,\delta) = L(n{+}m,\epsilon\delta)\oplus L(n{+}m,\epsilon\delta)\oplus\cdots\oplus L(|n{-}m|,\epsilon\delta)$

Figure 2.29 illustrates a few examples as string diagrams.

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Figure 2.29 - $L(2,+)\otimes L(2,-)$ and $L(3,-)\otimes L(1,-)$ representations of when q is not a root of unity

This algorithm can be applied to any finite-dimensional representation, to give the decomposition

$$\begin{aligned} V \quad \cong \quad & \bigoplus_{i=0}^{n} \left(\dim V_{q^i} - \dim V_{q^{i+1}} - \cdots - \dim V_{q^n}\right) L(i,+) \\ & \oplus \bigoplus_{j=0}^{m} \left(\dim V_{-q^j} - \dim V_{-q^{j+1}} - \cdots - \dim V_{-q^m}\right) L(i,-) \end{aligned}$$

where $n = \max\{i : V_{q^i} \neq 0\}$, $m = \max\{i : V_{-q^i} \neq 0\}$. Since this formula only involves the dimensions of weight spaces, the following is true:

Theorem. *Finite-dimensional representations of $U_q(\mathbf{sl_2})$ are determined up to isomorphism by two polynomials in q:*

$$\begin{aligned} \chi^+(V) \quad &= \quad \sum_{n=0}^{\infty} \dim V_{q^n} q^n \\ \chi^-(V) \quad &= \quad \sum_{m=0}^{\infty} \dim V_{-q^m} q^m \end{aligned}$$

(The notation is my own; there seems to be no standard definition in the literature)

From the classification Theorem 2.2:

$$\begin{aligned} \chi^+(L(n,+)) \quad &= \quad q^n + q^{n-2} + \cdots + q^{-n} = \tfrac{q^{n+1}-q^{-n-1}}{q-q^{-1}} \quad &= \quad \chi^-(L(n,-)) \\ \chi^-(L(n,-)) \quad &= \quad 0 \quad &= \quad \chi^+(L(n,+)) \end{aligned} \tag{2.31}$$

completely analogous to the classical case. Generalising (2.28) then gives this alternative formulation of the quantised Clebsch-Gordan rule:

$$\chi^+(V\otimes W) = \chi^+(V)\chi^+(W) = \chi^-(V)\chi^-(W) \tag{2.32}$$

$$\chi^-(V\otimes W) = \chi^+(V)\chi^-(W) = \chi^-(V)\chi^+(W)$$

Under this framework, identifying the irreducible components of a representation equates to expressing Laurent polynomials as linear combinations of $\left\{\frac{q^{n+1}-q^{-n-1}}{q-q^{-1}} : n \geq 0\right\}$.

3 Finite-dimensional representations of $U_q(\mathbf{sl_2})$ when q is a root of unity

3.1 Some examples

Before we develop the general theory, let's work through a couple of examples to get a feeling for some of the differences that arise when q is a root of unity. Throughout this subsection, we consider the simplest setting where $q^3 = 1$ ie $q = e^{2\pi i/3}$.

The matrices describing $L(n, \pm)$, as given in the classification Theorem 2.2, obey the quantised Serre relations whatever the value of q, so, when $q^3 = 1$, these matrices still define valid representations. For example, $L(4, +)$ is given by

$$\mathbf{E} = \begin{pmatrix} & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \alpha_3 & \\ & & & & \alpha_4 \\ & & & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & & & \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{pmatrix}; \mathbf{K} = \begin{pmatrix} q^4 & & & & \\ & q^2 & & & \\ & & 1 & & \\ & & & q^{-2} & \\ & & & & q^{-4} \end{pmatrix} \tag{3.1}$$

where

$$\alpha_r = \frac{q^r - q^{-r}}{(q - q^{-1})^2} \left(q^{5-r} - q^{r-5}\right) = \frac{q^r - q^{-r}}{(q - q^{-1})^2} \left(q^{2-r} - q^{r-2}\right) \tag{3.2}$$

$q^3 = 1$ means that $\alpha_2 = \alpha_3 = 0$, so the string diagram for $L(4, +)$ has two "missing links", as shown in Figure 3.3 below:

Figure 3.3 - The representation $L(4, +)$, when $q^3 = 1$, and its invariant submodules

Now $L(4, +)$ is not irreducible: the two left-most nodes in Figure 3.3 represent a $U_q(\mathbf{sl_2})$-invariant submodule (since no arrows leave this pair), and the same is true of the three left-most nodes. To be specific, $\alpha_2 = 0$ and $\alpha_3 = 0$ causes the third and fourth basis vectors respectively to become additional highest weight vectors, which each generate a proper submodules, as we saw in the proof of the classification Theorem 2.2.

More interestingly, neither submodule identified above has a $U_q(\mathbf{sl_2})$-invariant complement - from the form of the matrix $\mathbf{F}$, or from the string diagram, it is clear that, given any vector, some power of $\mathbf{F}$ sends this to a multiple of the last basis vector (the left-most node). Hence every $U_q(\mathbf{sl_2})$-invariant submodule contains this basis vector, so pairs of complementary submodules cannot exist. Contrary to the non-root-of-unity case, then, we have here a representation which is not completely reducible.

When we classified the irreducible representations in Section 2, a crucial consequence of q not being a root of unity is that repeated applications of $\mathbf{E}$ or $\mathbf{F}$ cannot send a weight space back to itself. This restriction disappears when $q^3 = 1$: given $v \in V_\lambda$ a highest weight vector, $\mathbf{F}^3 v \in V_{q^{-6}\lambda} = V_\lambda$, using (QSR2) as in (2.6). So $\mathbf{F}^3 v$ could well be a multiple of v. Indeed, assume $\mathbf{F}^3 v = v$, and continue our analysis in the usual way by working out the matrix $\mathbf{E}$ using (QSR3), as in (2.7). As usual, this calculation shows that $\mathbf{E}$ maps the span of $\{v, \mathbf{F}v, \mathbf{F}^2 v, ...\}$ to itself, so the linearly independent members of $\{v, \mathbf{F}v, \mathbf{F}^2 v, ...\}$ form a basis. In this example, the basis is $\{v, \mathbf{F}v, \mathbf{F}^2 v\}$, with respect to which (after some manipulation with the α_is,):

$$\mathbf{E} = \tfrac{1}{q-q^{-1}} \begin{pmatrix} & \lambda - \lambda^{-1} & \\ & & q\lambda^{-1} - q^{-1}\lambda \\ & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}; \mathbf{K} = \begin{pmatrix} \lambda & & \\ & q^{-2}\lambda & \\ & & q^{-4}\lambda \end{pmatrix} \tag{3.4}$$

In the non-root-of-unity case, our next step would be to set $\mathbf{E}\mathbf{F}^{n+1} v = 0$ and obtain a condition on λ. This is unnecessary here as α_3 contains $q^3 - q^{-3} = 0$ as a factor. Indeed, (3.4) satisfies the quantised Serre relations, for any $\lambda \in \mathbb{C}$, and it turns out to always be irreducible.

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Figure 3.5 - The representation $Z_1(\lambda)$, with $q^3 = 1$

In the pictorial description of Figure 3.5, we see that $\mathbf{E}, \mathbf{F}$ are "rotating" the weight spaces, compared to the "translation" we saw previously. We shall see that this "rotation" is typical of irreducible representations in the root-of-unity case.

Both phenomena exhibited above have analogues in the representations of classical $\boldsymbol{sl_2}$ over $\mathbb{F}_3$, the field of 3 elements. Reducing modulo 3 the usual five-dimensional complex representation of gives $\boldsymbol{sl_2}$:

$$\mathbf{E} = \begin{pmatrix} & 4 & & & \\ & & 6 & & \\ & & & 6 & \\ & & & & 4 \\ & & & & \end{pmatrix} = \begin{pmatrix} & 1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 1 \\ & & & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & & & \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{pmatrix}; \mathbf{H} = \begin{pmatrix} 4 & & & & \\ & 2 & & & \\ & & 0 & & \\ & & & -2 & \\ & & & & -4 \end{pmatrix} \tag{3.6}$$

As in (3.1), $\mathbf{E}$ has more zero entries than expected, which creates $\boldsymbol{sl_2}$-invariant proper submodules. Again, from the form of $\mathbf{F}$ we see that this representation is not completely reducible.

We also have the following three-dimensional representation where $\mathbf{F}$ "rotates" the weight spaces (check that this does satisfy the Serre relations):

$$\mathbf{E} = \begin{pmatrix} & 2 & \\ & & 2 \\ & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}; \mathbf{H} = \begin{pmatrix} \lambda & & \\ & \lambda - 2 & \\ & & \lambda - 4 \end{pmatrix} \tag{3.7}$$

Our naive viewpoint $K = q^H$ gives some explanation for the similarities between representations of $U_q(\mathbf{sl_2})$ when q is a pth root of unity and representations of $\boldsymbol{sl_2}$ over fields of characteristic p, where p is a prime - namely, $K^p = q^{pH}$is the identity under both these circumstances.

3.2 Classification of irreducible representations

As in Section 2, we now proceed to give a complete description of all irreducible $U_q(\mathbf{sl_2})$ representations, following 2.13 of [Jantzen]. Throughout this subsection, q^2 will be a primitive lth root of unity - if q is an even root of unity, then l is half of the multiplicative order of q; otherwise l is the order of q. The result we work towards is:

Theorem. *Let q^2 be a primitive lth root of unity. Then the irreducible finite-dimensional representations of $U_q(\mathbf{sl_2})$ are:*

- *$L(n,+)$ for $n \leq l-2$. These are $n+1$-dimensional:*

$$\mathbf{E} = \begin{pmatrix} & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \ddots & \\ & & & & \alpha_n \\ & & & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}; \mathbf{K} = \begin{pmatrix} q^n & & & & \\ & q^{n-2} & & & \\ & & \ddots & & \\ & & & q^{2-n} & \\ & & & & q^{-n} \end{pmatrix}$$

  where $\alpha_r = \frac{q^r - q^{-r}}{(q-q^{-1})^2}\left(q^{n-r+1} - q^{-n+r-1}\right)$

- *$L(n,-)$ for $n \leq l-2$. These are $n+1$-dimensional:*

$$\mathbf{E} = -\begin{pmatrix} & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \ddots & \\ & & & & \alpha_n \\ & & & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}; \mathbf{K} = -\begin{pmatrix} q^n & & & & \\ & q^{n-2} & & & \\ & & \ddots & & \\ & & & q^{2-n} & \\ & & & & q^{-n} \end{pmatrix}$$

  where $\alpha_r = \frac{q^r - q^{-r}}{(q-q^{-1})^2}\left(q^{n-r+1} - q^{-n+r-1}\right)$

- *$Z_0(\lambda)$ for $\lambda \in \mathbb{C}^\times \backslash \{\pm 1, \pm q, \pm q^2, ..., \pm q^{l-2}\}$, which are l-dimensional:*

$$\mathbf{E} = -\begin{pmatrix} & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \ddots & \\ & & & & \alpha_{l-1} \\ & & & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}; \mathbf{K} = -\begin{pmatrix} \lambda & & & & \\ & q^{-2}\lambda & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & q^{-2(l-1)}\lambda \end{pmatrix}$$

  where $\alpha_r = \frac{q^r - q^{-r}}{(q-q^{-1})^2}\left(q^{1-r}\lambda - q^{r-1}\lambda^{-1}\right)$

- *$Z_b(\lambda)$ for $\lambda \in \mathbb{C}^\times, b \in \mathbb{C}^\times$, with the extra identifications $Z_b(q^{r-1}) \cong Z_b(q^{-r-1})$, $Z_b(-q^{r-1}) \cong Z_b(-q^{-r-1})$ for $1 \leq r \leq l-1$. These are l-dimensional:*

$$\mathbf{E} = -\begin{pmatrix} & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \ddots & \\ & & & & \alpha_{l-1} \\ & & & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & & & b \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}; \mathbf{K} = -\begin{pmatrix} \lambda & & & & \\ & q^{-2}\lambda & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & q^{-2(l-1)}\lambda \end{pmatrix}$$

where $\alpha_r = \frac{q^r - q^{-r}}{(q-q^{-1})^2}\left(q^{1-r}\lambda - q^{r-1}\lambda^{-1}\right)$

- *${}^\omega Z_b(\lambda)$ for $\lambda \in \mathbb{C}^\times, b \in \mathbb{C}^\times$, with the extra identifications ${}^\omega Z_b(q^{1-r}) \cong^\omega Z_b(q^{1+r})$, ${}^\omega Z_b(-q^{1-r}) \cong^\omega Z_b(-q^{1+r})$ for $1 \leq r \leq l-1$. These are l-dimensional:*

$$\mathbf{E} = \begin{pmatrix} & & & & b \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}; \mathbf{F} = -\begin{pmatrix} & {}^\omega\alpha_1 & & & \\ & & {}^\omega\alpha_2 & & \\ & & & \ddots & \\ & & & & {}^\omega\alpha_{l-1} \\ & & & & \end{pmatrix}; \mathbf{K} = -\begin{pmatrix} \lambda & & & & \\ & q^{2}\lambda & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & q^{2(l-1)}\lambda \end{pmatrix}$$

where ${}^\omega\alpha_r = \frac{q^r - q^{-r}}{(q-q^{-1})^2}\left(q^{1-r}\lambda^{-1} - q^{r-1}\lambda\right)$

- *$W_{a,b}(\lambda)$, for $\lambda, b, a \in \mathbb{C}^\times$ with $\alpha_r \neq -ab \forall r \geq 1$, up to the relation $(\lambda, a, b) \sim (q^{-2r}\lambda, \alpha_r/b + a, b)$. These are l-dimensional:*

$$\mathbf{E} = -\begin{pmatrix} & \alpha_{1+ab} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \alpha_{l-1+ab} \\ a & & & & \end{pmatrix}; \mathbf{F} = \begin{pmatrix} & & & & b \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}; \mathbf{K} = -\begin{pmatrix} \lambda & & & & \\ & q^{-2}\lambda & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & q^{-2(l-1)}\lambda \end{pmatrix}$$

where $\alpha_r = \frac{q^r - q^{-r}}{(q-q^{-1})^2}\left(q^{1-r}\lambda - q^{r-1}\lambda^{-1}\right)$

($W_{a,b}(\lambda)$ is non-standard notation; these representations do not seem to have a name in the literature.)

–

Figure 3.8 - Some irreducible representations of $U_q(\mathbf{sl_2})$, *when* $q^5 = 1$

Observe that, topologically, the parametrising set of $W_{a,b}(\lambda)$ is a three-dimensional complex space whilst the other parametrising sets have lower dimension. Hence, a generic irreducible representation has the form $W_{a,b}(\lambda)$, and we should think of the other five scenarios as degeneracies of $W_{a,b}(\lambda)$. Figure 3.8 displays the associated string diagrams for the case $l = 5$.

This looks considerably more complicated than the non-root-of-unity case: not only do we have more types of irreducible representations, most types have uncountably many members. Another major contrast is that the dimension of any one of these irreducible representations is at most l, whereas previously the irreducible representations had arbitrarily large dimension. Indeed, when q is not a root of unity, infinite-dimensional irreducible representations exist (for example, analogues of the Verma modules of $\boldsymbol{sl_2}$, see 2.4 of [Jantzen]), but the proof which follows does not invoke the finite-dimensional hypothesis, which means there are no infinite-dimensional irreducible representations in the root-of-unity setting. These differences arise because the centre of $U_q(\mathbf{sl_2})$ is much larger when q is a root of unity, as we shall see two paragraphs below.

As with the non-root-of-unity case, we start the proof by checking that these matrices satisfy the quantised Serre relations, so they define valid representations; this is a straightforward, if slightly long, calculation. Next we show that all these representations are irreducible. In every case, $\mathbf{K}$ is diagonal with distinct eigenvalues, so, by the same reasoning as in Subsection 2.2 (when q was not a root of unity) we need only show "transitive permutation" of the weight spaces. (Strictly speaking, this is not a permutation action as weight spaces may be sent to 0, but what I mean is that it is possible to reach any weight space from any other via $U_q(\mathbf{sl_2})$ action.) In $Z_b(\lambda)$ and $W_{a,b}(\lambda)$, this is achieved by powers of $\mathbf{F}$ alone (as $b \neq 0$); powers of $\mathbf{E}$ play this role for ${}^{\omega}Z_b(\lambda)$. In the other three families, sending one weight space to another may require $\mathbf{E}$ or $\mathbf{F}$ (depending on the "direction"); these function as desired since the restrictions on n and λ ensure that all α_r are non-zero.

The rest of this subsection will be spent proving that the above list indeed exhausts all the irreducible representations, and is irredundant. Compared to the non-root-of-unity case, the only extra idea in this proof is the examination of central elements:

Lemma. *Let q^2 be a primitive lth root of unity (so $q^{2l} = 1$). Then E^l, F^l are central elements of $U_q(\mathbf{sl_2})$.*

It suffices to show that E^l, F^l commute with the generators E, F, K, and we do so simply by applying the quantised Serre relations repeatedly. By (QSR1) and (QSR2) respectively:

$$\begin{aligned} KE^rK^{-1} &= (KEK^{-1})^r = (q^2E)^r = q^{2r}E^r \\ &\Rightarrow KE^r = q^{2r}E^rK,\ K^{-1}F^r = q^{-2r}E^rK^{-1} \end{aligned}$$

$$\begin{aligned} KF^rK^{-1} &= (KFK^{-1})^r = (q^{-2}F)^r = q^{-2r}F^r \\ &\Rightarrow KF^r = q^{-2r}F^rK,\ K^{-1}F^r = q^{2r}F^rK^{-1} \end{aligned}$$

In particular, taking $r = l$ shows that E^l, F^l commute with K. Next, use (QSR3) to show by induction that

$$EF^r = \frac{K - K^{-1}}{q - q^{-1}} F^{r-1} + F \frac{K - K^{-1}}{q - q^{-1}} F^{r-2} + \cdots + F^{r-1} \frac{K - K^{-1}}{q - q^{-1}} + F^r E \quad (3.11)$$

$$FE^r = E^r F - E^{r-1} \frac{K - K^{-1}}{q - q^{-1}} - E^{r-2} \frac{K - K^{-1}}{q - q^{-1}} E + \cdots - \frac{K - K^{-1}}{q - q^{-1}} E^{r-1} \quad (3.12)$$

Using (3.9) and (3.10) to substitute for each term in the equations above:

$$EF^r = \left[q^{-2(r-1)} + \cdots + q^{-2} + 1\right] \frac{F^{r-1}K}{q - q^{-1}} - \left[q^{2(r-1)} + \cdots + q^2 + 1\right] \frac{F^{r-1}K^{-1}}{q - q^{-1}} + F^r E \quad (3.13)$$

$$FE^r = E^r F - \left[1 + q^2 + \cdots + q^{2(r-1)}\right] \frac{E^{r-1}K}{q - q^{-1}} + \left[1 + q^{-2} + \cdots + q^{-2(r-1)}\right] \frac{E^{r-1}K^{-1}}{q - q^{-1}} \quad 3.14$$

Again, set $r = l$. Each square bracket above is the sum of all the distinct lth roots of unity, which is zero, so E^l commutes with F, F^l commutes with E, as required. QED

It's worth remarking that, throughout the above proof, we applied identical arguments to E and to F - this symmetry between E and F is something we will exploit in case 3 below. Also notice that the commutation formulas (3.13) and (3.14) show that no power of E or F can be central if q is not a root of unity.

The point of the above lemma is that, given any irreducible representation V, Schur's lemma says that E^l, F^l must act as scalars. So four scenarios can occur:

Case 1 $\mathbf{E}^l, \mathbf{F}^l$ *are both the zero matrix :* $L(n, \pm)$ *and* $Z_0(\lambda)$

Here, $\ker \mathbf{E}$ is non-trivial, and (QSR1) shows that $\mathbf{K}(\ker \mathbf{E}) \subseteq \ker \mathbf{E}$. Hence $\ker \mathbf{E}$ contains some $\mathbf{K}$-eigenvector v, with some corresponding eigenvalue λ.

As we did in (2.6), for the non-root-of-unity case, repeated applications of (QSR2) give $\mathbf{F}^r v \in V_{q^{-2r}\lambda}$. Take n minimal with $\mathbf{F}^{n+1} v = 0$; since $\mathbf{F}^l$ is the zero map, such an n exists, and $n + 1 \leq l$. As $q^{-2r}\lambda$ are distinct for $0 \leq r < l$, and hence for $0 \leq r < n$, $\{v, \mathbf{F}v, ..., \mathbf{F}^n v\}$ is a set of linearly independent weight vectors. $\mathbf{K}$-action on this set is clear from the fact that $\mathbf{F}^r v \in V_{q^{-2r}\lambda}$.. To find the $\mathbf{E}$-action, reuse (2.7) and (2.8) without change - that is, use (QSR3) to show inductively that $\mathbf{E}\mathbf{F}^r v = \alpha_r \mathbf{F}^{r-1} v$ where

$$\alpha_r = \frac{q^r - q^{-r}}{(q - q^{-1})^2} \left(q^{1-r}\lambda - q^{r-1}\lambda^{-1}\right) \quad (3.15)$$

As in the non-root-of-unity setting, $\mathbf{E}\mathbf{F}^{n+1}v = 0$ implies that

$$\alpha_{n+1} = \frac{q^{n+1} - q^{-n-1}}{\left(q - q^{-1}\right)^2} \left(q^n\lambda - q^{-n}\lambda^{-1}\right) = 0 \tag{3.16}$$

Since $q^l = q^{-l} = \pm 1$ (depending on whether q has odd or even order), (3.16) is true for $n = l - 1$ irrespective of λ, and in this case we find no restrictions on our highest weight ; this is the representation $Z_0(\lambda)$. Otherwise we obtain the familiar condition $\lambda = \pm q^n$, which gives the previously seen modules $L(n, \pm)$. (Observe that $L(l-1, \pm) \cong Z_0(\pm q^{l-1})$, which is why n runs up to $l - 2$ only in the statement of the theorem.)

Note that, if $\alpha_r = 0$, then $\mathbf{E}\mathbf{F}^r v = 0$, so, as we saw in the first example of this section, $\{\mathbf{F}^r v, \mathbf{F}^{r+1}v, ..., \mathbf{F}^n v\}$ spans an $U_q(\mathbf{sl_2})$-invariant submodule. Hence we must require $\alpha_r \neq 0 \forall r, 1 \leq r < n+1$, or equivalently, $\lambda = \pm q^{r-1}$, by (3.15). This accounts for the removal of $\{\pm 1, \pm q, ..., \pm q^{l-2}\}$ from the range of λ for the representations $Z_0(\lambda)$.

Case 2 $\mathbf{E}^l = 0, \mathbf{F}^l = b\mathbf{I} \neq 0$: $Z_b(\lambda)$

As above, $\ker \mathbf{E}$ is non-trivial, so we can find a highest weight vector $v \in V_q$ for some $\lambda \in \mathbb{C}^\times$. $\mathbf{F}^l v = bv \neq 0$, so $\mathbf{F}^r v \neq 0 \forall r \geq 0$. By (QSR2), as in (2.6), $\mathbf{F}^r v \in V_{q^{-2r}\lambda}$, and these weight spaces are distinct for $0 \leq r < l$, so $\{v, \mathbf{F}v, ..., \mathbf{F}^{l-1}v\}$ is a linearly independent set. Since (2.7), the formula for $\mathbf{E}\mathbf{F}^r v$, still applies, $U_q(\mathbf{sl_2})$-action with respect to the basis $\{v, \mathbf{F}v, ..., \mathbf{F}^{l-1}v\}$ agrees with the matrices given in the statement of the theorem for $Z_b(\lambda)$.

Case 3 $\mathbf{E}^l = b\mathbf{I} \neq 0, \mathbf{F}^l = 0$: ${}^\omega Z_b(\lambda)$

Recalling what we said earlier about symmetry, we tackle this by essentially exchanging the roles of $\mathbf{E}$ and $\mathbf{F}$. So take $v \in \ker \mathbf{F}$ with $\mathbf{K}v = \lambda v$ ((QSR2) ensures that $\mathbf{K}$ preserves $\ker \mathbf{F}$). By (QSR1), as in (2.5), $\mathbf{E}^r v \in V_{q^{2r}\lambda}$, and these weight spaces are distinct for $0 \leq r < l$, so $\{v, \mathbf{E}v, ..., \mathbf{E}^{l-1}v\}$ is a linearly independent set. The inductive computation in (2.7), with $\mathbf{E}$ and $\mathbf{F}$ exchanged, gives

$$\mathbf{F}\mathbf{E}^r v = \frac{q^r - q^{-r}}{\left(q - q^{-1}\right)^2} \left(q^{1-r}\lambda^{-1} - q^{r-1}\lambda\right) \mathbf{E}^{r-1} v \forall r \geq 1 \tag{3.17}$$

so, with respect to the basis $\{v, \mathbf{E}v, ..., \mathbf{E}^{l-1}v\}$, $\mathbf{E}, \mathbf{F}, \mathbf{K}$ have the form shown in the theorem for ${}^\omega Z_b(\lambda)$. These representations are so named because they are "twisted" versions of $Z_b(\lambda)$, where $\mathbf{E}$ and $\mathbf{F}$ are exchanged and q^{-1} replaces q.

Case 4 $\mathbf{E}^l$ non-zero, $\mathbf{F}^l = b\mathbf{I} \neq 0$: $W_{a,b}(\lambda)$

Here we have no highest weight vectors, but there is no reason why we cannot still examine the $\mathbf{F}$-span of some $\mathbf{K}$-eigenvector $v \in V_\lambda$ - in fact, the usual argument shows that $\{v, \mathbf{F}v, ..., \mathbf{F}^{l-1}v\}$ are linearly independent with $\mathbf{F}^r v \in V_{q^{-2r}\lambda}$. What remains is to find out how $\mathbf{E}$ acts on this set.

Recall that the Casimir element

$$\Omega = FE + \frac{Kq - K^{-1}q^{-1}}{(q-q^{-1})^2} \tag{3.18}$$

is central in $U_q(\mathbf{sl_2})$; hence, by Schur's lemma, it must act on V as scalar multiplication by some $c \in \mathbb{C}$. So

$$\mathbf{FE}v = cv - \frac{\mathbf{K}q - \mathbf{K}^{-1}q^{-1}}{(q-q^{-1})^2}v = \text{some multiple of } v$$

$\mathbf{F}^l = b\mathbf{I} \neq 0$ allows us to invert $\mathbf{F}$: $\mathbf{F}^{-1} = b^{-1}\mathbf{F}^{l-1}$. So

$$\mathbf{E}v = \mathbf{F}^{-1}(\mathbf{FE}v) = b^{-1}\mathbf{F}^{l-1}(\text{some multiple of } v) = a\mathbf{F}^{l-1}v \tag{3.20}$$

for some $a \in \mathbb{C}^{\times}$. Now we apply the commutation relation (3.13) to find that

$$\begin{aligned} \mathbf{EF}^r v &= \sum_{i=0}^{r-1} \frac{q^{-2i}\lambda - q^{2i}\lambda^{-1}}{q-q^{-1}} \mathbf{F}^{r-1}v + \mathbf{F}^r\mathbf{E}v \\ &= \left[\frac{q^r - q^{-r}}{(q-q^{-1})^2}(q^{1-r}\lambda - q^{r-1}\lambda^{-1}) + ab\right]\mathbf{F}^{r-1}v \end{aligned}$$

Since $\mathbf{E}^l \neq 0$, these coefficients must all be non-zero. So we require $\alpha_r \neq -ab \forall r \geq 1$ (α_r as defined in the theorem) - this is $W_{a,b}(\lambda)$.

This completes the list of irreducible representations. As a result, the following algorithm allows us to determine the type and parameters of an unknown irreducible representation of $U_q(\mathbf{sl_2})$ (when $q^{2l} = 1$):

—

Observe that $W_{a,b}(\lambda)$ contains no distinguished weight spaces - its string diagram (Figure 3.8) is entirely symmetric. However, according to this algorithm, finding λ, a depends on a choice of weight. Hence these two parameters are not uniquely determined by the representation (unlike b, whose definition requires no choices). Indeed, assume we picked some other weight $q^{-2r}\lambda$(from the statement of the theorem we know all weights of $W_{a,b}(\lambda)$ have this form). Then our basis would be some multiple of the set $\{\mathbf{F}^r v, \mathbf{F}^{r+1}v, ..., \mathbf{F}^{l-1}v, bv, b\mathbf{F}v, ..., b\mathbf{F}^{r-1}v\}$, and we would name this representation is $W_{\alpha_r/b+a,b}(q^{-2r}\lambda)$. Hence we need to quotient out the parametrising set by the relation

$$(\lambda, a, b) \sim (q^{-2r}\lambda, \alpha_r/b + a, b) \tag{3.22}$$

A similar complication can occur with $Z_b(\lambda)$ and ${}^{\omega}Z_b(\lambda)$. In the generic case, the highest weight (or equivalent in ${}^{\omega}Z_b(\lambda)$) is unique, but this fails when $\alpha_r = 0$ (in $Z_b(\lambda)$) or (${}^{\omega}\alpha_r = 0$ in ${}^{\omega}Z_b(\lambda)$) for some r, $1 \leq r \leq l-1$. Take the first case, which happens exactly when $\lambda = \pm q^{r-1}$. As noted previously, this can hold for at most one value of r in this range. Then $q^{-2r}\lambda = \pm q^{-r-1}$ is another highest weight, so $Z_b(q^{r-1}) \cong Z_b(q^{-r-1})$ and $Z_b(-q^{r-1}) \cong Z_b(-q^{-r-1})$, as stated in the theorem. Figure 3.23 depicts the $r = 3$ case, with $q^5 = 1$. The analogous argument for shows that ${}^{\omega}Z_b(q^{1-r}) \cong^{\omega} Z_b(q^{1+r})$, and ${}^{\omega}Z_b(-q^{1-r}) \cong^{\omega} Z_b(-q^{1+r})$.

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Figure 3.23 - The irreducible representation $Z_b(q^2) \cong Z_b(q^4)$, with $q^5 = 1$

It follows from the algorithm that, up to these relations, all the representations listed in the theorem are distinct. Observe that the set of weights of any of the l-dimensional irreducible representations is precisely all the lth roots of some complex number λ^l. QED

3.3 Composition factors and tensor representations

In the absence of complete reducibility, there is no one-size-fits-all algorithm for identifying irreducible composition factors. We demonstrate a few useful techniques on the examples of $Z_0(\pm q^n)$, $L(n, \pm)$ and $W_{a,b}(\lambda) \otimes W_{c,d}(\mu)$.

Firstly, recall from Case 1 of the previous subsection that $Z_0(\pm q^n)$ is not irreducible for $n < l - 1$, since $\alpha_{n+1} = 0$. Let be a generating highest weight vector as usual, and write V^j for the span of the last $l - j$ basis vectors, ie $\{\mathbf{F}^j v, \mathbf{F}^{j+!} v, ..., \mathbf{F}^{l-1} v\}$ is a basis for V^j. Under this notation, V^{n+1} is $U_q(\mathbf{sl_2})$-invariant; to show that this is the only proper invariant submodule, we first need a

Lemma. *Suppose $v \in V^j$, $v \notin V^{j+1}$. Then any $U_q(\mathbf{sl_2})$-invariant submodule containing v must contain V^j.*

Apply induction on $l - j$. The assertion is clear if $j = l - 1$, since V^{l-1} is a one-dimensional subspace. If $j < l - 1$, then $\mathbf{F}v \in V^{j+1}$, $\mathbf{F}v \notin V^{j+2}$. Any $U_q(\mathbf{sl_2})$-invariant submodule containing v also contains $\mathbf{F}v$, hence, by inductive hypothesis, it contains V^{j+1}. So this submodule contains the span of v and V^{j+1}, which is precisely V^j. QED.

Now if $j \neq n + 1$, $\mathbf{E}(\mathbf{F}^j v) \in V^{j-1} \backslash V^j$, so a $U_q(\mathbf{sl_2})$-invariant submodule containing V^j must also contain V^{j-1}. Applying this argument iteratively shows that V^{n+1} is indeed the unique proper invariant submodule, therefore it must be irreducible. By dimension considerations, $V^{n+1} \cong L(l - n - 2, \pm)$, and the quotient $Z_0(\pm q^n)/V^{n+1} \cong L(n, \pm)$. To determine the signs, the flow diagram asks us to find the highest weights. The highest weight of $Z_0(\pm q^n)/V^{n+1}$ is that of $Z_0(\pm q^n)$, which is $\pm q^n$, and V^{n+1} has highest weight $\pm q^{n-2(n+1)} = \pm q^l q^{l-n-2}$. Recalling that $q^l = 1$ if q is an odd root of unity, and $q^l = -1$ otherwise, we have proved

Proposition. *Suppose q^2 is a primitive lth root of unity. Then:*

- *If q is an odd root of unity,*
  - *$Z_0(q^n)$ has unique proper invariant submodule $V^{n+1} \cong L(l-n-2,+)$ and the quotient $Z_0(q^n)/V^{n+1}$ is $L(n,+)$;*
  - *$Z_0(-q^n)$ has unique proper invariant submodule $V^{n+1} \cong L(l-n-2,-)$ and the quotient $Z_0(q^n)/V^{n+1}$ is $L(n,-)$;*
- *If q is an even root of unity,*
  - *$Z_0(q^n)$ has unique proper invariant submodule $V^{n+1} \cong L(l-n-2,-)$ and the quotient $Z_0(q^n)/V^{n+1}$ is $L(n,+)$;*
  - *$Z_0(-q^n)$ has unique proper invariant submodule $V^{n+1} \cong L(l-n-2,+)$ and the quotient $Z_0(q^n)/V^{n+1}$ is $L(n,-)$;*

Next, we generalise our example of $L(4,+)$from Subsection 3.1. Take $L(n,\pm)$ and write $n+1$ as $ml+s$ with $0 \le s < l$. So

$$\alpha_r = \frac{q^r - q^{-r}}{(q-q^{-1})^2}\left(q^{ml+s-r} - q^{-ml-s+r}\right) \tag{3.24}$$

which means $\alpha_l = \alpha_{2l} = \cdots = \alpha_{ml} = 0$, $\alpha_{l+s} = \alpha_{2l+s} = \cdots = \alpha_{ml+s} = 0$, and all other α_is are non-zero. Then, by the reasoning in the previous example, $V^l, V^{2l}, ..., V^{ml}, V^{l+s}, V^{2l+s}, ...V^{(m-1)l+s}$ are precisely the $U_q(\mathbf{sl_2})$-invariant submodules. So, if $s \neq 0$, the composition series for $L(n,\pm)$ reads

$$\{0\} \subseteq V^{ml} \subseteq V^{(m-1)l+s} \subseteq V^{(m-1)l} \subseteq \cdots \subseteq V^l \subseteq V^s \subseteq L(n,\pm) \tag{3.25}$$

Each composition factor has dimension $s < l$ or $l-s < l$, so the flow algorithm of the last subsection identifies them as $L(s-1,\pm)$ and $L(l-s-1,\pm)$. For example, when $q^3 = 1$, $L(4,+)$ has composition series $\{0\} \subseteq V^3 \subseteq V^2 \subseteq L(4,+)$, with alternating two- and one-dimensional quotients ($l = 3, m = 1, s = 2$), as in Figure 3.3. The signs of these representations again depend on the order of q. The highest weight of V^{rl}/V^{rl+s} is $\pm q^{ml+s-1-2rl} = \pm q^{s-1}(q^l)^m$; similarly, $V^{(r-1)l+s}/V^{rl}$ has highest weight $\pm q^{l-s-1}(q^l)^{m-1}$. This proves the following result, as illustrated in Figure 3.26:

Proposition. *Suppose q^2 is a primitive lth root of unity, and $n+1 = ml+s$ with $0 < s < l$. Then:*

- *If q is an odd root of unity, then*
  - *the composition factors of $L(n,+)$ are m copies of $L(s-1,+)$ alternated with $m-1$ copies of $L(l-s-1,+)$;*

- *the composition factors of $L(n,-)$ are m copies of $L(s-1,-)$ alternated with $m-1$ copies of $L(l-s-1,-)$;*

- *If q is an even root of unity, then*
  - *the composition factors of $L(n,+)$ are m copies of $L(s-1,\epsilon)$ alternated with $m-1$ copies of $L(l-s-1,-\epsilon)$;*
  - *the composition factors of $L(n,-)$ are m copies of $L(s-1,-\epsilon)$ alternated with $m-1$ copies of $L(l-s-1,\epsilon)$*

where ϵ is the sign of $(-1)^m$.

Figure 3.26 - The decomposition of $L(ml+s-1,\pm)$ when $q^{2l}=1$ and $s \neq 0$

Returning to $L(4,+)$ with $q^3=1$, its submodules $L(4,+)$, V^3 and V^2 have highest weights $q^4=q$, 1, $q^{-2}=q$ respectively, so the composition factors are $L(1,+)$, $L(0,+)$ and $L(1,+)$. If $q^6=1$, then $L(4,+)$ still has V^3 and V^2 as its proper invariant submodules (since l,m,s are unchanged), but now the highest weights are $q^4=-q$, 1, $q^{-2}=-q$, so the quotients become $L(1,-)$, $L(0,+)$ and $L(1,-)$ as guaranteed by the proposition.

Our final example investigates the structure of $W_{a,b}(\lambda)\otimes W_{c,d}(\mu)$, the tensor product of two generic irreducible representations. We will give an argument for complete reducibility in the generic case, then study in detail one of the "singular" cases.

For simplicity, take $q^4=1$, so $l=2$. Let v be any weight vector in $W_{a,b}(\lambda)$, w any weight vector in $W_{c,d}(\mu)$, and write $wt(v)$ for the weight of v. Then, as in (2.28):

$$\tilde{\mathbf{K}}(v\otimes w)=\mathbf{K}v\otimes\mathbf{K}w=wt(v)wt(w)v\otimes w \tag{3.27}$$

so $\{v\otimes w\}$ form a basis of weight vectors. (As before, $\tilde{\mathbf{E}},\tilde{\mathbf{F}},\tilde{\mathbf{K}}$ refer to $U_q(\mathbf{sl_2})$-action on $W_{a,b}(\lambda)\otimes W_{c,d}(\mu)$.) Hence the weights of $W_{a,b}(\lambda)\otimes W_{c,d}(\mu)$ are $\lambda\mu$ and $-\lambda\mu$, each with multiplicity 2.

Finding $\tilde{\mathbf{F}}$ is a similarly straightforward calculation. Since comultiplication Δ is an algebra homomorphism,

$$\begin{aligned}\Delta(F^2) &= (1\otimes F+F\otimes K^{-1})^2\\ &= 1\otimes F^2+F\otimes(FK^{-1}+K^{-1}F)+F^2\otimes K^{-2}\\ &= 1\otimes F^2+F^2\otimes K^{-2}\end{aligned}$$

using (QSR2) in the last equality. Therefore

$$\tilde{\mathbf{F}}^2=v\otimes dw+bv\otimes wt(w)^{-2}=(d+b\mu^{-2})v\otimes w \tag{3.29}$$

since all weights of $W_{c,d}(\mu)$ square to μ^2. As $\{v \otimes w\}$ is a basis for $W_{a,b}(\lambda) \otimes W_{c,d}(\mu)$, this shows that $\tilde{\mathbf{F}}^2 = \tilde{b}\tilde{\mathbf{I}}$ where $\tilde{b} = d + b\mu^{-2}$ .To generalise this to larger l, we can calculate $\Delta(F^r)$ inductively and find

$$\Delta(F^l) = 1 \otimes F^l + F^l \otimes K^{-1} \tag{3.30}$$

so $\tilde{\mathbf{F}} = (d + b\mu^{-l})\tilde{\mathbf{I}}$.

Encouraged by Subsection 2.3, the proof of complete reducibility in the non-root-of-unity case, we look for eigenspaces of the Casimir operator

$$\tilde{\mathbf{\Omega}} = \tilde{\mathbf{F}}\tilde{\mathbf{E}} + \frac{q\tilde{\mathbf{K}} - q^{-1}\tilde{\mathbf{K}}^{-1}}{(q - q^{-1})^2}$$

Since $\tilde{\mathbf{\Omega}}$ commutes with $\tilde{\mathbf{K}}$, $\tilde{\mathbf{\Omega}}$ preserves each weight space, so we can study its action on each weight space separately. Returning to $l = 2$, take weight vectors $v_0 \in [W_{a,b}(\lambda)]_\lambda$, $w_0 \in [W_{c,d}(\mu)]_\mu$ and set $v_1 = \tilde{\mathbf{F}}v_0$, $w_1 = \tilde{\mathbf{F}}w_0$. We have

$$\begin{aligned} \Delta(FE) &= (1 \otimes F + F \otimes K^{-1})(K \otimes E + E \otimes 1) \\ &= K \otimes FE + E \otimes F + FK \otimes K^{-1}E + FE \otimes K^{-1} \end{aligned}$$

so, with respect to the basis $\{v_0 \otimes w_0, v_1 \otimes w_1\}$ of the weight space $[W_{a,b}(\lambda) \otimes W_{c,d}(\mu)]_{\lambda\mu}$, $\tilde{\mathbf{\Omega}}$ is represented by

$$\begin{pmatrix} \lambda cd + \mu^{-1}ab - \frac{i\lambda\mu - i\lambda^{-1}\mu^{-1}}{4} & \left(\frac{\lambda-\lambda^{-1}}{2i} + ab\right)d - \lambda\mu^{-1}b\left(\frac{\mu-\mu^{-1}}{2i} + cd\right) \\ a - \lambda c\mu^{-1} & -\lambda\left(\frac{\mu-\mu^{-1}}{2i} + cd\right) - \mu^{-1}\left(\frac{\lambda-\lambda^{-1}}{2i} + ab\right) + \frac{i\lambda\mu - i\lambda^{-1}\mu^{-1}}{4} \end{pmatrix} \tag{3.33}$$

Generically, the characteristic polynomial of this matrix will have two distinct roots ν_1 and ν_2. Then the corresponding eigenvectors x_1, x_2 give a basis of $[W_{a,b}(\lambda) \otimes W_{c,d}(\mu)]_{\lambda\mu}$. Make the additional assumption that $\tilde{\mathbf{F}}^2 = (d + b\mu^{-2})\tilde{\mathbf{I}} \neq 0$, so $\tilde{\mathbf{F}}$ is an isomorphism. Then $\tilde{\mathbf{F}}x_1, \tilde{\mathbf{F}}x_2$ inherit linear independence from x_1, x_2 , so is a basis of $[W_{a,b}(\lambda) \otimes W_{c,d}(\mu)]_{-\lambda\mu}$.

We see from the definition of $\tilde{\mathbf{\Omega}}$,

$$\tilde{\mathbf{E}}x_1 = \tilde{b}^{-1}\tilde{\mathbf{F}}^2\tilde{\mathbf{E}}x_1 = \tilde{b}^{-1}\tilde{\mathbf{F}}\left(\nu_1 x_1 - \frac{q\tilde{\mathbf{K}} + q^{-1}\tilde{\mathbf{K}}^{-1}}{q - q^{-1}}x_1\right) = \tilde{a}_1\tilde{\mathbf{F}}x_1 \tag{3.34}$$

for some $\tilde{a}_1 \in \mathbb{C}$, and

$$\tilde{\mathbf{E}}(\tilde{\mathbf{F}}x_1) = \tilde{\mathbf{F}}\tilde{\mathbf{E}}x_1 + \frac{\tilde{\mathbf{K}} + \tilde{\mathbf{K}}^{-1}}{q - q^{-1}}x_1 = \tilde{a}_1\tilde{b}x_1 + \frac{\lambda\mu + \lambda^{-1}\mu^{-1}}{q - q^{-1}}x_1 \tag{3.35}$$

using (3.33) in the second equality. So $\{x_1, \tilde{\mathbf{F}}x_1\}$ form the basis of a $U_q(\mathbf{sl_2})$-module, and the same is true for $\{x_2, \tilde{\mathbf{F}}x_2\}$, as we can find $\tilde{a}_2 \in \mathbb{C}$ with $\tilde{\mathbf{E}}x_2 =$

$\tilde{a}_2\tilde{\mathbf{F}}x_2$. Since $\{x_1, \tilde{\mathbf{F}}x_1, x_2, \tilde{\mathbf{F}}x_2\}$ is a basis for $W_{a,b}(\lambda)\otimes W_{c,d}(\mu)$, these two submodules are disjoint. Hence, in the generic case where $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{a}_1\tilde{b} + \frac{\lambda\mu+\lambda^{-1}\mu^{-1}}{q-q^{-1}}$, $\tilde{a}_2\tilde{b} + \frac{\lambda\mu+\lambda^{-1}\mu^{-1}}{q-q^{-1}}$ are all non-zero,

$$W_{a,b}(\lambda)\otimes W_{c,d}(\mu) = W_{\tilde{a}_1,\tilde{b}}(\lambda\mu)\oplus W_{\tilde{a}_2,\tilde{b}}(\lambda\mu) \tag{3.36}$$

where $W_{\tilde{a}_i,\tilde{b}}(\lambda\mu)$ has basis $\{x_i, \tilde{\mathbf{F}}x_i\}$. (If some of the quantities above are zero, then we still have two irreducible two-dimensional summands, but they may be of type $Z_b(\lambda)$.)

Applying the ideas above to general gives the following

Theorem A. *If $\tilde{\mathbf{\Omega}}$ has linearly independent eigenvectors $\{x_1, x_2, ..., x_l\} \in [W_{a,b}(\lambda)\otimes W_{c,d}(\mu)]_{\lambda\mu}$ then $W_{a,b}(\lambda)\otimes W_{c,d}(\mu) = \bigoplus_{i=1}^{l} X^i$ where X^i has basis $\{x_i, \tilde{\mathbf{F}}x_i, ..., \tilde{\mathbf{F}}^{l-1}x_i\}$. Furthermore, if $\tilde{b} := d + b\mu^{-l}$ is non-zero, then every X^i is irreducible.*

A generalisation of (3.34) will compute $\tilde{\mathbf{E}}x_i$ and thus identify the type and parameters of X^i. This theorem has a partial converse

Theorem B. *Suppose $\tilde{b} := d+b\mu^{-l} \neq 0$ and $W_{a,b}(\lambda)\otimes W_{c,d}(\mu)$ is completely reducible. Then $\tilde{\mathbf{\Omega}}$ has l linearly independent eigenvectors in $[W_{a,b}(\lambda)\otimes W_{c,d}(\mu)]_{\lambda\mu}$.*

This is because $\tilde{\mathbf{\Omega}}$ acts as a scalar on each irreducible summand; since $\tilde{b} =\neq 0$, these all have l weight spaces of distinct weights and hence must each contain a linear subspace of $[W_{a,b}(\lambda)\otimes W_{c,d}(\mu)]_{\lambda\mu}$.

Theorem B states that, if $\tilde{\mathbf{\Omega}}$ is not diagonalisable, then this tensor product representation is not completely reducible. We now construct one such example where $q^4 = 1$, $\lambda = 1$, $\mu = -1$ - this will greatly simplify our calculations, since the α_is of both factors, which are $\frac{\lambda-\lambda^{-1}}{q-q^{-1}}$ and $\frac{\mu-\mu^{-1}}{q-q^{-1}}$, vanish, as does $i\lambda\mu - i\lambda^{-1}\mu^{-1}$. This last condition means that $[W_{a,b}(1)\otimes W_{c,d}(-1)]_{-1} \subseteq \ker \frac{q\tilde{\mathbf{K}}-q^{-1}\tilde{\mathbf{K}}^{-1}}{(q-q^{-1})^2}$, so $\tilde{\mathbf{\Omega}}$ and $\tilde{\mathbf{F}}\tilde{\mathbf{E}}$ agree on this subspace, and is given by

$$\begin{pmatrix} cd-ab & abd+bcd \\ a+c & -cd+ab \end{pmatrix} \tag{3.37}$$

This has characteristic polynomial $\xi^2 + (b+d)(a^2b + c^2d)$, which gives repeated roots precisely when $b + d = 0$ or $a^2b + c^2d = 0$. Assume that the latter holds, and that $b + d \neq 0$, so $\tilde{\mathbf{F}}$ is again an isomorphism. This forces $cd - ab \neq 0$, $a + c \neq 0$. Thus $x_1 := (cd-ab)v_0\otimes w_0 + (a+c)v_1\otimes w_1$, $x_2 := v_0\otimes w_0$ is a basis for $[W_{a,b}(1)\otimes W_{c,d}(-1)]_{-1}$.

By direct computation, $\tilde{\mathbf{F}}\tilde{\mathbf{E}} = \tilde{\mathbf{\Omega}}$ has the form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ with respect to the basis $\{x_1, x_2\}$. Since $\tilde{\mathbf{F}}$ is injective,$\tilde{\mathbf{F}}\tilde{\mathbf{E}}x_1 = 0$ implies $\tilde{\mathbf{E}}x_1 = 0$. By (QSR3)

$$\tilde{\mathbf{E}}(\tilde{\mathbf{F}}x_1) = \tilde{\mathbf{F}}\tilde{\mathbf{E}}x_1 + \frac{\tilde{\mathbf{K}}+\tilde{\mathbf{K}}^{-1}}{q-q^{-1}}x_1 = 0 + 0 \tag{3.38}$$

So $\{x_1, \tilde{\mathbf{F}}x_1\}$spans a $U_q(\mathbf{sl_2})$-invariant submodule where $\tilde{\mathbf{E}}$-action is zero. Since x_1 belongs to the -1-weight space, this submodule is $Z_{\tilde{b}}(-1) \cong Z_{\tilde{b}}(1)$.

As we saw earlier, linear independence of x_1, x_2 passes through to $\tilde{\mathbf{F}}x_1, \tilde{\mathbf{F}}x_2$, so $\{\tilde{\mathbf{F}}x_1, \tilde{\mathbf{F}}x_2\}$ is a basis for the 1-weight space of $W_{a,b}(1) \otimes W_{c,d}(-1)$. So, to complete our analysis, we need to find how $\tilde{\mathbf{E}}$ acts on x_2 and $\tilde{\mathbf{F}}x_2$. Since $\tilde{\mathbf{F}}\tilde{\mathbf{E}}x_2 = x_1$,

$$\tilde{\mathbf{E}}(x_2) = \tilde{b}^{-1}\tilde{\mathbf{F}}^2\tilde{\mathbf{E}}x_2 + \tilde{b}^{-1}\tilde{\mathbf{F}}x_1 \tag{3.39}$$

and, using (QSR3) as in (3.37):

$$\tilde{\mathbf{E}}(\tilde{\mathbf{F}}x_2) = \tilde{\mathbf{F}}\tilde{\mathbf{E}}x_2 + \frac{\tilde{\mathbf{K}} + \tilde{\mathbf{K}}^{-1}}{q - q^{-1}}x_2 = x_1 \tag{3.40}$$

Hence, with respect to the basis $\{x_1, \tilde{\mathbf{F}}x_1, x_2, \tilde{\mathbf{F}}x_2\}$, the representation $W_{a,b}(1) \otimes W_{c,d}(-1)$ is given by the matrices:

$$\tilde{\mathbf{E}} = \begin{pmatrix} & 0 & & 1 \\ 0 & & \tilde{b}^{-1} & \\ & & & 0 \\ & & 0 & \end{pmatrix}; \tilde{\mathbf{F}} = \begin{pmatrix} & \tilde{b} & & \\ 1 & & & \\ & & & \tilde{b} \\ & & 1 & \end{pmatrix}; \tilde{\mathbf{K}} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \tag{3.41}$$

From either the matrices (3.41) or the string diagram Figure 3.42, we deduce that the two composition factors of $W_{a,b}(1) \otimes W_{c,d}(-1)$ are both $Z_{\tilde{b}}(-1) \cong Z_{\tilde{b}}(1)$.

Figure 3.42 - The representation $W_{a,b}(1) \otimes W_{c,d}(-1)$, *when* $a^2b + c^2d = 0$ *and* $b + d \neq 0$, *which is not completely reducible*

Conclusion

Despite restricting our attention to $U_q(\mathbf{sl_2})$, we have actually glimpsed many of the important properties of general quantised enveloping algebras. Such an algebra is created by taking a simple Lie algebra and suitably "bending" its Serre relations - I will not reproduce the eyesore of these formulae here, but the spirit is no different from our Section 1. So $U_q(\mathbf{g})$ is generated by $E_1, E_2, ..., E_n, F_1, F_2, ...F_n, K_1, K_2, ...K_n$, where n is the number of simple roots in $\mathbf{g}$; this is again a Hopf algebra.

We can generalise the concept of weight vector to denote vectors which are simultaneously eigenvectors of all the $\mathbf{K}_i$; a highest weight vector is then one which is annihilated by all $\mathbf{E}_i$. Then, in the non-root-of-unity setting, the $U_q(\mathbf{sl_2})$-theory we developed shows (via a modified classical argument) that any finite-dimensional irreducible representation of $U_q(\mathbf{g})$ has a basis of weight vectors. In particular, as we saw with $U_q(\mathbf{sl_2})$, each such representation is uniquely specified by its highest weight. In Section 2, we labelled these highest weights by a positive integer and a sign $\epsilon \in \{+, -\}$ which relates to $\mathbf{K}$-action. In $U_q(\mathbf{g})$, there are n analogues of $\mathbf{K}$, so n signs are needed to describe the irreducible $U_q(\mathbf{g})$-modules, in addition to a dominant weight of $\mathbf{g}$ which plays the role of the positive integers. The usual proof of this is directly adapted from that of the classical version (that dominant weights index the finite-dimensional irreducible representations of $\mathbf{g}$). The fact that this combinatorial data specifies finite-dimensional irreducible representations uniquely means we can define analogous character formulae and use them to decompose arbitrary finite-dimensional representations, since these are again completely reducible.

As for the root-of-unity case, things are again complicated by the larger centre, which contains lth powers of all the $\mathbf{E}_i$s and $\mathbf{F}_i$s. As we saw with $U_q(\mathbf{sl_2})$, the dimension of irreducible representations is bounded (by l^N, where N is the number of positive roots in $\mathbf{g}$) and the great majority of them have the maximal dimension.

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