Monomial Bases for Combinatorial Hopf Algebras

C.Y. Amy Pang
Department of Mathematics, Hong Kong Baptist University

based on “Hopf algebras of parking functions and decorated planar trees”, joint work with Nantel Bergeron, Rafael Gonzalez d’Leon, Shu Xiao Li, Yannic Vargas

presented at AlCoVE, 15 June 2021
The Monomial basis of QSym

\[ M_{2,1,2} = \sum_{i<j<k} x_i^2 x_j x_k^2 = x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + \ldots + x_3^2 x_1 x_4 + \ldots + x_2^2 x_4 x_7 + \ldots \]

indexed by compositions

The degree of a composition is the number of squares.
The Monomial basis of QSym

\[ M_{2,1,2} = \sum_{i<j<k} x_i^2 x_j^1 x_k^2 = x_1^2 x_2^1 x_3^2 + x_1^2 x_2^1 x_4^2 + \ldots x_1^2 x_3^1 x_4^2 + \ldots x_2^1 x_4^1 x_7^2 + \ldots \]

indexed by compositions

The degree of a composition is the number of squares.

The product (combining of compositions) in the M basis expands positively - it is quasishuffle of blocks:

\[ M_{1,2} M_1 = \sum_{i<j} x_i^1 x_j^2 \sum_k x_k^1 = \]

\[ = M_{1,2,1} + M_{1,3} + M_{1,1,2} + M_{2,2} + M_{1,1,2} \]

Indexed by compositions:

- \( i < j < k \)
- \( i < j = k \)
- \( i < k < j \)
- \( i = k < j \)
- \( k < i < j \)
The Monomial basis of QSym

QSym is a Hopf algebra, i.e. it has a coproduct $\Delta : \text{QSym} \rightarrow \text{QSym} \otimes \text{QSym}$ (breaking of compositions), compatible with its product.

Given $f(x_1, x_2, \ldots)$, let $f(y_1, y_2, \ldots, z_1, z_2, \ldots) = \sum_i g_i(y_1, y_2, \ldots) h_i(z_1, z_2, \ldots)$. Let $\Delta(f) = \sum_i g_i \otimes h_i$, and $\Delta+(f) = \Delta(f) - 1 \otimes f - f \otimes 1$.

The coproduct in the M basis

$$\Delta_+(M_{1,2,1}) = \Delta_+ \left( \sum_{i<j<k} x_i^1 x_j^2 x_k^1 \right) = M_1 \otimes M_{2,1} + y_i^1 z_j^2 z_k^1$$
The Monomial basis of QSym

QSym is a Hopf algebra, i.e. it has a coproduct \( \Delta : \text{QSym} \rightarrow \text{QSym} \otimes \text{QSym} \) (breaking of compositions), compatible with its product.

Given \( f(x_1, x_2, \ldots) \), let \( f(y_1, y_2, \ldots, z_1, z_2, \ldots) = \sum_i g_i(y_1, y_2, \ldots) h_i(z_1, z_2, \ldots) \).
Let \( \Delta(f) = \sum_i g_i \otimes h_i \), and \( \Delta^+(f) = \Delta(f) - 1 \otimes f - f \otimes 1 \).

The coproduct in the M basis is deconcatenate between blocks

\[
\Delta^+(M_{1,2,1}) = \Delta^+ \left( \sum_{i<j<k} x_i^1 x_j^2 x_k^1 \right) = M_1 \otimes M_{2,1} + M_{1,2} \otimes M_1
\]

i.e. compositions have a “unique factorisation” and the coproduct deconcatenates the factors – i.e. this coproduct is cofree (i.e. the dual basis in the dual Hopf algebra is free)
Other bases of QSym

M-basis of QSym → combining compositions by quasishuffle of blocks; breaking compositions by deconcatenation between blocks.

?-basis of QSym → combining compositions by ???; breaking compositions by ???.

Fundamental basis: \( F_\alpha = \sum_{\beta \geq \alpha} M_\beta \) using the refinement order

e.g. \( F_{3,1} = M_{3,1} + M_{2,1,1} + M_{1,2,1} + M_{1,1,1,1} \)
The Fundamental basis of $\text{QSym}$

The product in the $F$ basis is the shuffle of squares:

$$F_{1,2}F_1 = F_{1,3} + F_{1,2,1} + F_{2,2} + F_{1,1,2}$$

$$F_{1,1}F_2 = F_{1,3} + F_{2,2} + F_{1,1,2} + F_{3,1} + F_{1,2,1} + F_{2,1,1}$$
The Fundamental basis of QSym

The product in the F basis is the shuffle of squares:

\[
F_{1,2}F_1 = F_{1,3} + F_{1,2,1} + F_{2,2} + F_{1,1,2}
\]

\[
F_{1,1}F_2 = F_{1,3} + F_{2,2} + F_{1,1,2} + F_{3,1} + F_{1,2,1} + F_{2,1,1}
\]

The coproduct in the F basis is deconcatenate between squares - which produces one term in each degree:

\[
\Delta_+(F_{3,1}) = \Delta_+(M_{3,1} + M_{2,1,1} + M_{1,2,1} + M_{1,1,1,1})
\]

\[
= M_1 \otimes M_{2,1} + M_1 \otimes M_{1,1,1}
\]

\[
+ M_2 \otimes M_{1,1} + M_{1,1} \otimes M_{1,1}
\]

\[
+ M_3 \otimes M_1 + M_{2,1} \otimes M_1 + M_{1,2} \otimes M_1 + M_{1,1,1} \otimes M_1
\]

\[
= F_1 \otimes F_{2,1} + F_2 \otimes F_{1,1} + F_3 \otimes F_1
\]
Other Hopf algebras

- Many other Hopf algebras have a F-like basis:
  - The product is some shuffling of the ground set;
  - The coproduct is deconcatenation of the ground set, producing one term of each degree.

- Often, $\exists$ a poset on the underlying objects, and we can define a M-like basis by $F_\alpha = \sum_{\beta \geq \alpha} M_\beta$:
  - The coproduct in the M basis is cofree, given by deconcatenation “between factors” of a unique factorisation - this is proved ad-hoc;
  - The product is ???.

We distill the Aguiar-Sottile approach into axioms: check that shuffling, deconcatenation and the poset interact in these correct ways, and you are guaranteed a M basis with positive product and cofree coproduct.
Axioms for coproduct

**Δ1.** Coproduct in fundamental basis is “deconcatenate everywhere”

\[
\Delta_+(F_f) = \sum_{i=1}^{\deg f - 1} F_i f \otimes F_f i; \quad \deg f = i
\]

Example: a new Hopf algebra $\text{PSym}$ of parking functions, viewed as binary trees labelled with a permutation satisfying some conditions

\[
\Delta_+ \left( \begin{array}{c} 5 \\ 3 \\ 4 \\ 2 \\ 1 \end{array} \right) = 1 \otimes \begin{array}{c} 3 \\ 4 \\ 2 \\ 1 \end{array} + 2 \otimes \begin{array}{c} 1 \\ 3 \\ 2 \\ 1 \end{array} + 3 \otimes \begin{array}{c} 1 \\ 2 \\ 3 \\ 2 \end{array} + 4 \otimes \begin{array}{c} 1 \\ 2 \\ 3 \\ 2 \end{array}
\]
Axioms for coproduct

\[ \Delta_+(F_f) = \sum_{i=1}^{\deg f - 1} F_i f \otimes F_{fi}; \quad \deg^i f = i \]

\[ \Delta_+(\begin{pmatrix} 5 & 3 \\ 4 & 2 \\ 1 \end{pmatrix}) = 1 \otimes \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} + 2 \otimes \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} + 3 \otimes \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} + 4 \otimes \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + 5 \otimes \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \]

\[ \Delta_2. \text{Deconcatenation is order-preserving: if } f \leq f', \text{ then } i f \leq i f' \text{ and } f_i \leq f_{i'} \text{.} \]

\[ f \leq f' \text{ if their trees are comparable in Tamari order and their permutations are comparable in weak order.} \]
Axioms for coproduct (cont’d)

**Theorem**: If $\Delta_{1-3}$ are satisfied, and we define $M$ basis by $F_f = \sum_{g \geq f} M_g$, then

$$\Delta_+(M_f) = \sum_{i \in \text{GDes}(f)} M_{if} \otimes M_{fi}. \quad \text{(deconcatenate “between blocks”) \quad \text{)}$$
Axioms for coproduct (cont’d)

Δ3. “Maximal concatenation” is well defined:

Given $g, h$, \( \exists \text{ unique} \ \max\{f | ^i f = g, f^i = h\} := g/h; \)

\[ \begin{array}{c}
\begin{array}{c}
\text{e.g. } \begin{array}{c}
\begin{array}{c}
\text{1, 1}
\end{array}
\begin{array}{c}
\text{2}
\end{array}
\end{array}
= \max\{\begin{array}{c}
\begin{array}{c}
\text{1, 1, 2}
\end{array}
\begin{array}{c}
\text{1, 3}
\end{array}
\end{array}\} = \begin{array}{c}
\begin{array}{c}
\text{1, 1, 2}
\end{array}
\end{array}
\end{array}
\end{array} \]

\textbf{Theorem} : If Δ1-3 are satisfied, and we define M basis by $F_f = \sum_{g \geq f} M_g$, then

\[ \Delta_+(M_f) = \sum_{i \in \text{GDes}(f)} M_{i_f} \otimes M_{f_i}. \quad \text{(deconcatenate “between blocks”)} \]
Axioms for coproduct (cont’d)

\[ \Delta 3. \text{“Maximal concatenation” is well defined:} \]

Given \( g, h \), \( \exists \) unique \( \max \{ f | i f = g, f^i = h \} := g/h \);

\[ \text{e.g. } \begin{array}{c|c|c} 1 & 1 & 2 \\ \hline 1 & 1 & 2 \\ \hline 1 & 3 \\ \hline 1, 1, 2 \end{array} = \max \{ \begin{array}{c|c|c} 1 & 1 & 2 \\ \hline 1, 1, 2 \end{array}, \begin{array}{c|c|c} 1 & 1 & 2 \\ \hline 1, 3 \end{array} \} = \begin{array}{c|c|c} 1 & 1 & 2 \end{array} \]

So we can define “between blocks” to be “positions of maximal concatenation”, also called global descents \( G\text{Des}(f) := \{ i : f = i f^i / f^i \} \)

**Theorem**: If \( \Delta 1-3 \) are satisfied, and we define \( M \) basis by \( F_f = \sum g \geq f M_g \), then

\[ \Delta_+(M_f) = \sum_{i \in G\text{Des}(f)} M_{if} \otimes M_{fi} \text{. (deconcatenate “between blocks”) } \]

E.g. in Monomial Basis:

\[ \Delta_+ \left( \begin{array}{c} 5 \\ 4 \\ 2 \\ 1 \\ 3 \end{array} \right) = \begin{array}{c} 1 \\ 4 \\ 2 \\ 1 \\ 3 \end{array} \otimes \begin{array}{c} 1 \\ 4 \\ 2 \\ 1 \\ 3 \end{array} \]
Axioms for product

m1. Product in fundamental basis is a sum of shuffles $\zeta(f, g)$:
\[
F_f F_g = \sum_{\zeta \in \text{Sh}(f, g)} F_{\zeta(f, g)}
\]
e.g.

m2. Shuffles are order-preserving: if $f \leq f', g \leq g'$, then $\zeta(f, g) \leq \zeta(f', g')$.

m3. Shuffles are join-preserving: $\zeta(f_1 \lor f_2, g_1 \lor g_2) \leq \zeta(f_1, g_1) \lor \zeta(f_2, g_2)$.

**Theorem**: If m1-3 are satisfied, then the coefficient of $M_h$ in $M_f M_g$ is the number of shuffles $\zeta$ satisfying
- $\zeta(f, g) \leq h$;
- if $f' \geq f, g' \geq g$ satisfy $\zeta(f', g') \leq h$, then $f' = f, g' = g$. 

Applications

- To prove that a Hopf algebra is cofree, and have an explicit basis that shows cofreeness, i.e. shows the “unique factorisation” of the combinatorial objects;
- To construct isomorphisms:
  - Vargas’s self-duality isomorphism: $WQSym \rightarrow WQSym^*$
    (make a monomial basis for $WQSym$ and for $WQSym^*$, and show their products match)
  - An isomorphism: $PSym$ (our new algebra) $\rightarrow PQSym$ (Novelli-Thibon) ??
    obstacle: known bases on $PQSym$ do not satisfy the axioms, but Hugo Mlodecki has a basis that conjecturally does

Under additional axioms, we can give a cancellation free formula for the antipode in the monomial basis