You are expected to be familiar with the course content of MATH 2207, Linear Algebra, as written in the first 12 weeks of `http://www.math.hkbu.edu.hk/~amypang/2207/linalbook.pdf`. (This contains slightly more material than the Fall 2020 course.)

To succeed in this class, you should **easily** be able to solve the following problems. This is NOT an exhaustive list: the class may also require techniques and concepts not on this list. (Tip: some of these questions may be future homework problems.)

1. Let

$$A = \begin{pmatrix} 2 & -3 & -1 & 5 \\ 1 & -2 & 0 & 3 \\ 2 & 0 & -4 & 2 \\ 1 & -5 & 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -4 \\ -3 \\ 2 \\ -9 \end{pmatrix}.$$

a) Find all solutions $X \in \mathbb{R}^4$ to $AX = B$. Please show all steps in your computation.

b) Find, with justification, a basis for the column space of A.

c) Show that $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ 3 \\ -2 \end{pmatrix} \right\}$ is a basis for the null space of A. (You may wish to use the basis theorem.)

d) Let $\sigma : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by $\sigma(\alpha) = A\alpha$ (i.e. the standard matrix of σ is A). Let $\mathscr{B}$ be the basis of $\mathbb{R}^4$ given by

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

Write down $[\sigma]_{\mathscr{B}}$, the matrix for σ relative to $\mathscr{B}$, as the product of three matrices and/or their inverses. (You do **not** need to invert or multiply the three matrices.)

2. a) Determine whether $\left\{ \begin{pmatrix} 1 \\ 0 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \\ 6 \end{pmatrix} \right\}$ is linearly independent.

b) Find, with explanation, the dimension of Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \\ 6 \end{pmatrix} \right\}$.

c) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^4$, such that $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$. Show that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

d) Let $M_{2\times 2}$ be the vector space of 2×2 matrices. Prove that the set of upper-triangular 2×2 matrices is a subspace of $M_{2\times 2}$.

3. Let $P_{<3}(\mathbb{R})$ be the set of polynomials over $\mathbb{R}$ of degree less than 3. Consider the function $\sigma : P_{<3}(\mathbb{R}) \to P_{<3}(\mathbb{R})$ given by $\sigma(a + bx + cx^2) = (a - b) + (b + c)x^2$.

   a) Show that σ is a linear transformation.

   b) Find the matrix representing σ relative to the standard basis $\{1, x, x^2\}$ of $P_{<3}(\mathbb{R})$.

   c) Find a linearly independent set of polynomials that span the kernel of σ.

   d) What is the codomain of σ?

4. Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -2 & 2 & -1 \end{pmatrix}$, and hence diagonalise A, i.e. find a P and a diagonal D such that $A = PDP^{-1}$. You do **not** need to compute P^{-1}. (You may use an online RREF calculator, but remember you only have an ordinary calculator in the exams.)

5. a) Find the eigenvalues of $\begin{pmatrix} 5 & 2 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

   Suppose A is a 3×3 matrix whose only eigenvalues are 1 and 2.

   b) If A is diagonalisable, then what are the possibilities for the dimensions of its eigenspaces? (Answer in this way: "$\dim E_1 =$? and $\dim E_2 =$?, or $\dim E_1 =$? and $\dim E_2 =$?, or ...", and then give your reasons.)

   c) If A is not diagonalisable, then what are the possibilities for the dimensions of its eigenspaces? (Answer in the same way as in part b.)

6. Consider the subspace W of $\mathbb{R}^3$:

$$W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

   a) Find an orthogonal basis for W.

   b) Find a basis for $W^\perp$, the orthogonal complement of W.

7. Consider

$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}.$$

   Note that $\{\alpha_1, \alpha_2, \alpha_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

   a) Find the length of β.

   b) By computing dot products, express β as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

8. Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix} \right\} \subseteq \mathbb{R}^4$.

   a) Calculate the orthogonal projection of $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ onto W.

   b) Find the closest point in W to $\begin{pmatrix} 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}$.

   c) Find the distance from $\begin{pmatrix} 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}$ to W.

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