Last time:

Th 6.5.5 a) $W_1 + \cdots + W_k = \mathrm{Span}(W_1 \cup \cdots \cup W_k)$

b) if $W_i = \mathrm{Span}(A_i)$, then $W_1 + \cdots + W_k = \mathrm{Span}(A_1 \cup \cdots \cup A_k)$

Proof: a): $W_1 + W_2 \supseteq \mathrm{Span}(W_1 \cup W_2)$

(k=2)

$\because W_1 + W_2$ is a subspace and contains W_1 and W_2 $\therefore$ contains $W_1 \cup W_2$.

$\mathrm{Span}(W_1 \cup W_2) \supseteq W_1 + W_2$

Take $\alpha \in W_1 + W_2$ $\therefore$ $\alpha = \alpha_1 + \alpha_2$ with $\alpha_i \in W_i$

$\alpha_1 \in W_1 \subseteq W_1 \cup W_2 \subseteq \mathrm{Span}(W_1 \cup W_2)$

$\alpha_2 \in W_2 \subseteq W_1 \cup W_2 \subseteq \mathrm{Span}(W_1 \cup W_2)$

$\mathrm{Span}(W_1 \cup W_2)$ is closed under addition,

so $\alpha = \alpha_1 + \alpha_2 \in \mathrm{Span}(W_1 \cup W_2)$.

b. We show $\mathrm{Span}(W_1 \cup W_2) = \mathrm{Span}(\mathcal{A}_1 \cup \mathcal{A}_2)$

- $\mathrm{Span}(W_1 \cup W_2) \supseteq \mathrm{Span}(\mathcal{A}_1 \cup \mathcal{A}_2)$:
  $\because W_1 \cup W_2 \supseteq \mathcal{A}_1 \cup \mathcal{A}_2$
  (use HW1 Q4c)
- $\mathrm{Span}(\mathcal{A}_1 \cup \mathcal{A}_2) \supseteq \mathrm{Span}(W_1 \cup W_2)$:
  $\because \mathrm{Span}(\mathcal{A}_1 \cup \mathcal{A}_2)$ is a subspace,
  so it's enough to show (by 6.3.8)
  $\mathrm{Span}(\mathcal{A}_1 \cup \mathcal{A}_2) \supseteq W_1 \cup W_2$.
  i.e. $\mathrm{Span}(\mathcal{A}_1 \cup \mathcal{A}_2) \supseteq W_1$ and $\supseteq W_2$.

$W_1 = \mathrm{Span}(\mathcal{A}_1) \subseteq \mathrm{Span}(\mathcal{A}_1 \cup \mathcal{A}_2)$
$\uparrow$
($\because$ HW1 Q4c, $\because \mathcal{A}_1 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$)

$W_2 = \mathrm{Span}(\mathcal{A}_2) \subseteq \mathrm{Span}(\mathcal{A}_1 \cup \mathcal{A}_2)$
$\because \mathcal{A}_2 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$
$\therefore W_1 \cup W_2 \subseteq \mathrm{Span}(\mathcal{A}_1 \cup \mathcal{A}_2)$. ▤

To see b) in a previous example:

$$\begin{array}{ccccc} V_1 & + & V_2 & = & \mathbb{R}^3 \\ \mathrm{Span}\{e_1,e_2\} & + & \mathrm{Span}\{e_2,e_3\} & = & \mathrm{Span}\{e_1,e_2,e_2,e_3\} \\ \mathrm{Span}\{e_1,e_2\} & + & \mathrm{Span}\{2e_2,e_3\} & = & \mathrm{Span}\{e_1,e_2,2e_2,e_3\} \end{array}$$

Notice: if $\mathcal{A}_i$ is a basis of W_i, then $\mathcal{A}_1 \cup \mathcal{A}_2$ is NOT always a basis of $W_1 + W_2$.
"$\because$ of overlap in W_i"
($\therefore$ to get a basis of $W_1 + W_2$, use casting-out algorithm)

More precisely:

Th 6.5.6: $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

in example above:

$$\begin{array}{ccccccc} \dim \mathbb{R}^3 & = & \dim V_1 & + & \dim V_2 & - & \dim \mathrm{Span}\{e_2\} \\ 3 & = & 2 & + & 2 & - & 1 \end{array}$$

(Compare: $|\mathcal{A}_1 \cup \mathcal{A}_2| = |\mathcal{A}_1| + |\mathcal{A}_2| - |\mathcal{A}_1 \cap \mathcal{A}_2|$)

Note: how to write proofs about dimensions — see also RNT

- given $\dim U = d$ — "let $\{\alpha_1, \cdots, \alpha_d\}$ be a basis of U"
- to prove $\dim U = d$ — make a basis of U with d vectors.

(or use theorems)

Proof: Let $\dim(W_1 \cap W_2) = r$

$\dim W_1 = r + s$ (i.e. let $s = \dim W_1 - \dim(W_1 \cap W_2) \geq 0$

$\dim W_2 = r + t$ $\because W_1 \cap W_2$ is a subspace of W_1.)

We will build a basis of $W_1 + W_2$ with $(r+s)+(r+t)-r = r+s+t$ vectors,

based on (A)

Let $\{\alpha_1, \cdots, \alpha_r\}$ be a basis of $W_1 \cap W_2$.

Extend to $\{\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_s\}$ a basis of W_1

$\{\alpha_1, \cdots, \alpha_r, \gamma_1, \cdots, \gamma_t\}$ a basis of W_2.

(this is possible $\because \{\alpha_1, \cdots, \alpha_r\}$ is linearly independent.)

We show that $A = \{\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_s, \gamma_1, \cdots \gamma_t\}$ is a basis of $W_1 + W_2$.

- Span $A = W_1 + W_2$ by 6.5.5b.
- check linear independence

Suppose $a_1\alpha_1 + \cdots + a_r\alpha_r + b_1\beta_1 + \cdots b_s\beta_s + c_1\gamma_1 + \cdots + c_t\gamma_t = \vec{0}$

① $\underbrace{c_1\gamma_1 + \cdots + c_t\gamma_t}_{\in W_2} = \underbrace{-a_1\alpha_1 - \cdots - a_r\alpha_r - b_1\beta_1 - \cdots - b_s\beta_s}_{\in W_1}$.

$\therefore$ both sides are in $W_1 \cap W_2$.

$\therefore c_1\gamma_1 + \cdots + c_t\gamma_t = d_1\alpha_1 + \cdots + d_r\alpha_r$

$\because \{\alpha_1, \cdots, \alpha_r\}$ spans $W_1 \cap W_2$.

$c_1\gamma_1 + \cdots + c_t\gamma_t - d_1\alpha_1 - \cdots - d_r\alpha_r = \vec{0}$

$\{\alpha_1, \cdots, \alpha_r, \gamma_1, \cdots \gamma_t\}$ is a basis for W_2,

$\therefore$ linearly independent $\therefore c_1 = \cdots = c_t = d_1 = \cdots = d_r = 0$.

Back to ①. $\vec{0} = -a_1\alpha_1 - \cdots - a_r\alpha_r - b_1\beta_1 - \cdots - b_s\beta_s$

$\{\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_s\}$ is a basis for W_1 $\therefore$ linearly independent

$\therefore a_1 = \cdots = a_r = b_1 = \cdots = b_s = 0$ ▤