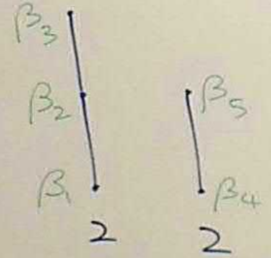
Last time: Ex 8.3.7 $\chi_B(x) = -(x-2)^5$

$$B - 2I = \begin{pmatrix} -1 & 0 & -1 & 1 & 0 \\ -4 & -1 & -3 & 2 & 1 \\ -2 & -1 & -2 & 1 & 1 \\ -3 & -1 & -3 & 2 & 1 \\ -8 & -2 & -7 & 5 & 2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad (B-2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Step 1: diagram:



Step 2: to find β_3.

- basis of $\mathrm{Nul}(B-2I)^3 = \{e_1, \cdots, e_5\}$
- their images under $(B-2I)^2$ = columns of $(B-2I)^2$
- a linearly independent subset = e.g. column 1 $\therefore \beta_3 = e_1$

Step 4:
- Let $\{\beta_{i_1}, \beta_{i_2}, \cdots\}$ be the vectors in the bottom m' levels of existing eigenstrings, extend to $\{\beta_{i_1}, \beta_{i_2}, \cdots \alpha_1', \alpha_2', \cdots\}$ a spanning set of $\mathrm{Ker}(\sigma - \lambda_i)^{m'}$.
- Consider $\{(\sigma-\lambda_i)^{m'-1}(\beta_{i_1}), (\sigma-\lambda_i)^{m'-1}(\beta_{i_2}), \cdots (\sigma-\lambda_i)^{m'-1}(\alpha_1'), (\sigma-\lambda_i)^{m'-1}(\alpha_2'), \cdots\}$ these α' are the
- Take a linearly independent subset containing existing eigenstring bottoms and some $(\sigma-\lambda_i)^{m'-1}(\alpha')$ — new eigenstring tops

Ex: Continue with B: $m' = 2$

Bottom m' levels of existing eigenstrings $= \{\beta_1, \beta_2\}$

- A basis of $\mathrm{Nul}(B-2I)^2$: from matrix earlier: $-x_1 - x_3 + x_4 = 0 \mapsto$

$$\begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \alpha_4' \\ \left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \right. & \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, & \left. \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \end{array}$$

- Apply $B-2I$ to the spanning set of $\{\beta_1, \beta_2, \alpha_1', \alpha_2', \alpha_3', \alpha_4'\}$:

can stop here

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \\ 0 & -1 & -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 1 & -3 & 2 \end{array}\right) \xrightarrow{\text{row reduction}} \begin{pmatrix} 0 & \boxed{-1} & 1 & 0 & -1 & 1 \\ 0 & 0 & \boxed{-1} & 1 & -2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$(B-2I)(\beta_1) = \vec{0}$ $\quad (B-2I)(\beta_2) = \beta_1$ $\quad (B-2I)(\alpha_1')$ $\quad (B-2I)(\alpha_2')$ ...

$\therefore$ columns 2, 3 have pivots

$\therefore$ choose new top (β_5) to correspond to column 3

ie. $\beta_5 = \alpha_1' = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\beta_4 = (B-2I)\alpha_1'$

So: $P = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & -4 & 0 & -1 & 1 \\ -1 & -2 & 0 & -1 & 0 \\ -1 & -3 & 0 & -1 & 0 \\ -1 & 8 & 0 & -2 & 0 \end{pmatrix}$ $\quad J = \begin{pmatrix} 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & 2 & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 2 \end{pmatrix}$

Shortcut: we can stop at $|$,

$\because$ we know from diagram we only need one new eigenstring top, i.e. only one α_i' so that $(B-2I)\alpha_i' \cup \{\text{previous eigenstring bottoms}\}$ is linearly independent, i.e. need an α_i' so that $(B-2I)\alpha_i'$ is not a multiple of β_1. And $\alpha_1' = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies this.

9.1/9.2 Linear forms and the dual space

From §7.1: $L(V,W)$ is a vector space.

Def 9.1.1: A linear form or linear functional on V is a linear transformation: $V \to \mathbb{F}$.

The set of all linear forms on V is the dual space of V: $\hat{V} = L(V, \mathbb{F})$

(V^* in some books)

a bijection / isomorphism

Ex: if $V = \mathbb{R}^3$, then $\hat{V} = L(\mathbb{R}^3, \mathbb{R}) \xrightarrow[\text{matrix}]{\text{take standard}} M_{1,3}(\mathbb{R})$

i.e. every $\phi \in \hat{V}$ has some standard matrix $\begin{pmatrix} a & b & c \end{pmatrix}$

i.e. $\phi\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax+by+cz.$ $\left(\text{i.e. } \phi(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \alpha \text{ – see §10.2}\right)$

A basis of $M_{1,3}(\mathbb{R}) = \{E^{1,1}, E^{1,2}, E^{1,3}\}$

$= \{(1\ 0\ 0), (0\ 1\ 0), (0\ 0\ 1)\}$

$\therefore$ A basis of $\hat{V} = \left\{ \phi_1\begin{pmatrix}x\\y\\z\end{pmatrix} = x, \phi_2\begin{pmatrix}x\\y\\z\end{pmatrix} = y, \phi_3\begin{pmatrix}x\\y\\z\end{pmatrix} = z \right\}$

To similarly find a basis of $\hat{V}$ for other V, notice:

$$\phi_i(e_i) = 1 \quad \text{and} \quad \phi_i(e_j) = 0 \text{ if } i \neq j.$$

Def 9.1.3/Th 9.1.2: If $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}$ is a basis of V, then the <u>dual basis to $\mathcal{A}$</u> is $\hat{\mathcal{A}} = \{\phi_1, \cdots, \phi_n\} \subseteq \hat{V}$ is defined by $\phi_i(\alpha_i) = 1$ and $\phi_i(\alpha_j) = 0$ if $i \neq j$.

In particular, $\dim \hat{V} = \dim V$.

(The order of the basis vectors in A is important!)

Ex: $V = P_{<2}(\mathbb{R})$, $\mathcal{A} = \{1, x, x^2\}$

then $\hat{\mathcal{A}} = \{\phi_1, \phi_2, \phi_3\}$ where

$\phi_1(1) = 1$, $\phi_1(x) = 0$, $\phi_1(x^2) = 0$.

To get a formula for ϕ_1:

$$\phi_1(a + bx + cx^2)$$
$$= a\,\phi_1(1) + b\,\phi_1(x) + c\,\phi_1(x^2)$$
$$= a \cdot 1 + b \cdot 0 + c \cdot 0 = a.$$

By same calculation:

$$\phi_2(a + bx + cx^2) = b$$
$$\phi_3(a + bx + cx^2) = c$$

i.e. ϕ_i is the function that takes the coefficient of α_i.

i.e. if $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$, then $\phi_i(\alpha) = a_i$.

Another view: we can find a_i by evaluating ϕ_i:

$$\alpha = \phi_1(\alpha)\alpha_1 + \cdots + \phi_n(\alpha)\alpha_n.$$

Interesting application: Lagrange interpolation.