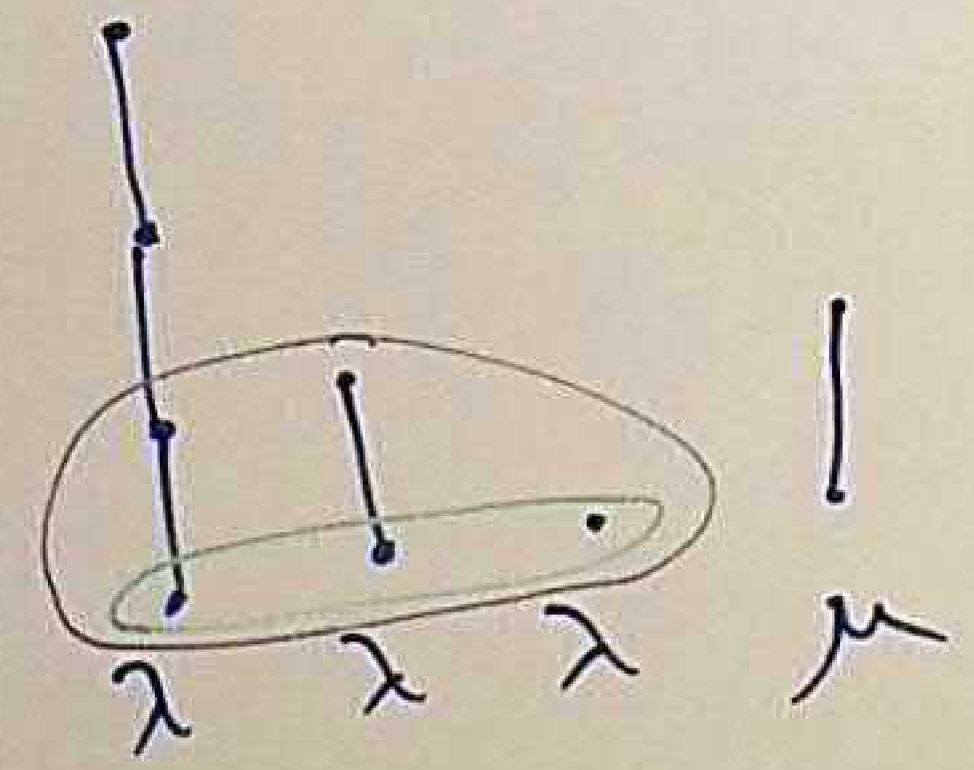
Section 8.3: Algorithm to find Jordan form

Make your own examples: choose P, J, let $A=PJP^{-1}$. Use algorithm to find P', J', check $A=P'J'P'^{-1}$ (remember P is not unique).

Step 1: build the eigenstring diagram horizontally

Ex:



$\dim(\operatorname{Ker}(\sigma-\lambda_c)) = 3$

$\dim(\operatorname{Ker}(\sigma-\lambda_c)^2) = 5$

$\dim(\operatorname{Ker}(\sigma-\lambda_c)^3) = 6$

$\dim(\operatorname{Ker}(\sigma-\lambda_c)^4) = 7$

$\dim(\operatorname{Ker}(\sigma-\lambda_c)^5) = 7$

Key: the bottom i "levels" of the λ-eigenstrings is a basis for $\operatorname{Ker}(\sigma-\lambda_c)^i$ (or $\operatorname{Nul}(A-\lambda I)^i$)

$\therefore$ calculate $\dim \operatorname{Ker}(\sigma-\lambda_c)^i$ for $i=1,2,\dots,$

number of vectors in level i

$= \dim \operatorname{Ker}(\sigma-\lambda_c)^i - \dim \operatorname{Ker}(\sigma-\lambda_c)^{i-1}$

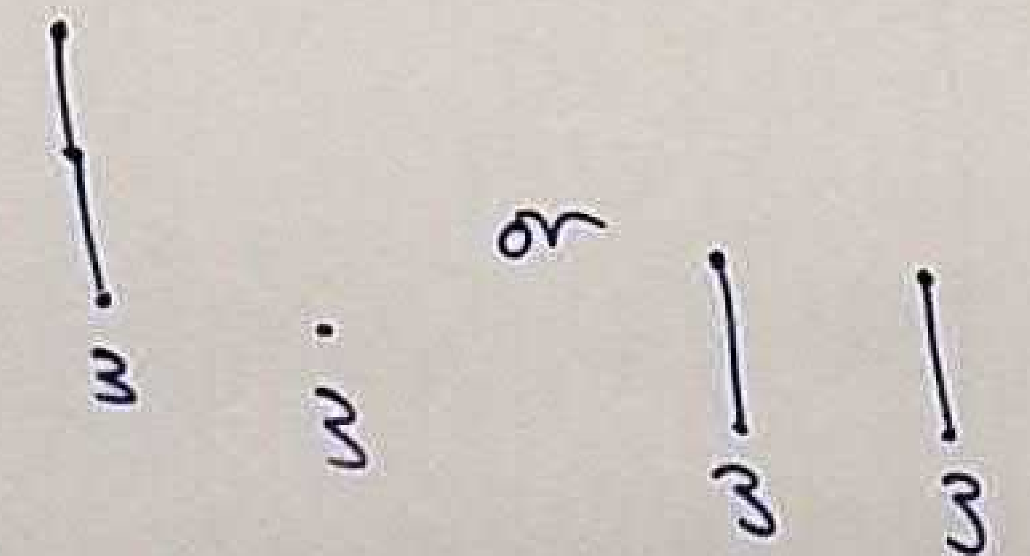
stop when $\dim \operatorname{Ker}(\sigma-\lambda_c)^{i+1} = \underbrace{\dim \operatorname{Ker}(\sigma-\lambda_c)^i}_{\text{number of free variables in } (A-\lambda I)^i}$.

Ex: $A = \begin{pmatrix} 3 & -2 & 2 & -1 \\ 0 & 6 & -3 & 2 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ Given: $\chi_A(x) = (x-3)^4$

$A - 3I = \begin{pmatrix} 0 & -2 & 2 & -1 \\ 0 & 3 & -3 & 2 \\ 0 & 3 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 0 & \boxed{-2} & 2 & -1 \\ 0 & 0 & 0 & \boxed{1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$

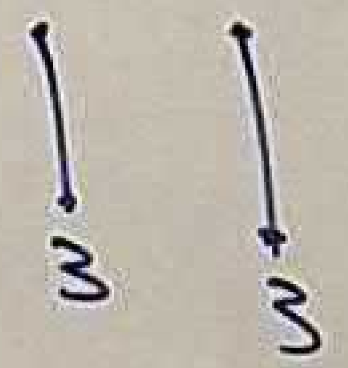
2 free variables,
so $\dim \mathrm{Nul}(A-3I) = 2$
$\therefore$ bottom level:

possible diagrams:



$(A-3I)^2 =$ zero matrix — $\dim \mathrm{Nul}((A-3I)^2) = 4$

$\therefore$ bottom two levels:



$J = \begin{pmatrix} 3 & 1 & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot \\ \cdot & \cdot & 3 & 1 \\ \cdot & \cdot & \cdot & 3 \end{pmatrix}$

Not necessary here, $\because$ thinking:

$(A-3I)^3 =$ zero matrix $\therefore \dim \mathrm{Nul}((A-3I)^3) = 4 = \dim \mathrm{Nul}((A-3I)^2)$

so the diagram is complete.

Warning: if there is more than one eigenvalue, then $(A-\lambda I)^i$ will not be the zero matrix.

Information from the Jordan form:

- $\chi_A(x) = \pm(x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}$ λ_i distinct

  m_i = multiplicity of λ_i

  = sum of the sizes of λ_i-blocks

  = $\dim \mathcal{K}_{\lambda_i}$ (generalised λ_i-eigenspace)

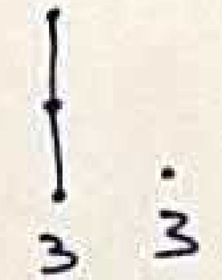
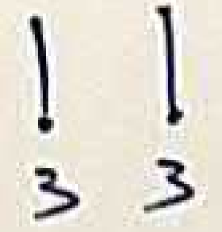
- $\dim E_\lambda = \dim \operatorname{Nul}(A-\lambda I)$

  = number of vectors in bottom level of λ-eigenstrings

  = number of λ-blocks.

Th:

- $m_A(x) = (x-\lambda_1)^{d_1} \cdots (x-\lambda_k)^{d_k}$ $\qquad d_i$ = size of the biggest λ_i-block.

e.g.  : $m(x) = (x-3)^3$;  : $m(x) = (x-3)^2$

In particular: if $m_A(x) = (x-\lambda_1) \cdots (x-\lambda_k)$, then all blocks have size $1 \Rightarrow A$ is diagonalisable.

<u>Proof</u>: i) why σ satisfies $(x-\lambda_1)^{d_1} \cdots (x-\lambda_k)^{d_k}$. $\qquad$ (ii why σ does not satisfy a polynomial of lower degree.)

i.e. why $(\sigma-\lambda_1)^{d_1} \cdots (\sigma-\lambda_k)^{d_k}(\beta) = 0$ for each β in the Jordan basis.

suppose β is on the jth level of the λ_k-eigenstring, so $(\sigma-\lambda_k\iota)^j(\beta) = 0$

$\therefore j \leqslant d_k$ so $(\sigma-\lambda_k\iota)^{d_k}(\beta) = 0$

so $(\sigma-\lambda_1)^{d_1} \cdots (\sigma-\lambda_{k-1})^{d_{k-1}}(\sigma-\lambda_k)^{d_k}(\beta) = 0$ $\qquad$ and similarly for other λ_i.

ii) Let $f(x) = \underbrace{(x-\lambda_1)^{a_1} \cdots (x-\lambda_{k-1})^{a_{k-1}}}_{g(x)} (x-\lambda_k)^{a_k}$

$= g(x) \, (x-\lambda_k)^{a_k}$

where $a_k < d_k$

Then $\exists \beta$ in level $a_k + 1$ of a λ_k-eigenstring

so $(\sigma - \lambda_k)^{a_k}(\beta) = \beta'$ is a λ_k-eigenvector.

so $[f(\sigma)](\beta) = g(\sigma) \circ (\sigma - \lambda_k)^{a_k}(\beta)$

$= g(\sigma)\beta'$

$= g(\lambda_k)\beta'$ HW3 Q1

so σ does not satisfy f.

$\neq \vec{0}$ $\because \beta' \neq \vec{0}$

and $g(\lambda_k) \neq 0$

$\because \lambda_k$ is not a solution to $(x-\lambda_1)^{a_1} \cdots (x-\lambda_{k-1})^{a_{k-1}}$