How to describe images:

Lemma: $\sigma(\mathrm{Span}(\mathcal{A})) = \mathrm{Span}\{\sigma(\alpha) \mid \alpha \in \mathcal{A}\}$

In particular: if U' has a spanning set, then $\sigma(U')$ is a subspace. (Th. 7.1.10, Cor 7.1.11)

$\leftarrow$ exercise: remove this condition by checking subspace axioms.

Proof: $\supseteq$: if $\beta \in \mathrm{Span}\{\sigma(\alpha) \mid \alpha \in \mathcal{A}\}$

then
$$\beta = a_1\sigma(\alpha_1) + \cdots + a_n\sigma(\alpha_n) \quad \text{for some } \alpha_i \in \mathcal{A}.$$
$$= \sigma(\underbrace{a_1\alpha_1 + \cdots + a_n\alpha_n}_{\in\, \mathrm{Span}(\mathcal{A})}) \in \sigma(\mathrm{Span}(\mathcal{A})).$$

$\subseteq$: if $\beta \in \sigma(\mathrm{Span}(A))$

then $\beta = \sigma(a_1\alpha_1 + \cdots + a_n\alpha_n)$ for some $\alpha_i \in A$

$= a_1\sigma(\alpha_1) + \cdots + a_n\sigma(\alpha_n)$

$\in \mathrm{Span}\{\sigma(\alpha) \mid \alpha \in A\}$

Prop. 7.1.14, Cor 7.1.15:

Preimages of subspaces are subspaces

Proof: HW.

Th. 7.1.17 Rank-Nullity Theorem (RNT):

if $\dim U < \infty$ and $\sigma \in L(U,V)$

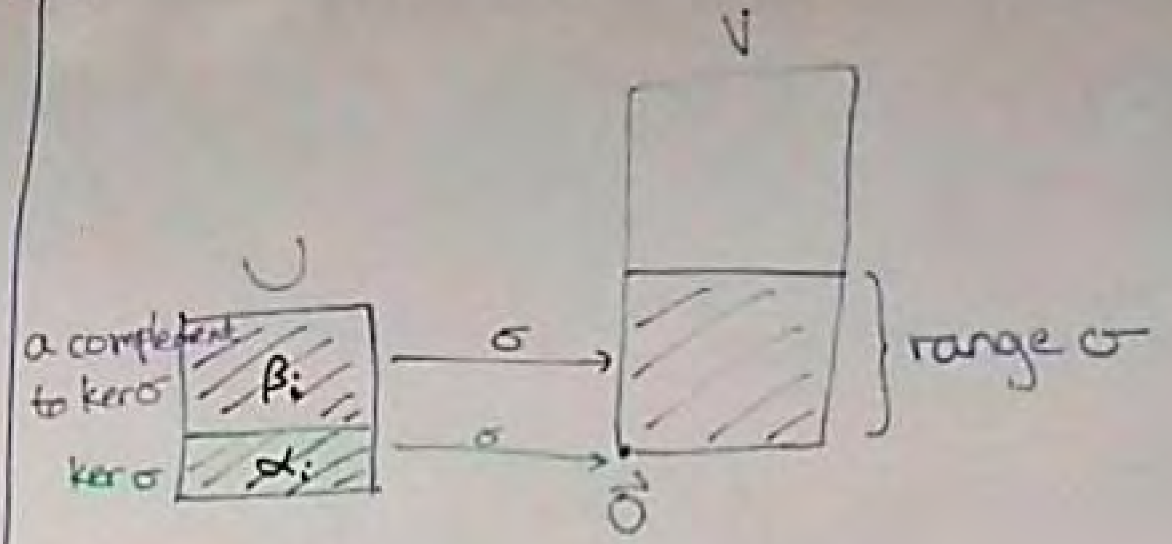
then $\underset{\text{dim range } \sigma}{\underset{\|}{\text{rank } \sigma}} + \underset{\text{dim ker } \sigma}{\underset{\|}{\text{nullity } \sigma}} = \dim U$

(Elsewhere in algebra — First Isomorphism Theorem — e.g. Th. 4.10, 5.15 in Abstract Algebra)

Proof: Let $\dim U = n$. $\ker\sigma \subseteq U$, so $\dim \ker \sigma \leq n$.

Let $\dim \ker \sigma = k$.

Take bases $\{\alpha_1, \dots, \alpha_k\}$ of $\ker\sigma$,



$\{\beta_1, \cdots, \beta_{n-k}\}$ of a complement of $\ker\sigma$:

(i.e. $\{\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_{n-k}\}$ is a basis of U)

We show $\{\sigma(\beta_1), \cdots, \sigma(\beta_{n-k})\}$ is a basis for range σ — that would mean rank σ $=$ dim range $\sigma = n-k = \dim U - \text{nullity } \sigma$, done.

— see HW.

See webpage for different proof -- also works for infinite-dimensional U.

Th. 7.1.18 Let $\sigma \in L(U,V)$:

σ is injective (i.e. one-to-one) $\Longleftrightarrow$ $\ker \sigma = \{\vec{0}\}$ $\Longleftrightarrow$ nullity $\sigma = 0$

see 2207 week 4 p24 or textbook

the only subspace of dimension 0 is $\{\vec{0}\}$.

σ is surjective (i.e. onto) $\Longleftrightarrow$ range $\sigma = V$ $\Longleftrightarrow$ rank $\sigma = \dim V$

definition of surjective

the only subspace of V, with the same dimension as V, is V.

Combine: σ is an isomorphism (i.e. injective and surjective)

$\Longleftrightarrow$ nullity $\sigma = 0$, rank $\sigma = \dim V$

$\Longrightarrow$ $\dim U = \dim V$.

Th. 7.1.19: (Invertible Matrix Theorem):

if $\dim U = \dim V$ $(< \infty)$ and $\sigma \in L(U,V)$

then nullity $\sigma = 0$

$\iff$ rank $\sigma = \dim V$ (by RNT)

$\therefore$ σ is injective $\iff$ σ is surjective.

$\therefore$ to check if σ is an isomorphism
only need to check one of

Warning: this is false if $\dim U = \infty$

e.g. differentiation: $\mathbb{R}[x] \to \mathbb{R}[x]$
is surjective, but not injective.

exercise: find a $\sigma: V \to V$ that is injective but not surjective