- $\sigma : \mathbb{F}[x] \longrightarrow \mathbb{F}[x]$

  multiplication by some fixed $p(x) \in \mathbb{F}[x]$.

  i.e. $[\sigma(f)](x) = f(x)p(x)$.

e.g. $\sigma : P_{<2}(\mathbb{R}) \longrightarrow P_{<4}(\mathbb{R})$

multiplication by $2+x^2 .= p(x)$

$$\sigma(a+bx) = (a+bx)(2+x^2)$$
$$= 2a+2bx+ax^2+bx^3.$$

- $\sigma : C^0(\mathbb{R}) \longrightarrow \mathbb{R}$

  evaluation at some fixed $a \in \mathbb{R}$.

  i.e. $\sigma(f) = f(a)$.

e.g. evaluation at $2 = a$

$\sigma : P_{<3}(\mathbb{R}) \longrightarrow \mathbb{R}$ $\sigma(f) = f(2)$

$$\sigma(a+bx+cx^2)$$
$$= a+b2+c4$$

exercise: check σ is linear

(also $\sigma : \mathbb{F}[x] \longrightarrow M_{n,n}(\mathbb{F})$
by evaluating at a $n \times n$ matrix: $\sigma(f) = f(A)$)

To make more linear transformations from these basic ones:

- Prop. 7.1.8 and Def: $\mathcal{L}(U,V)$ is the set of linear transformations from U to V. It is a vector space with operations

$$(\sigma+\tau)(\alpha) = \sigma(\alpha) + \tau(\alpha)$$
$$(a\sigma)(\alpha) = a(\sigma(\alpha))$$

Prop. 7.1.7

- A composition of linear transformations is linear (proof in HW)

∴ e.g. $f(x) \longmapsto f''(x) + (x+2)f'$ is linear

e.g. Fourier transforms

- Th. 7.1.4 If $\sigma \in L(U,V)$ is invertible, then $\sigma^{-1} \in L(V,U)$

<u>Def</u> if $\sigma \in L(U,V)$ and $W \subseteq U$ is a subspace,

then the <u>restriction</u> $\sigma|_W \in L(W,V)$ is

defined by $\sigma|_W(\alpha) = \sigma(\alpha)$

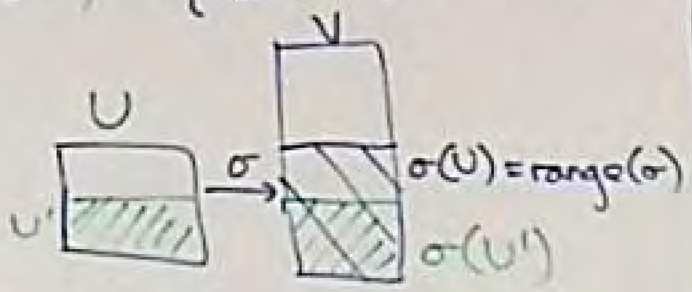
See evaluation example: $\sigma \in L(C^0(\mathbb{R}), \mathbb{R})$

we considered $\sigma|_{P_{<3}(\mathbb{R})}$.

Subspaces related to linear transformations:

Def: given $\sigma \in L(U,V)$, subspaces $U' \subseteq U$, $V' \subseteq V$.

- The image of U', $\sigma(U') = \{\sigma(\alpha) \mid \alpha \in U'\} \subseteq V$



In particular:

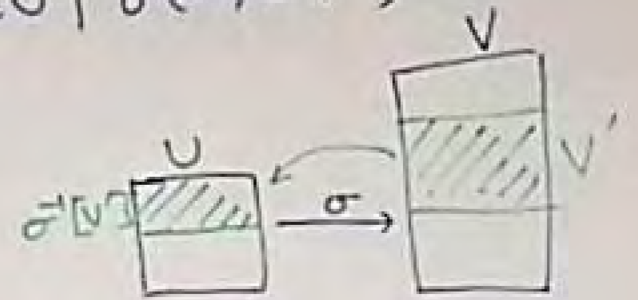
the range of σ is $\sigma(U) = \{\sigma(\alpha) \mid \alpha \in U\}$.

$\sigma(U') = \text{range}(\sigma|_{U'})$

- The preimage of V' is

$\sigma^{-1}[V'] = \{\alpha \in U \mid \sigma(\alpha) \in V'\} \subseteq U$.

↑ ! σ might not be invertible



In particular:

the kernel is $\ker(\sigma) = \sigma^{-1}[\{\vec{0}\}]$
$= \{\alpha \in U \mid \sigma(\alpha) = \vec{0}\}$

Ex: $\sigma : \mathbb{R}[x] \to \mathbb{R}[x]$, $\sigma(f) = f'$

$\sigma(P_{<3}(\mathbb{R})) = P_{<2}(\mathbb{R})$

$\sigma(\{ax \mid a \in \mathbb{R}\}) = \{a \mid a \in \mathbb{R}\} = P_{<1}(\mathbb{R})$

$\sigma^{-1}[P_{<1}(\mathbb{R})] = P_{<2}(\mathbb{R})$

Note: $\sigma^{-1}[\sigma(U')] \neq U'$

Question: is $\sigma(\sigma^{-1}[V']) = V'$ for this σ?
for other σ?