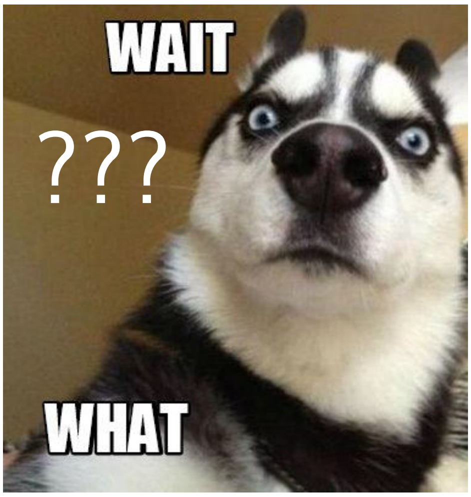
Linear Algebra

June 2020

Much of the material here comes from previous LinAl teachers at HKBU, and from our official textbook, "Linear Algebra and its Applications" by Lay. Thank you also to my many teachers, co-teachers and students; your views and ideas are incorporated in this retelling.

What is Linear Algebra?



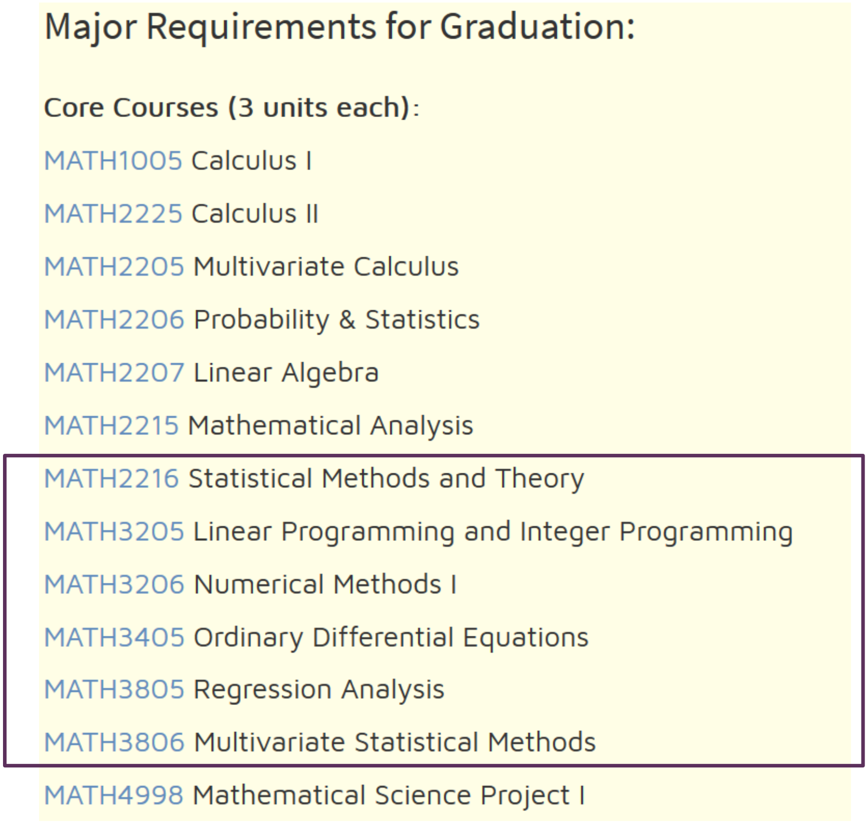
Linear algebra is the study of “adding things”.

In mathematics, there are many situations where we need to “add things” (e.g. numbers, functions, shapes), and linear algebra is about the properties that are common to all these different “additions”. This means we only need to study these properties once, not separately for each type of “addition” (better explanation in Week 7).

Because so many problems require “adding things”, linear algebra is one of the best tools in mathematics.

(picture from mememaker.net)

The concepts in linear algebra are important for many branches of mathematics:



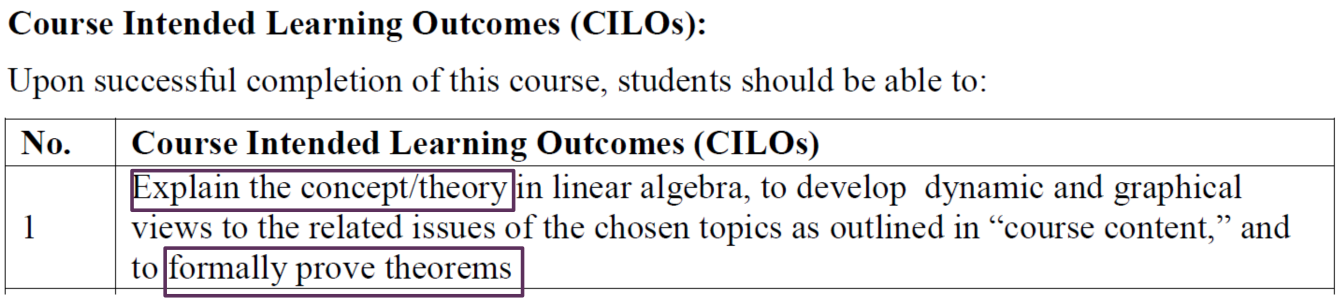
Major Requirements for Graduation:

Core Courses (3 units each):

MATH1005 Calculus I
MATH2225 Calculus II
MATH2205 Multivariate Calculus
MATH2206 Probability & Statistics
MATH2207 Linear Algebra
MATH2215 Mathematical Analysis
MATH2216 Statistical Methods and Theory
MATH3205 Linear Programming and Integer Programming
MATH3206 Numerical Methods I
MATH3405 Ordinary Differential Equations
MATH3805 Regression Analysis
MATH3806 Multivariate Statistical Methods
MATH4998 Mathematical Science Project I

All these classes list Linear Algebra as a prerequisite
(Info from math department website)

This class is about more than calculations. From the official syllabus:



Course Intended Learning Outcomes (CILOs):

Upon successful completion of this course, students should be able to:

No.	**Course Intended Learning Outcomes (CILOs)**
1	Explain the concept/theory in linear algebra, to develop dynamic and graphical views to the related issues of the chosen topics as outlined in "course content," and to formally prove theorems

Linear algebra is used in future courses in entirely different ways. So it's not enough to know routine calculations; you need to understand the concepts and ideas, to solve problems you haven't seen before on the exam. This will require words and not just formulae.

For many people, this is different from their previous math classes, and will require a lot of study time.
(Week 1 is straightforward computation; the abstract theory starts in Week 2.)

§1.1: Systems of Linear Equations

Linear Algebra starts with linear equations.

Example: $y = 5x + 2$ is a linear equation. We can take all the variables to the left hand side and rewrite this as $(-5)x + (1)y = 2$.

Example: $3(x_1 + 2x_2) + 1 = x_1 + 1 \longrightarrow (2)x_1 + (6)x_2 = 0$

Example: $x_2 = \sqrt{2}(\sqrt{6} - x_1) + x_3 \longrightarrow \sqrt{2}x_1 + (1)x_2 + (-1)x_3 = 2\sqrt{3}$

The following two equations are not linear, why?

$x_2 = 2\sqrt{x_1}$ $\qquad$ $xy + x = e^5$ $\qquad$ The problem is that the variables are not only multiplied by numbers.

In general, a linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

$x_1, x_2, \ldots x_n$ are the variables. $\qquad$ $a_1, a_2, \ldots a_n$ are the coefficients.

A linear equation has the form $a_1x_1 + a_2x_2 + \ldots a_nx_n = b$.

Definition: A *system of linear equations* (or a *linear system*) is a collection of linear equations involving the same set of variables.

Example:

$$\begin{array}{cccccc} x & +y & & = & 3 \\ 3x & & +2z & = & -2 \end{array}$$

is a system of 2 equations in 3 variables, x, y, z. Notice that not every variable appears in every equation.

Definition: A *solution* of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$ respectively.

Definition: The *solution set* of a linear system is the set of all possible solutions.

Example: One solution to the above system is $(x, y, z) = (2, 1, -4)$, because $2 + 1 = 3$ and $3(2) + 2(-4) = -2$.

Question: Is there another solution? How many solutions are there?

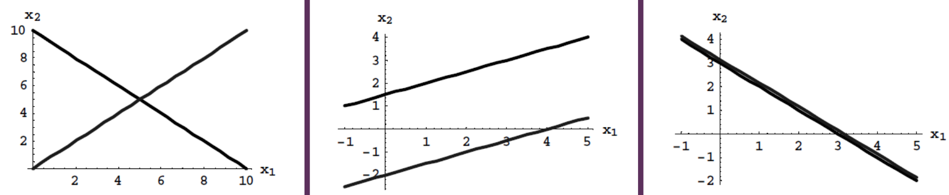
Definition: A linear system is *consistent* if it has a solution,
and *inconsistent* if it does not have a solution.

Fact: (which we will prove in the next class) A linear system has either

- exactly one solution consistent
- infinitely many solutions consistent
- no solutions inconsistent

EXAMPLE Two equations in two variables:

$$\begin{array}{rcrcr} x_1 &+& x_2 &=& 10\\ -x_1 &+& x_2 &=& 0\end{array} \qquad \begin{array}{rcrcr} x_1 &-& 2x_2 &=& -3\\ 2x_1 &-& 4x_2 &=& 8\end{array} \qquad \begin{array}{rcrcr} x_1 &+& x_2 &=& 3\\ -2x_1 &-& 2x_2 &=& -6\end{array}$$



one unique solution consistent

no solution inconsistent

infinitely many solutions consistent

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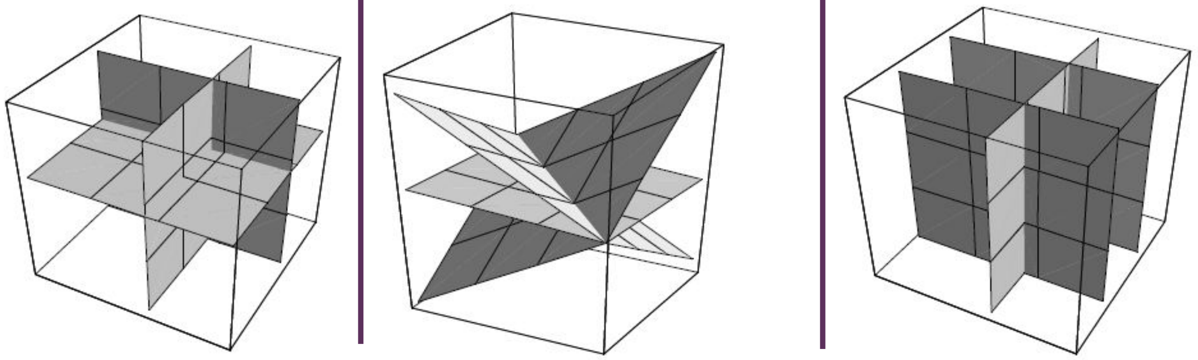
i.e. $ax + by + cz = d$

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

i) The planes intersect in one point. *(one solution)*

ii) The planes intersect in one line. *(infinitely many solutions)*

iii) There is no point in common to all three planes. *(no solution)*



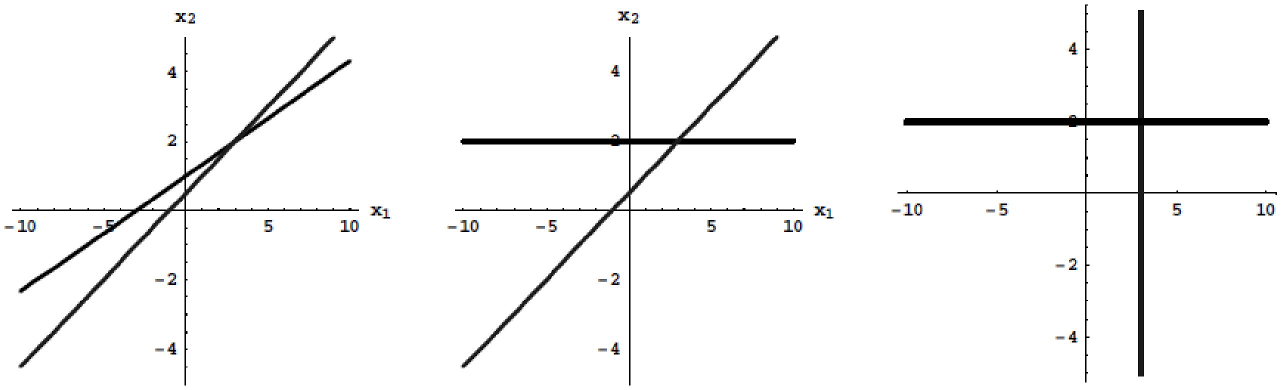
Which of these cases are consistent?

consistent consistent inconsistent

Our goal for this week is to develop an efficient algorithm to solve a linear system.

Example:

$$\begin{array}{lrcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \quad \begin{array}{r} \longrightarrow \\ R_2 + R_1 \to \end{array} \quad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \quad \begin{array}{r} R_1 + 2R_2 \to \\ \end{array} \quad \begin{array}{rcr} x_1 & = & 3 \\ x_2 & = & 2 \end{array}$$



Definition: Two linear systems are *equivalent* if they have the same solution set.

So the three linear systems above are different but equivalent.

A general strategy for solving a linear system: replace one system with an equivalent system that is easier to solve.

We simplify the writing by using matrix notation, recording only the coefficients and not the variables.

$$\begin{array}{lrcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \quad \begin{array}{r} \longrightarrow \\ R_2 + R_1 \to \end{array} \quad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \quad \begin{array}{r} R_1 + 2R_2 \to \\ \end{array} \quad \begin{array}{rcr} x_1 & = & 3 \\ x_2 & = & 2 \end{array}$$

$$\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

coefficient of x_1 coefficient of x_2 right hand side

The augmented matrix of a linear system contains the right hand side: $\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$

The coefficient matrix of a linear system is the left hand side only: $\left[\begin{array}{rr} 1 & -2 \\ -1 & 3 \end{array}\right]$

(The textbook does not put a vertical line between the coefficient matrix and the right hand side, but I strongly recommend that you do to avoid confusion.)

$$\begin{array}{lrcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \quad \begin{array}{rrcrcr} \to & x_1 & - & 2x_2 & = & -1 \\ R_2+R_1 \to & & & x_2 & = & 2 \end{array} \quad \begin{array}{rrcr} R_1 \underset{\to}{+} 2R_2 \to & x_1 & = & 3 \\ & x_2 & = & 2 \end{array}$$

$$\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

In this example, we solved the linear system by applying elementary row operations to the augmented matrix (we only used 1. above, the others will be useful later):

1. Replacement: add a multiple of one row to another row. $R_i \to R_i + cR_j$
2. Interchange: interchange two rows. $R_i \to R_j,\ R_j \to R_i$
3. Scaling: multiply all entries in a row by a nonzero constant. $R_i \to cR_i,\ c \neq 0$

Definition: Two matrices are *row equivalent* if one can be transformed into the other by a sequence of elementary row operations.

Fact: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set, i.e. they are equivalent linear systems.

General strategy for solving a linear system: do row operations to its augmented matrix to get an equivalent system that is easier to solve.

EXAMPLE:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ & & & \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ & & & \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ & & & \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & & & & & = & \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} & & & \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution: $(x_1,x_2,x_3)=(29,16,3)$

Check: Is $(29, 16, 3)$ a solution of the ***original*** system?

Warning: Do not do multiple elementary row operations at the same time, except adding multiples of the same row to several rows.

These are NOT equivalent systems: in the system on the right, x_1 can take any value, which is not true for the system on the left.

$$\cancel{\begin{array}{l} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{array} \qquad \begin{array}{l} x_2 = 2 \\ x_2 = 2 \end{array}}$$

$$\cancel{\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right]} \begin{array}{l} \leftarrow R_1 + R_2 \\ \leftarrow R_2 + R_1 \end{array}$$

$$\begin{array}{rcrcl} x_1 & - & 2x_2 & = & -3 \\ & & x_2 & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \leftarrow R_1 - R_3 \\ \leftarrow R_2 + 4R_3 \\ \end{array}$$

$$\begin{array}{rcl} x_1 & = & 29 \\ x_2 & = & 16 \\ x_3 & = & 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Sometimes we are not interested in the exact value of the solutions, just the number of solutions. In other words:

1. Existence of solutions: is the system consistent?
2. Uniqueness of solutions: if a solution exists, is it the only one?

Answering this requires less work than finding the solution.

Example:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We can stop here: back-substitution shows that we can find a unique solution.

EXAMPLE: Is this system consistent?

$$x_1 - 2x_2 + 3x_3 = -1$$

$$5x_1 - 7x_2 + 9x_3 = 0$$

$$3x_2 - 6x_3 = 8$$

EXAMPLE: For what values of h will the following system be consistent?

$$x_1 - 3x_2 = 4$$

$$-2x_1 + 6x_2 = h$$

Section 1.2: Row Reduction and Echelon Forms

Motivation: it is easy to solve a linear system whose augmented matrix is in reduced echelon form

Echelon form (or **row echelon form):**

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

EXAMPLE: Echelon forms

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$

Reduced echelon form: Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

EXAMPLE (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

EXAMPLE: Are these matrices in echelon form, reduced echelon form, or neither?

$$\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Here is the example from p10. Notice that we use row operations to first put the matrix into echelon form, and then into reduced echelon form.

Can we always do this for any linear system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \boxed{\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]}$$

echelon form

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & & & & & = & 29 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \boxed{\left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]}$$

reduced echelon form

Theorem: Any matrix A is row-equivalent to exactly one reduced echelon matrix, which is called its reduced echelon form and written $\text{rref}(A)$.

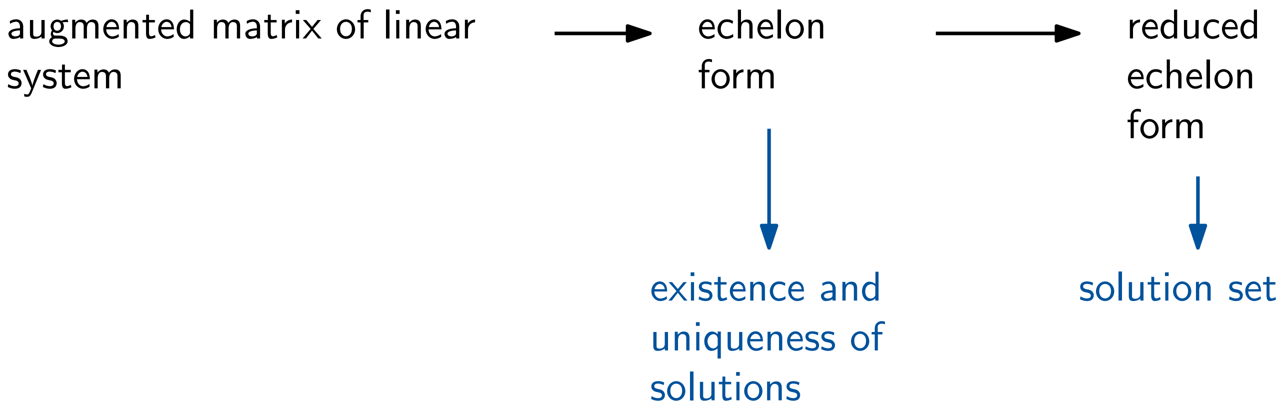
So our general strategy for solving a linear system is: apply row operations to its augmented matrix to obtain its rref.

And our general strategy for determining existence/uniqueness of solutions is: apply row operations to its augmented matrix to obtain an echelon form, i.e. a row-equivalent echelon matrix.

Warning: an echelon form is not unique. Its entries depend on the row operations we used. But its pattern of ■ and $*$ is unique.

These processes of row operations (to get to echelon or reduced echelon form) are called row reduction.

Row reduction:



augmented matrix of linear system → echelon form → reduced echelon form

echelon form ↓ existence and uniqueness of solutions

reduced echelon form ↓ solution set

The rest of this section:

- The row reduction algorithm (p21-25);
- Getting the solution, existence/uniqueness from the (reduced) echelon form (p26-29).

Important terms in the row reduction algorithm:

- pivot position: the position of a leading entry in a row-equivalent echelon matrix.
- pivot: a nonzero entry of the matrix that is used in a pivot position to create zeroes below it.
- pivot column: a column containing a pivot position.

The black squares are the pivot positions.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

Row reduction algorithm:

EXAMPLE:

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{array}\right]$$

1. The top of the leftmost nonzero column is a pivot position.
2. Put a pivot in this position, by scaling or interchanging rows.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right] \quad \begin{array}{c} R_3 \\ \\ R_1 \end{array}$$

3. Create zeroes in all positions below the pivot, by adding multiples of the top row to each row.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ & & & & & \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

4. Ignore this row and all rows above, and repeat steps 1-3.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

1. The top of the leftmost nonzero column is a pivot position.

2. Put a pivot in this position, by scaling or interchanging rows.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ & & & & & \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

3. Create zeroes in all positions below the pivot, by adding multiples of the top row to each row.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ & & & & & \end{array}\right]$$

4. Ignore this row and all rows above, and repeat steps 1-3.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

1. The top of the leftmost nonzero column is a pivot position.
2. Put a pivot in this position, by scaling or interchanging rows.
3. Create zeroes in all positions below the pivot, by adding multiples of the top row to each row.

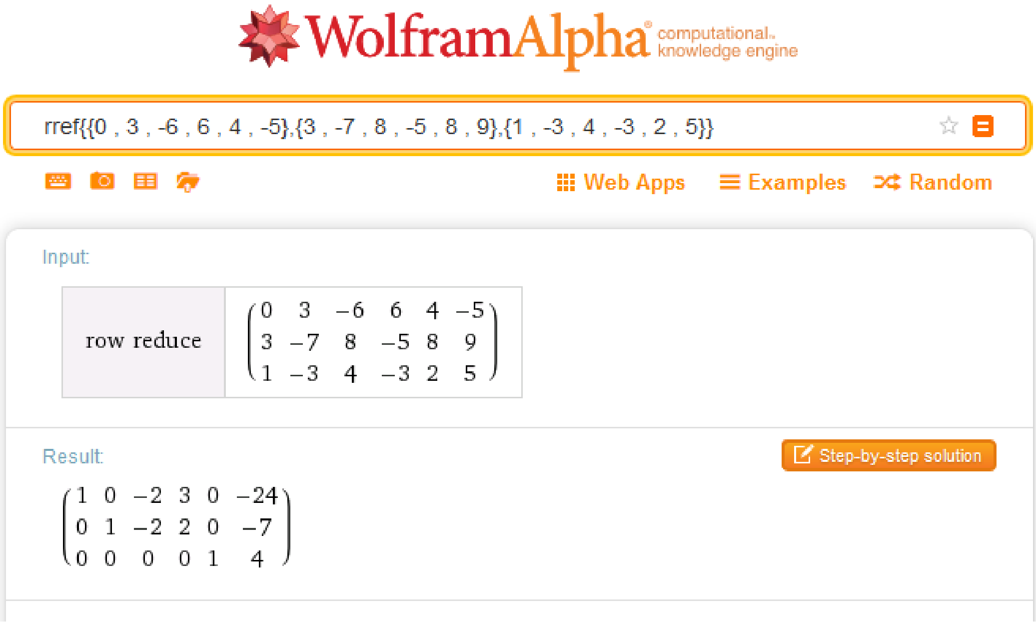
We are at the bottom row, so we don't need to repeat anymore. We have arrived at an echelon form.

5. To get from echelon to reduced echelon form (back substitution):
   Starting from the bottom row: for each pivot, add multiples of the row with the pivot to the other rows to create zeroes above the pivot.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{c} R_1 - 2R_3 \\ R_2 - R_3 \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} & & & & & \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Check your answer: www.wolframalpha.com



rref{{0 , 3 , -6 , 6 , 4 , -5},{3 , -7 , 8 , -5 , 8 , 9},{1 , -3 , 4 , -3 , 2 , 5}}

Input:

row reduce $\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{pmatrix}$

Result:

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Getting the solution set from the reduced echelon form:

A basic variable is a variable corresponding to a pivot column.
All other variables are free variables.

6. Write each row of the augmented matrix as a linear equation.

Example:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \qquad \begin{array}{cccccc} x_1 & & -2x_3 & +3x_4 & & = -24 \\ & x_2 & -2x_3 & +2x_4 & & = -7 \\ & & & & x_5 & = 4 \end{array}$$

basic variables: x_1, x_2, x_5, free variables: x_3, x_4.

The free variables can take any value. These values then uniquely determine the basic variables.

7. Take the free variables in the equations to the right hand side, and add equations of the form "free variable = itself", so we have equations for each variable in terms of the free variables.

Example:

$$\begin{array}{lcccc} x_1 = & -24 & +2x_3 & -3x_4 \\ x_2 = & -7 & +2x_3 & -2x_4 \\ x_3 = & & x_3 & \\ x_4 = & & & x_4 \\ x_5 = & 4 & & \end{array}$$

So the solution set is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -24 +2s -3t \\ -7 +2s -2t \\ s \\ t \\ 4 \end{pmatrix}$$

where s and t can take any value.

What this means: for every choice of s and t, we get a different solution:
e.g. $s = 0, t = 1$: $(x_1, x_2, x_3, x_4, x_5) = (-27, -9, 0, 1, 4)$
$s = 1, t = -1$: $(x_1, x_2, x_3, x_4, x_5) = (-19, -3, 1, -1, 4)$
and infinitely many others. (Exercise: check these two are solutions.)

We will see a better way to write the solution set next week (Week 2 p29-31, §1.5).

Answering existence and uniqueness of solutions from the echelon form

Example: On p14 we found
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \\ 0 & 3 & -6 & 8 \end{array}\right] \xrightarrow{\text{row-reduction}} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 3 & -1 \\ 0 & \boxed{3} & -6 & 5 \\ 0 & 0 & 0 & \boxed{3} \end{array}\right]$$

The last equation says $0x_1 + 0x_2 + 0x_3 = 3$, so this system is inconsistent. Generalising this observation gives us "half" of the following theorem:

Theorem 2: Existence and Uniqueness:
A linear system is consistent if and only if an echelon form of its augmented matrix has no row of the form $[0 \ldots 0|\blacksquare]$ with $\blacksquare \neq 0$.

Be careful with the logic here: this theorem says "if and only if", which means it claims two different things:

- If a linear system is consistent, then an echelon form of its augmented matrix cannot contain $[0 \ldots 0|\blacksquare]$ with $\blacksquare \neq 0$.
  This is the observation from the example above.

- If there is no row $[0 \ldots 0|\blacksquare]$ with $\blacksquare \neq 0$ in an echelon form of the augmented matrix, then the system is consistent.
  This is because we can continue the row-reduction to the rref, and then the solution method of p26-27 will give us solutions.

As for the uniqueness of solutions:

Theorem 2: Existence and Uniqueness:
If a linear system is consistent, then:
- it has a unique solution if there are no free variables;
- it has infinitely many solutions if there are free variables.

In particular, this proves the fact we saw earlier, that a linear system has either a unique solution, infinitely many solutions, or no solutions.

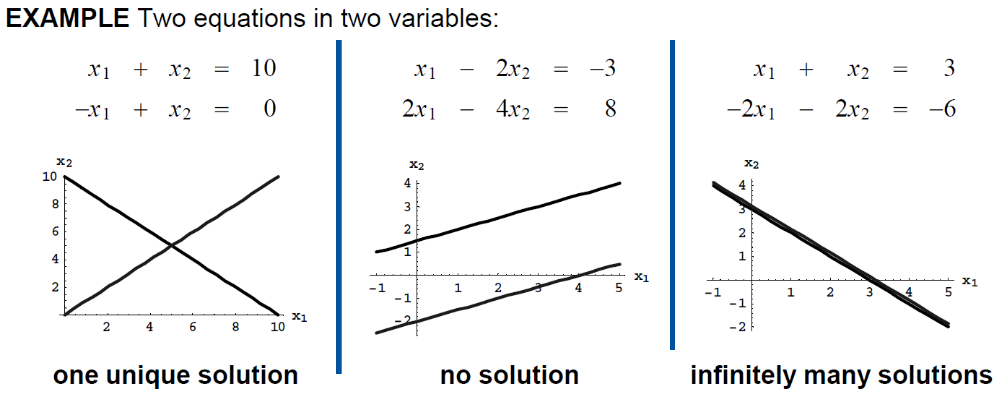
Warning: In general, the existence of solutions is unrelated to the uniqueness of solutions. (We will meet an important exception in §2.3.)

Remember from last week:

Fact: A linear system has either

- exactly one solution
- infinitely many solutions
- no solutions

We gave an algebraic proof via row reduction, but the picture, although not a proof, is useful for understanding this fact.



EXAMPLE Two equations in two variables:

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned} \qquad \begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$

one unique solution **no solution** **infinitely many solutions**

This week and next week, we will think more geometrically about linear systems.

1.4 Span - related to existence of solutions
1.5 A geometric view of solution sets (a detour)
1.7 Linear independence - related to uniqueness of solutions

We are aiming to understand the two key concepts in three ways:

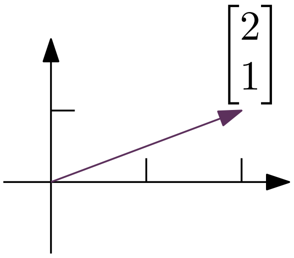
- The related computations: to solve problems about a specific linear system with numbers (Week 2 p10, Week 3 p9-10).
- The rigorous definition: to prove statements about an abstract linear system (Week 2 p15, Week 3 p13).
- The conceptual idea: to guess whether statements are true, to develop a plan for a proof or counterexample, and to help you remember the main theorems (Week 2 p13-14, Week 3 p3-5). This informal view is for thinking only, NOT for answering problems on homeworks and exams.

§1.3: Vector Equations

A column vector is a matrix with only one column.

Until Chapter 4, we will say "vector" to mean "column vector".

A vector $\mathbf{u}$ is in $\mathbb{R}^n$ if it has n rows, i.e. $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$



Example: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are vectors in $\mathbb{R}^2$.

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have a geometric meaning: think of $\begin{bmatrix} x \\ y \end{bmatrix}$ as the point (x, y) in the plane.

There are two operations we can do on vectors:

addition: if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$.

scalar multiplication: if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and c is a number (a scalar), then $c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$.

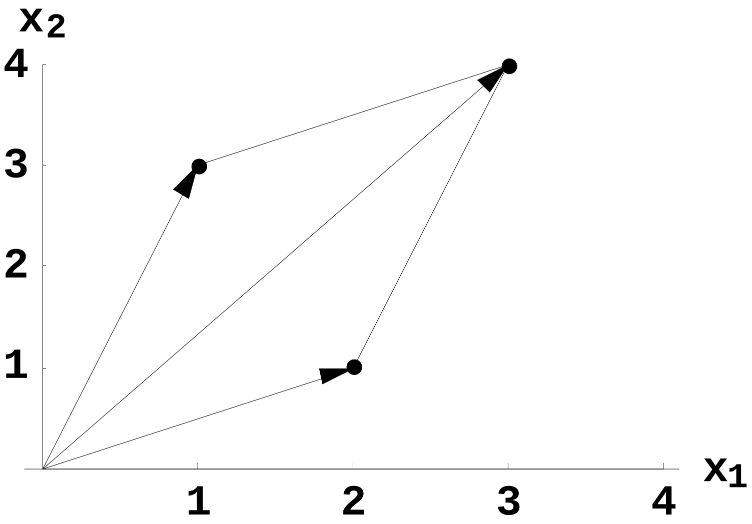
These satisfy the usual rules for arithmetic of numbers, e.g.

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}, \quad c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}, \quad 0\mathbf{u} = \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

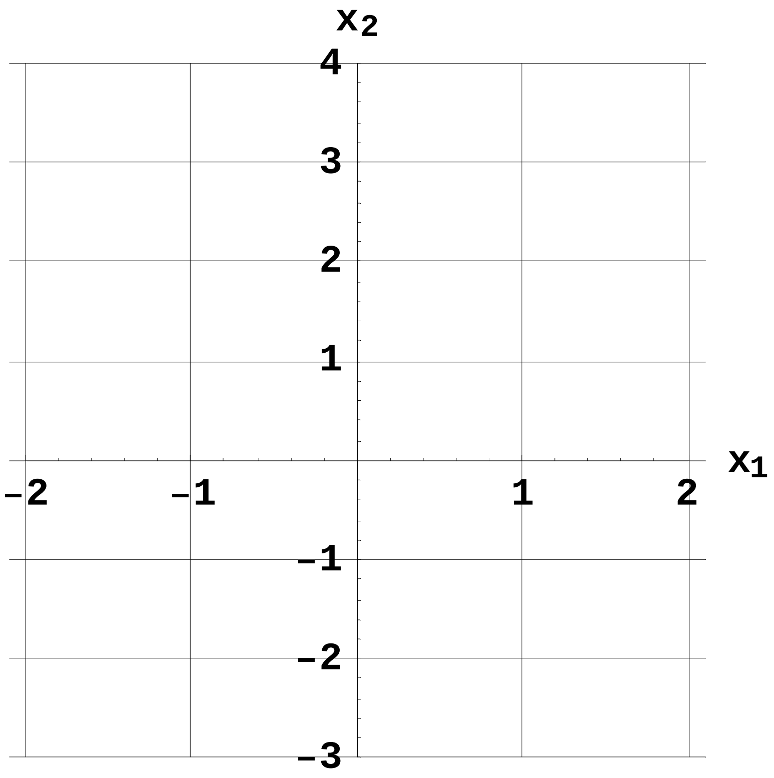
Parallelogram rule for addition of two vectors:

If **u** and **v** in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **0**, **u** and **v**. (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.



Combining the operations of addition and scalar multiplication:

Definition: Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_p$, the vector

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ with *weights* $c_1, c_2, \ldots, c_p$.

Example: $\mathbf{u} = \begin{bmatrix}1\\3\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}2\\1\end{bmatrix}$. Some linear combinations of $\mathbf{u}$ and $\mathbf{v}$ are:

$$3\mathbf{u} + 2\mathbf{v} = \begin{bmatrix}7\\11\end{bmatrix}. \qquad \tfrac{1}{3}\mathbf{u} + 0\mathbf{v} = \begin{bmatrix}1/3\\1\end{bmatrix}.$$

$$\mathbf{u} - 3\mathbf{v} = \begin{bmatrix}-5\\0\end{bmatrix}. \qquad \mathbf{0} = 0\mathbf{u} + 0\mathbf{v} = \begin{bmatrix}0\\0\end{bmatrix}.$$

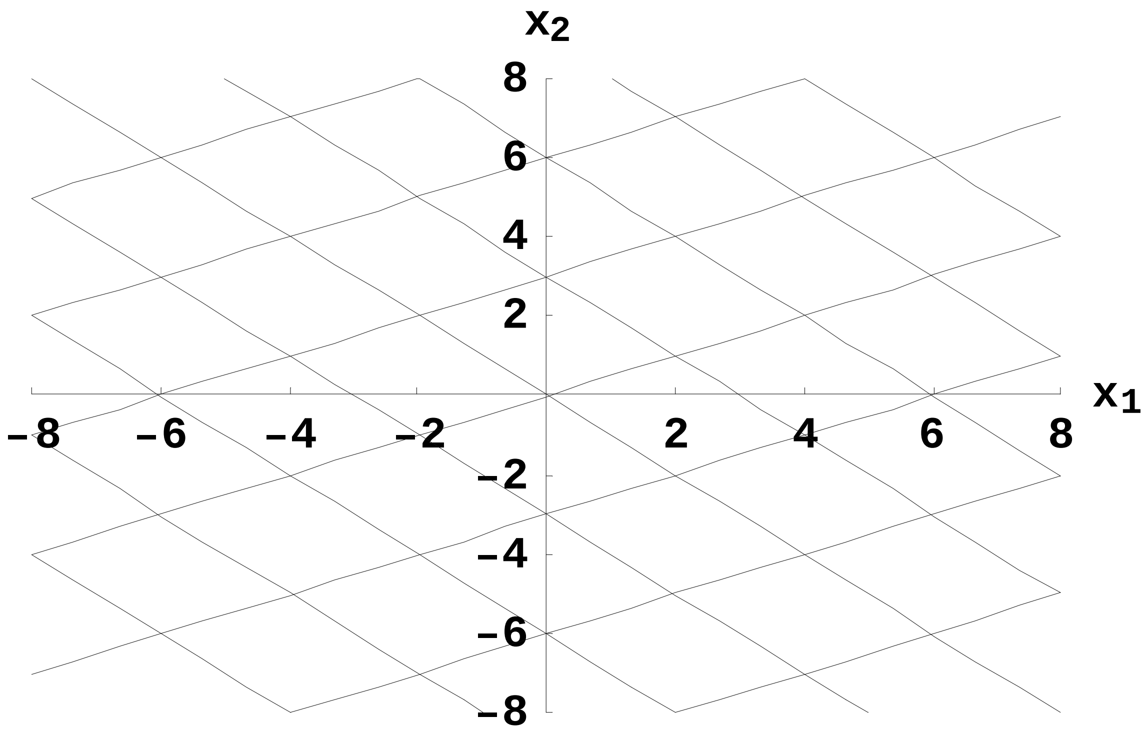
(i.e. $\mathbf{u} + (-3)\mathbf{v}$)

Study tip: an "example" after a definition does NOT mean a calculation example. These more theoretical examples are objects (vectors, in this case) that satisfy the definition, to help you understand what the definition means. You should also make your own examples when you see a definition.

Geometric interpretation of linear combinations: "all the points you can go to if you are only allowed to move in the directions of $\mathbf{v}_1, \ldots, \mathbf{v}_p$".

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



When we don't have the grid paper:

EXAMPLE: Let $\mathbf{a}_1 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}$.

Express **b** as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Solution: Vector **b** is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$ if ______________________________

Vector equation:

Corresponding linear system:

Corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 4 & 3 & -2 \\ 2 & 6 & 8 \\ 14 & 10 & -8 \end{array}\right]$$

Reduced echelon form:

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Exercise: Use this algebraic method on the examples on the previous page and check that you get the same answer.

What we learned from the previous example:

1. Writing $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_p$ is the same as solving the vector equation
$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b};$$
2. This vector equation has the same solution set as the linear system whose augmented matrix is
$$\left[\begin{array}{cccc|c} | & | & & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_p & \mathbf{b} \\ | & | & & | & | \end{array}\right].$$

In particular, it is not always possible to write $\mathbf{b}$ as a linear combination of given vectors: in fact, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p$ if and only if there is a solution to the linear system with augmented matrix
$$\left[\begin{array}{cccc|c} | & | & & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_p & \mathbf{b} \\ | & | & & | & | \end{array}\right].$$

Definition: Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$. The *span* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$, written
$$\text{Span}\left\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\right\},$$
is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$.

In other words, Span $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is the set of all vectors which can be written as $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p$ for any choice of weights $x_1, x_2, \dots, x_p$.

In set notation:

the $\in$ sign means "is in"

$\notin$ means "is not in"

$$\text{Span}\left\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\right\} = \left\{\underbrace{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p} \;\middle|\; \underbrace{x_1, \dots, x_p \in \mathbb{R}}\right\}.$$

the set of

vectors of the form $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p$

such that

$x_1, \dots x_p$ are real numbers (i.e. they can take any value)

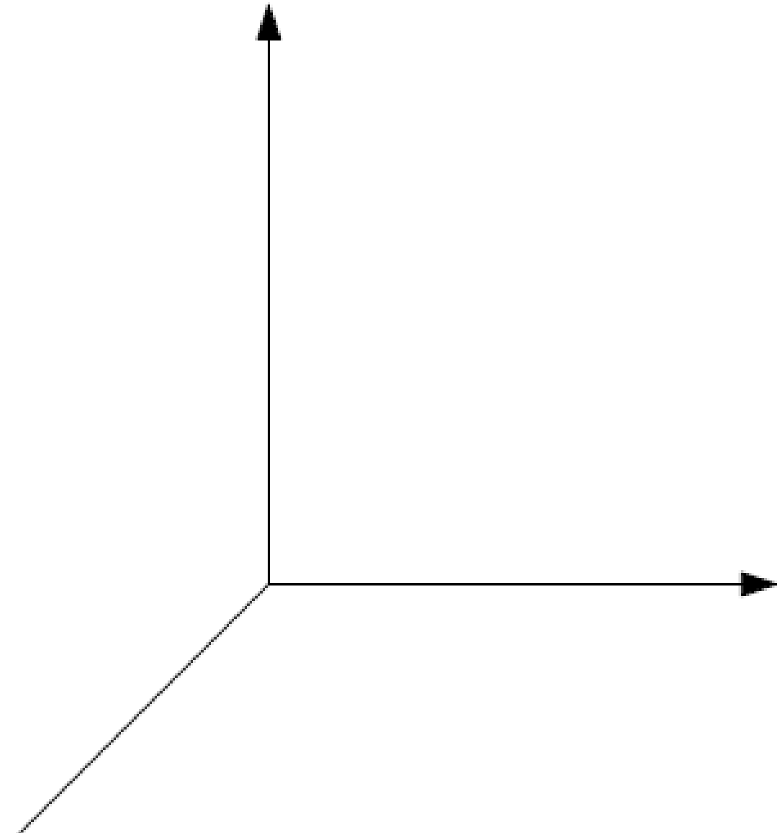
DEFINITION: $\mathbf{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = \{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p \mid x_1, \ldots, x_p \in \mathbb{R}\}$

EXAMPLE: Span of one vector in $\mathbb{R}^3$:

When $p = 1$, the definition says $\mathrm{Span}\,\{\mathbf{v}_1\} = \{x_1\mathbf{v}_1 \mid x_1 \in \mathbb{R}\}$,

i.e. $\mathrm{Span}\,\{\mathbf{v}_1\}$ is all scalar multiples of $\mathbf{v}_1$.

- $\mathrm{Span}\,\{\mathbf{0}\} = \{\mathbf{0}\}$, because $x_1\mathbf{0} = \mathbf{0}$ for all scalars x_1.
- If $\mathbf{v}_1$ is not the zero vector, then $\mathrm{Span}\,\{\mathbf{v}_1\}$ is ________________________________

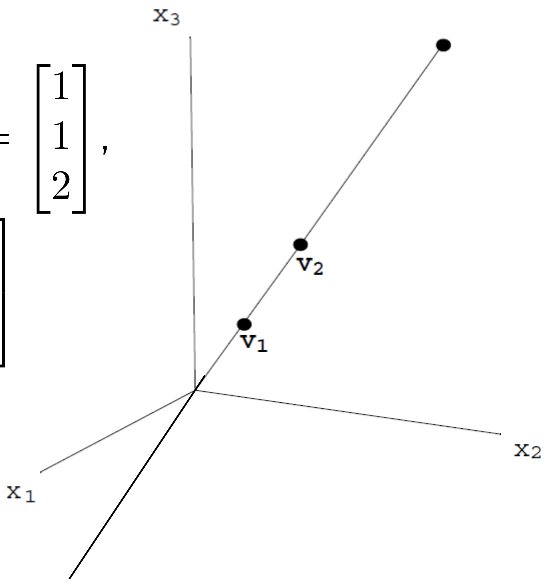


Example: Span of two vectors in $\mathbb{R}^3$:

When $p = 2$, the definition says Span $\{\mathbf{v}_1, \mathbf{v}_2\} = \{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \mid x_1, x_2 \in \mathbb{R}\}$.

e.g. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$,

$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$

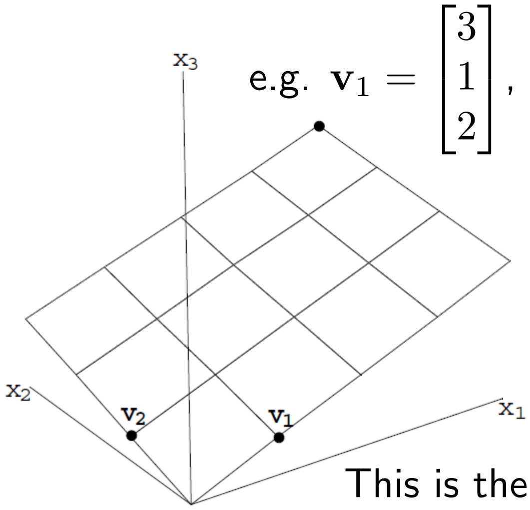


$\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ =**Span**$\{\mathbf{v}_1\}$ =**Span**$\{\mathbf{v}_2\}$

(line through the origin)

e.g. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$



This is the plane spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\mathbf{v}_2$ is **not** a multiple of $\mathbf{v}_1$

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ =plane through the origin

A first exercise in writing proofs.

Each proof is different. Here are some general guidelines, but not every proof is like this. In particular, do NOT memorise and copy the equations in a particular proof, it will NOT work for a different question.

EXAMPLE: Prove that, if $\mathbf{u}$ is in $\mathrm{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then $2\mathbf{u}$ is in $\mathrm{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

STRATEGY:

Step 1: Find the conclusion of the proof, i.e. what the question is asking for. Using definitions (ctrl-F in the notes if you don't remember), write it out as formulas:

Step 2: **on a separate piece of paper**, use definitions to write out the given information as formulas. Be careful to use different letters in different formulas.

Step 3: If the required conclusion (from Step 1) is an equation: start with the left hand side, and calculate/reorganise it using the information in Step 2 to obtain the right hand side.

(More examples: week 3 p13, week 4 p22, week 5 p17, week 5 p19, many exercise sheets.)

The professional way to write this (which may be confusing for beginners):

In more complicated proofs, you may want to use theorems (see week 5 p26).

To improve your proofs:

- **Memorise your definitions**, i.e. how to translate a technical term into a formula.
- After finishing a proof, think about why that strategy works, and why other strategies that you tried didn't work.
- Come to office hours with questions from homework or the textbook, we can do them together.

Recall from page 10 that writing $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_p$ is equivalent to solving the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b},$$

and this has the same solution set as the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} | & | & & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_p & \mathbf{b} \\ | & | & & | & | \end{array}\right].$$

In particular, $\mathbf{b}$ is in Span $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p\}$ if and only if the above linear system is consistent.

We now develop a different way to write this linear system.

---

§1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$

We can think of the weights $x_1, x_2, \dots, x_p$ as a vector.

m rows, p columns

The product of an $m \times p$ matrix A and a vector $\mathbf{x}$ in $\mathbb{R}^p$ is the linear combination of the columns of A using the entries of $\mathbf{x}$ as weights:

$$A\mathbf{x} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_p \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p.$$

Example: $\begin{bmatrix} 4 & 3 \\ 2 & 6 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}.$

Example: $$\begin{bmatrix} 4 & 3 \\ 2 & 6 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}.$$

There is another, faster way to compute $A\mathbf{x}$, one row of A at a time:

Example: $$\begin{bmatrix} 4 & 3 \\ 2 & 6 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4(-2) + 3(2) \\ 2(-2) + 6(2) \\ 14(-2) + 10(2) \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}.$$

It is easy to check that $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ and $A(c\mathbf{u}) = cA\mathbf{u}$.

Warning: The product $A\mathbf{x}$ is only defined if the number of columns of A equals the number of rows of $\mathbf{x}$. The number of rows of $A\mathbf{x}$ is the number of rows of A.

Warning: Always write $A\mathbf{x}$, with the matrix on the left and the vector on the right - $\mathbf{x}A$ has a different meaning. And do not write $A \cdot \mathbf{x}$, that has a different meaning.

We have three ways of viewing the same problem:

1. The system of linear equations with augmented matrix $[A|\mathbf{b}]$,
2. The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b}$,
3. The matrix equation $A\mathbf{x} = \mathbf{b}$.

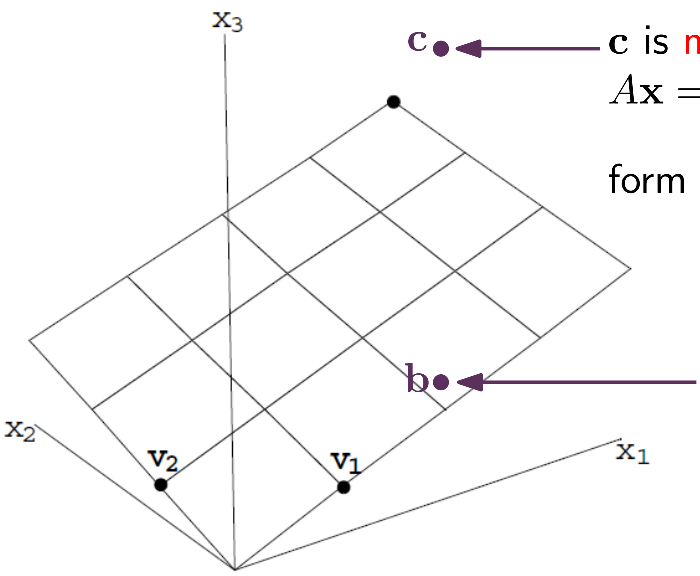
These three problems have the same solution set, so the following three things are the same (they are simply different ways to say "the above problem has a solution"):

1. The system of linear equations with augmented matrix $[A|\mathbf{b}]$ has a solution,
2. $\mathbf{b}$ is a linear combination of the columns of A (or $\mathbf{b}$ is in the span of the columns of A),
3. The matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

Another way of saying this: The span of the columns of A is the set of vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

The span of the columns of A is the set of vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

Example: If $A = \begin{bmatrix} 3 & 1 \\ 1 & 4 \\ 2 & 1 \end{bmatrix}$, then the relevant vectors are $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$.



$\mathbf{c}$ is not on the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, so $A\mathbf{x} = \mathbf{c}$ does not have a solution. The echelon form of $[A|\mathbf{c}]$ is $\left[\begin{array}{cc|c} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{array}\right]$.

$\mathbf{b}$ is on the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, so $A\mathbf{x} = \mathbf{b}$ has a solution. The echelon form of $[A|\mathbf{b}]$ is $\left[\begin{array}{cc|c} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{array}\right]$.

Warning: If A is an $m \times n$ matrix, then the pictures on the previous page are for the right hand side $\mathbf{b} \in \mathbb{R}^m$, not for the solution $\mathbf{x} \in \mathbb{R}^n$ (as we were drawing in Week 1, and also in p29-31 later this week). In this example, we cannot draw the solution sets on the same picture, because the solutions $\mathbf{x}$ are in $\mathbb{R}^2$, but our picture is in $\mathbb{R}^3$.

Because $\mathbf{b} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$, the way to see a solution $\mathbf{x}$ on this $\mathbb{R}^3$ picture is like on p9: $\mathbf{x}$ gives the location of $\mathbf{b}$ relative to the gridlines drawn by $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e x_i tells you how far $\mathbf{b}$ is in the $\mathbf{v}_i$ direction (see week 8 p22). For example, for the lower purple dot, $x_1 \sim 2.2$ and $x_2 \sim 0.2$.

So these three things are the same:

1. The system of linear equations with augmented matrix $[A|\mathbf{b}]$ has a solution,
2. $\mathbf{b}$ is a linear combination of the columns of A (or $\mathbf{b}$ is in the span of the columns of A),
3. The matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

One question of particular interest: when are the above statements true for all vectors $\mathbf{b}$ in $\mathbb{R}^m$? i.e. when is $A\mathbf{x} = \mathbf{b}$ consistent for all right hand sides $\mathbf{b}$, and when is Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\} = \mathbb{R}^m$?

Example: $(m = 3)$ Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then Span $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \mathbb{R}^3$, because $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

But for a more complicated set of vectors, the weights will be more complicated functions of x, y, z. So we want a better way to answer this question.

Theorem 4: Existence of solutions to linear systems: For an $m \times n$ matrix A, the following statements are logically equivalent (i.e. for any particular matrix A, they are all true or all false):

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$ (i.e. Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\} = \mathbb{R}^m$).
d. A has a pivot position in every row.

You may view d) as a computation (reduction to echelon form) to check for a), b) or c).

Warning: the theorem says nothing about the uniqueness of the solution.

Proof: (outline): By the previous discussion, (a), (b) and (c) are logically equivalent. So, to finish the proof, we only need to show that (a) and (d) are logically equivalent, i.e. we need to show that,

- if (d) is true, then (a) is true;
- if (d) is false, then (a) is false. (This is the same as "if (a) is true, then (d) is true".)

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
d. A has a pivot position in every row.

Proof: (continued)

Suppose (d) is true. Then, for every $\mathbf{b}$ in $\mathbb{R}^m$, the augmented matrix $[A|\mathbf{b}]$ row-reduces to $[\text{rref}(A)|\mathbf{d}]$ for some $\mathbf{d}$ in $\mathbb{R}^m$. This does not have a row of the form $[0\dots0|\blacksquare]$, so, by the Existence of Solutions Theorem (Week 1 p27), $A\mathbf{x} = \mathbf{b}$ is consistent. So (a) is true.

Suppose (d) is false. We want to find a counterexample to (a): i.e. we want to find a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ has no solution.

(This last part of the proof, written on the next page, is hard, and is not something you are expected to think of by yourself. But you should try to understand the part of the proof on this page.)

---

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
d. A has a pivot position in every row.

Proof: (continued) Suppose (d) is false. We want to find a counterexample to (a): i.e. we want to find a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ has no solution.

A does not have a pivot position in every row, so the last row of $\text{rref}(A)$ is $[0\dots0]$.

Let $\mathbf{d} = \begin{bmatrix} * \\ \vdots \\ * \\ 1 \end{bmatrix}$.

Then the linear system with augmented matrix $[\text{rref}(A)|\mathbf{d}]$ is inconsistent.

Now we apply the row operations in reverse to get an equivalent linear system $[A|\mathbf{b}]$ that is inconsistent.

Example:

$$\left[\begin{array}{rr|r} 1 & -3 & 1 \\ -2 & 6 & -1 \end{array}\right] \xrightleftharpoons[R_2\to R_2-2R_1]{R_2\to R_2+2R_1} \left[\begin{array}{rr|r} 1 & -3 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

Theorem 4: Existence of solutions to linear systems: For an $m \times n$ matrix A, the following statements are logically equivalent (i.e. for any particular matrix A, they are all true or all false):

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$ (i.e. Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\} = \mathbb{R}^m$).
d. A has a pivot position in every row.

We will add more statements to this theorem throughout the course.

Observe that A has at most one pivot position per column (condition 5 of a reduced echelon form, or think about how we perform row-reduction). So if A has more rows than columns (a "tall" matrix), then A cannot have a pivot position in every row, so the statements above are all false.
In particular, a set of fewer than m vectors cannot span $\mathbb{R}^m$.

Warning/Exercise: It is not true that any set of m or more vectors span $\mathbb{R}^m$: can you think of an example?

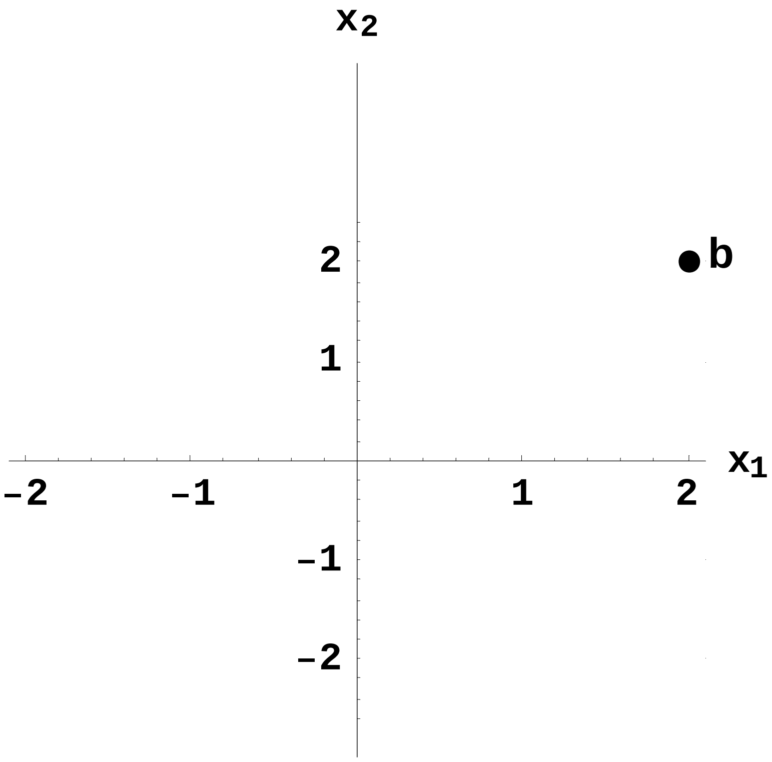
Remember that the solutions to $A\mathbf{x} = \mathbf{b}$ are the weights for writing $\mathbf{b}$ as a linear combination of the columns of A.

$$A\mathbf{x} = \mathbf{b} \quad \longleftrightarrow \quad \mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p.$$

The "linear combination of columns" viewpoint gives us a picture way to understand existence of solutions (p20).

Here is a picture about uniqueness of solutions: what does it mean for the weights x_i to be non-unique.

EXAMPLE: $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$



Informally, the non-uniqueness of weights happens because we can use our given vectors to "walk in a circle back to $\mathbf{0}$" - this is the idea of linear dependence (week 3).

§1.5: Solution Sets of Linear Systems

Goals:
- use vector notation to give geometric descriptions of solution sets (in parametric form: $\{\mathbf{p} + s\mathbf{v} + t\mathbf{w} + \dots | s, t, \dots \in \mathbb{R}\}$).
- to compare the solution sets of $A\mathbf{x} = \mathbf{b}$ and of $A\mathbf{x} = \mathbf{0}$.

Definition: A linear system is *homogeneous* if the right hand side is the zero vector, i.e.

$$A\mathbf{x} = \mathbf{0}.$$

When we row-reduce $[A|\mathbf{0}]$, the right hand side stays $\mathbf{0}$, so the reduced echelon form does not have a row of the form $[0 \dots 0|\blacksquare]$.
So a homogeneous system is always consistent.

In fact, $\mathbf{x} = \mathbf{0}$ is always a solution, because $A\mathbf{0} = \mathbf{0}$. The solution $\mathbf{x} = \mathbf{0}$ called the trivial solution.

A non-trivial solution $\mathbf{x}$ is a solution where at least one x_i is non-zero.

If there are non-trivial solutions, what does the solution set look like?

EXAMPLE:

$$2x_1 + 4x_2 - 6x_3 = 0$$

$$4x_1 + 8x_2 - 10x_3 = 0$$

Corresponding augmented matrix:

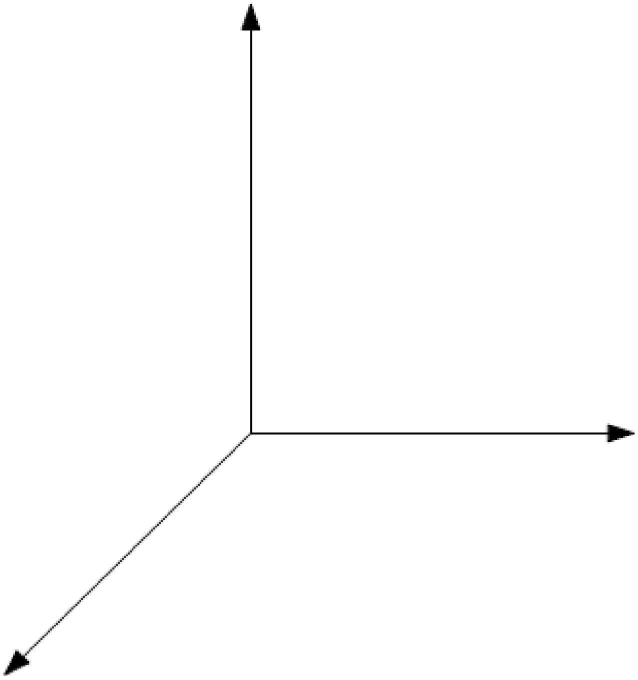
$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 0 \end{array}\right]$$

Corresponding reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Solution set:

Geometric representation:



EXAMPLE: (same left hand side as before)

$$2x_1 + 4x_2 - 6x_3 = 0$$

$$4x_1 + 8x_2 - 10x_3 = 4$$

Corresponding augmented matrix:

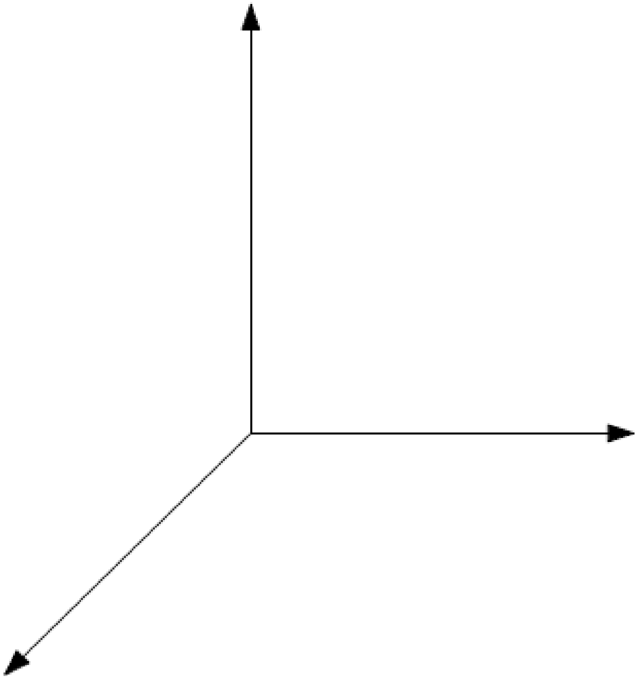
$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{array}\right]$$

Corresponding reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution set:

Geometric representation:



EXAMPLE: Compare the solution sets of:

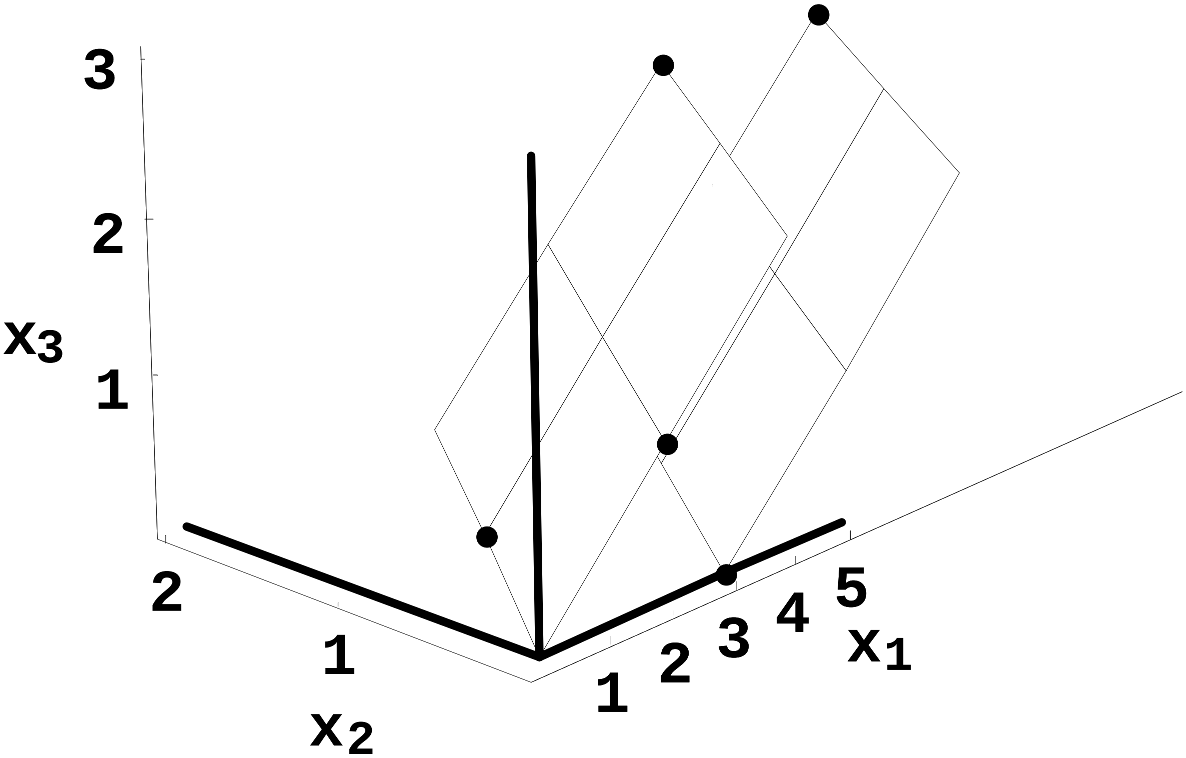
$$x_1 - 2x_2 - 2x_3 = 0 \qquad\qquad x_1 - 2x_2 - 2x_3 = 3$$

Corresponding augmented matrices:

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \end{array}\right] \qquad\qquad \left[\begin{array}{ccc|c} 1 & -2 & -2 & 3 \end{array}\right]$$

These are already in reduced echelon form.
Solution sets:

Geometric representation:



Parallel Solution Sets of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

In our first example:

- The solution set of $A\mathbf{x} = \mathbf{0}$ is a line through the origin parallel to $\mathbf{v}$.
- The solution set of $A\mathbf{x} = \mathbf{b}$ is a line through $\mathbf{p}$ parallel to $\mathbf{v}$.

In our second example:

- The solution set of $A\mathbf{x} = \mathbf{0}$ is a plane through the origin parallel to $\mathbf{u}$ and $\mathbf{v}$.
- The solution set of $A\mathbf{x} = \mathbf{b}$ is a plane through $\mathbf{p}$ parallel to $\mathbf{u}$ and $\mathbf{v}$.

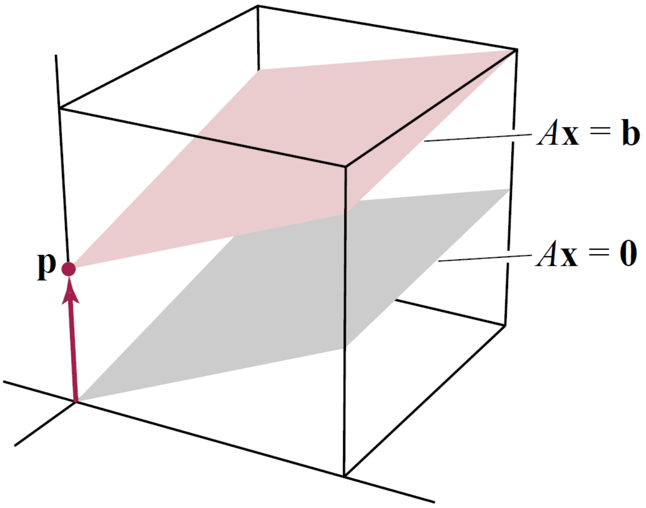
In both cases: to get the solution set of $A\mathbf{x} = \mathbf{b}$, start with the solution set of $A\mathbf{x} = \mathbf{0}$ and translate it by $\mathbf{p}$.

$\mathbf{p}$ is called a particular solution (one solution out of many).

In general:

Theorem 6: Solutions and homogeneous equations: Suppose $\mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Then the solution set to $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem 6: Solutions and homogeneous equations: Suppose $\mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Then the solution set to $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Theorem 6: Solutions and homogeneous equations: Suppose $\mathbf{p}$ is a solution to $A\mathbf{x}=\mathbf{b}$. Then the solution set to $A\mathbf{x}=\mathbf{b}$ is the set of all vectors of the form $\mathbf{w}=\mathbf{p}+\mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x}=\mathbf{0}$.

Proof: (outline)
We show that $\mathbf{w}=\mathbf{p}+\mathbf{v_h}$ is a solution:

$$\begin{aligned}&A(\mathbf{p}+\mathbf{v_h})\\=&A\mathbf{p}+A\mathbf{v_h}\\=&\mathbf{b}+\mathbf{0}\\=&\mathbf{b}.\end{aligned}$$

We also need to show that all solutions are of the form $\mathbf{w}=\mathbf{p}+\mathbf{v_h}$ - see q25 in Section 1.5 of the textbook.

Two typical applications of this theorem:

1. If you write the solutions to $A\mathbf{x}=\mathbf{b}$ in parametric form, then the part with free variables is the solution to $A\mathbf{x}=\mathbf{0}$, e.g. on week1 p26, we found that the

$$\text{solutions to } \begin{bmatrix}0&3&-6&6&4\\3&-7&8&-5&8\\1&-3&4&-3&2\end{bmatrix}\mathbf{x}=\begin{bmatrix}-5\\9\\5\end{bmatrix} \text{ is } \boxed{\begin{bmatrix}-24\\-7\\0\\0\\4\end{bmatrix}}+\boxed{s\begin{bmatrix}2\\2\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-3\\-2\\0\\1\\0\end{bmatrix}}$$

particular solution →

$$\text{solutions to } \begin{bmatrix}0&3&-6&6&4\\3&-7&8&-5&8\\1&-3&4&-3&2\end{bmatrix}\mathbf{x}=\mathbf{0}$$

2. If you already have the solutions to $A\mathbf{x}=\mathbf{0}$ and you need to solve $A\mathbf{x}=\mathbf{b}$, then you don't need to row-reduce again: simply find one particular solution (e.g. by guessing) and then add it to the solution set to $A\mathbf{x}=\mathbf{0}$ (example on next page).

How this theorem is useful: a shortcut to Q1b on ex. sheet #5:

Example: Let $A = \begin{bmatrix} | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$.

In Q1a, you found that the solution set to $A\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} t$, where r, s, t can take any value.

In Q1b, you want to solve $A\mathbf{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Now $\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 0\mathbf{a}_1 + 1\mathbf{a}_2 + 0\mathbf{a}_3 + 0\mathbf{a}_4 = A \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution. So the solution set is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} t$, where r, s, t can take any value.

Notice that this solution looks different from the solution obtained from row-reduction:

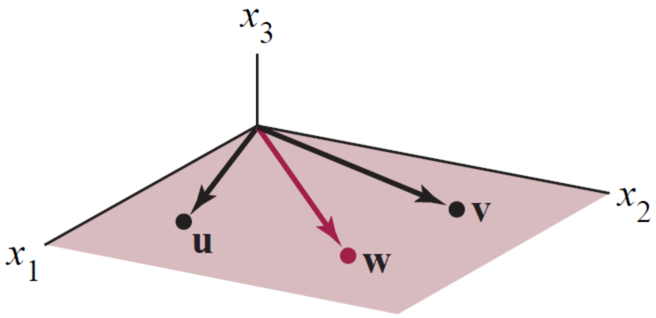
rref $\left(\left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 3 \\ 2 & 6 & 0 & -8 & 6 \end{array}\right] \right) = \left[\begin{array}{cccc|c} 1 & 3 & 0 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, which gives a different particular solution $\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

But the solution sets are the same:

$$\begin{aligned} \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} t &= \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} (r-1) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} t \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} (r-1) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} t, \end{aligned}$$

and r, s, t taking any value is equivalent to $r-1, s, t$ taking any value.

§1.7: Linear Independence



In this picture, the plane is Span $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ = Span $\{\mathbf{u}, \mathbf{v}\}$, so we do not need to include $\mathbf{w}$ to describe this plane.
We can think that $\mathbf{w}$ is "too similar" to $\mathbf{u}$ and $\mathbf{v}$ - and linear dependence is the way to make this idea precise.

Definition: A set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ is *linearly independent* if the only solution to the vector equation

$$x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

is the trivial solution $(x_1 = \cdots = x_p = 0)$.

The opposite of linearly independent is linearly dependent:

Definition: A set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ is *linearly dependent* if there are weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p} = \mathbf{0}.$$

The equation $c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p} = \mathbf{0}$ is a linear dependence relation.

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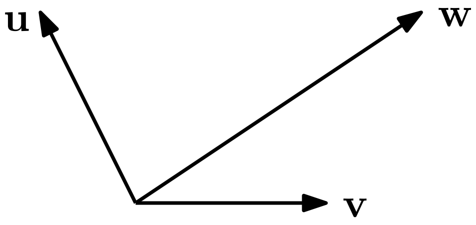
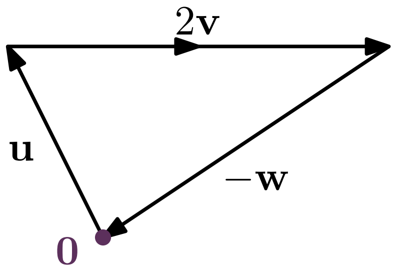
The equation $c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p} = \mathbf{0}$ is a linear dependence relation.

A picture of a linear dependence relation: " you can use the given directions to move in a circle".



$$\mathbf{u} + 2\mathbf{v} - \mathbf{w} = \mathbf{0}$$



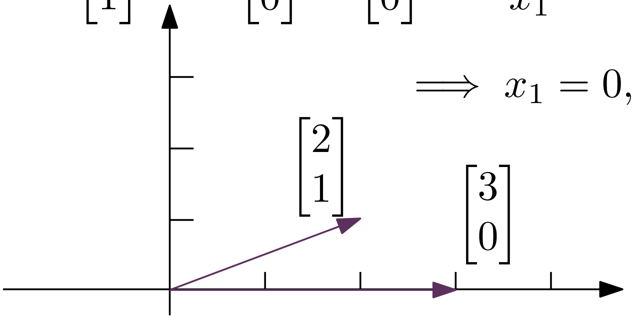
$$x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

The only solution is $x_1 = \cdots = x_p = 0$
→ linearly independent

Example: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$ is linearly independent because

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{aligned} 2x_1 + 3x_2 &= 0 \\ x_1 \quad\quad &= 0 \end{aligned}$$

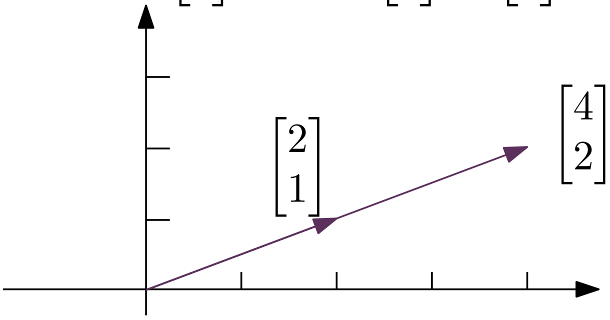
$$\Longrightarrow x_1 = 0, x_2 = 0.$$



There is a solution with some $x_i \neq 0$
→ linearly dependent

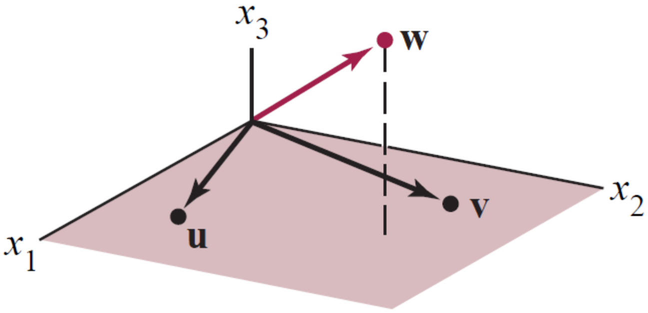
Example: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$ is linearly dependent because

$$2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$



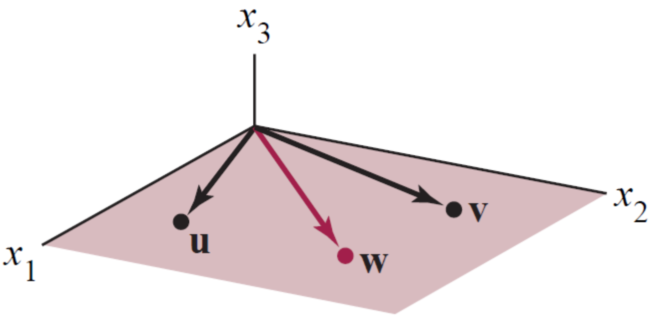
$$x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

The only solution is $x_1 = \cdots = x_p = 0$ (i.e. unique solution)
$\rightarrow$ linearly independent



Informally: $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in "totally different directions"; there is "no relationship" between $\mathbf{v}_1, \ldots \mathbf{v}_p$.

There is a solution with some $x_i \neq 0$ (i.e. infinitely many solutions)
$\rightarrow$ linearly dependent



Informally: $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in "similar directions"

Some easy cases:

- Sets containing the zero vector $\{\mathbf{0}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$: then the linear dependence equation is
$$x_1\mathbf{0} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0}.$$
A non-trivial solution is
$$(1)\mathbf{0} + (0)\mathbf{v_2} + \cdots + (0)\mathbf{v_p} = \mathbf{0},$$
so such a set is linearly dependent (it doesn't matter what $\mathbf{v}_2, \ldots, \mathbf{v}_p$ are).

- Sets containing one vector $\{\mathbf{v}\}$: then the linear dependence equation is
$$x\mathbf{v} = \mathbf{0} \quad \text{i.e.} \begin{bmatrix} xv_1 \\ \vdots \\ xv_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$
If some $v_i \neq 0$, then $x = 0$ is the only solution. So $\{\mathbf{v}\}$ is linearly independent if $\mathbf{v} \neq \mathbf{0}$.

Some easy cases:

- Sets containing two vectors $\{\mathbf{u}, \mathbf{v}\}$: then the linear dependence equation is

$$x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}.$$

Using the same argument as in the example on p4, we can show that, if $\mathbf{v} = c\mathbf{u}$ for any c, then $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent:

$$\mathbf{v} = c\mathbf{u} \quad \text{means} \quad c\mathbf{u} + (-1)\mathbf{v} = \mathbf{0}.$$

so (-1) is a nonzero weight. The same argument applies if $\mathbf{u} = d\mathbf{v}$ for any d. Is this the only way in which two vectors can be linearly dependent?

The answer is yes: Two vectors are linearly dependent if and only if one vector is a multiple of the other, i.e. they have the same or opposite direction.

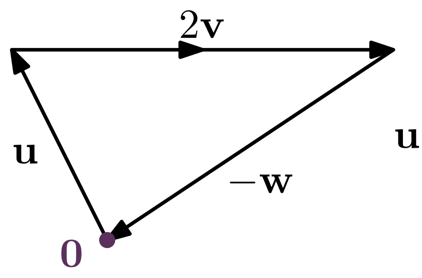
Here's the proof for the "only if" part: suppose $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$ and x_1, x_2 are not both zero.

If $x_1 \neq 0$, then we can divide by it: $\mathbf{u} = \dfrac{-x_2}{x_1}\mathbf{v}$.

Similarly, if $x_2 \neq 0$, then $\mathbf{v} = \dfrac{-x_1}{x_2}\mathbf{u}$.

When there are more vectors, it is hard to tell quickly if a set is linearly independent or dependent.

As shown in this example from p3, three vectors can be linearly dependent without any of them being a multiple of any other vector.



$$\mathbf{u} + 2\mathbf{v} - \mathbf{w} = \mathbf{0}$$

The correct generalisation of the two-vector case is the following: a set of vectors is linearly dependent if and only if one of the vectors is a linear combination of the others. (More specifically: if the weight x_i in the linear dependency relation $x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$ is non-zero, then $\mathbf{v}_i$ is a linear combination of the other $\mathbf{v}$s, by the same argument as in the case of two vectors.)

How to determine if $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p\}$ is linearly independent:

EXAMPLE Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}.$

a. Determine if $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is linearly independent.
b. If possible, find a linear dependence relation among $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$.

Solution: (a) $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is linearly independent if ____

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{array}\right] \quad \text{row reduces to} \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -18 & 0 \\ & & & \end{array}\right]$$

x_3 is a free variable

⇒____⇒ there are nontrivial solutions.

(Alternative explanation:

⇒____⇒ there are nontrivial solutions.)

$\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is ______________________

(b) Reduced echelon form: $\left[\begin{array}{ccc|c} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Let $x_3 =$ _____ (any nonzero number). Then $x_1 =$ _____ and $x_2 =$ _____.

$$\underline{\hspace{2em}}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \underline{\hspace{2em}}\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \underline{\hspace{2em}}\begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\underline{\hspace{2em}}\mathbf{v}_1 + \underline{\hspace{2em}}\mathbf{v}_2 + \underline{\hspace{2em}}\mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

A non-trivial solution to $A\mathbf{x} = \mathbf{0}$ is a linear dependence relation between the columns of A: $A\mathbf{x} = \mathbf{0}$ means $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

Theorem: Uniqueness of solutions for linear systems: For a matrix A, the following are equivalent:

a. $A\mathbf{x} = \mathbf{0}$ has no non-trivial solution (i.e. $\mathbf{x} = \mathbf{0}$ is the only solution).
b. If $A\mathbf{x} = \mathbf{b}$ is consistent, then it has a unique solution.
c. The columns of A are linearly independent.
d. A has a pivot position in every column (i.e. all variables are basic).

In particular: the row reduction algorithm produces at most one pivot position in each row of A. So, if A has more columns than rows (a "fat" matrix), then A cannot have a pivot position in every column.

So a set of more than n vectors in $\mathbb{R}^n$ is always linearly dependent.

Exercise: Combine this with the Theorem of Existence of Solutions (Week 2 p23) to show that a set of n linearly independent vectors span $\mathbb{R}^n$.

Theorem: Uniqueness of solutions for linear systems: For a matrix A, the following are equivalent:

a. $A\mathbf{x} = \mathbf{0}$ has no non-trivial solution (i.e. $\mathbf{x} = \mathbf{0}$ is the only solution).
b. If $A\mathbf{x} = \mathbf{b}$ is consistent, then it has a unique solution.
c. The columns of A are linearly independent.
d. A has a pivot position in every column (i.e. all variables are basic).

Study tip: now that we're working with different types of mathematical objects (matrices, vectors, equations, numbers), you should be careful which properties apply to which objects: e.g. linear independence applies to a set of vectors, not to a matrix (at least not until Chapter 4). Do not say "$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix}$ is linearly dependent" when

one object with 9 numbers

you mean "$\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} \right\}$ are linearly dependent".

three objects each with 3 numbers

Tip: in proofs and in computations, linear dependence and independence are handled differently (see p4).

dependence: find ONE nonzero solution to $x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$.

independence: SOLVE $x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$ and show there are no nonzero solutions.

Another way to say the definition of linear independence:

$$\text{if } x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}, \text{ then } x_1 = \cdots = x_p = 0.$$

A WRONG definition of linear independence:

$$x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}, \text{ where } x_1 = \cdots = x_p = 0.$$

i.e. if we choose $x_1 = \cdots = x_p = 0$

The wrong definition is saying, if $x_1 = \cdots = x_p = 0$, then $x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$. This is always true, no matter what $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are, so it doesn't give any information about the vectors.

The words between the formulas are important, they explain how the formulas are related to each other.

A second exercise in writing proofs.

EXAMPLE: Suppose $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Show that $\{\mathbf{u}, \mathbf{u}+\mathbf{v}\}$ is linearly independent.

STRATEGY 1: direct proof

Step 1: Write out the conclusion as formulas:

Step 2: **on a separate piece of paper**, use definitions to write out the given information as formulas. Be careful to use different letters in different formulas.

Step 3: if the required conclusion (from Step 1) is about "the **only** solutions": solve the required equations using the information in Step 2.

STRATEGY 2: proof by contradiction / contrapositive:

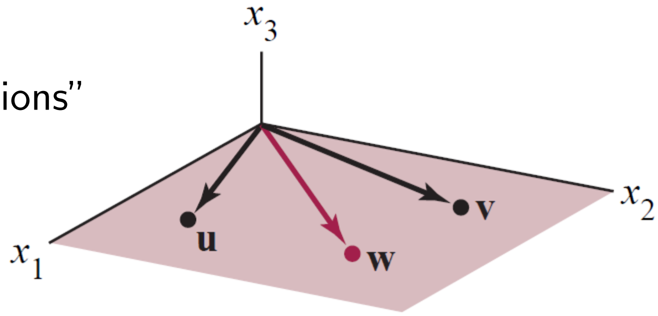
Partial summary of linear dependence:

The definition: $x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$ has a non-trivial solution (not all x_i are zero); equivalently, it has infinitely many solutions.

Equivalently: one of the vectors is a linear combination of the others (see p8, also Theorem 7 in textbook). But it might not be the case that every vector in the set is a linear combination of the others (see ex. sheet #5 q2b).

Computation: rref $\left(\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_p \\ | & & | \end{bmatrix}\right)$ has at least one free variable.

Informal idea: the vectors are in "similar directions"



Partial summary of linear dependence (continued):

Easy examples:

- Sets containing the zero vector;
- Sets containing "too many" vectors (more than n vectors in $\mathbb{R}^n$);
- Multiples of vectors: e.g. $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}4\\2\end{bmatrix}\right\}$ (this is the only possibility if the set has two vectors);
- Other examples: e.g. $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$. Make your own examples!

Adding vectors to a linearly dependent set still makes a linearly dependent set (see ex. sheet #5 q2c).
Equivalent: removing vectors from a linearly independent set still makes a linearly independent set (because P implies Q is equivalent to (not Q) implies (not P) - this is the contrapositive).

Study tips:

- Linear independence will appear again in many topics throughout the class, so I suggest you add to this summary throughout the semester, so you can see the connections between linear independence and the other topics.
- Topic summaries like this one is useful for exam revision, but even more useful is making these summaries yourself. I encourage you to use my summary as a template for your own summaries of the other topics.
- Examples can be useful for solving true/false questions: if a true/false question is about a linear dependent set, try it on the examples on the previous page. Try to make a counterexample, and if you can't, it will give you some idea of why the statement is true.

§1.8-1.9: Linear Transformations

This week's goal is to think of the equation $A\mathbf{x} = \mathbf{b}$ in terms of the "multiplication by A" function: its input is $\mathbf{x}$ and its output is $\mathbf{b}$.

Primary One:

$2^2 = 4$
$3^2 = 9$

Primary Four:

Think of this as:

$$\begin{tikzcd} 2 \arrow[r, "\text{squaring}"'] & 4 \\ 3 \arrow[r] & 9 \end{tikzcd}$$

Last week:

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Today:

Think of this as:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \xrightarrow{\text{multiply by } A} \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

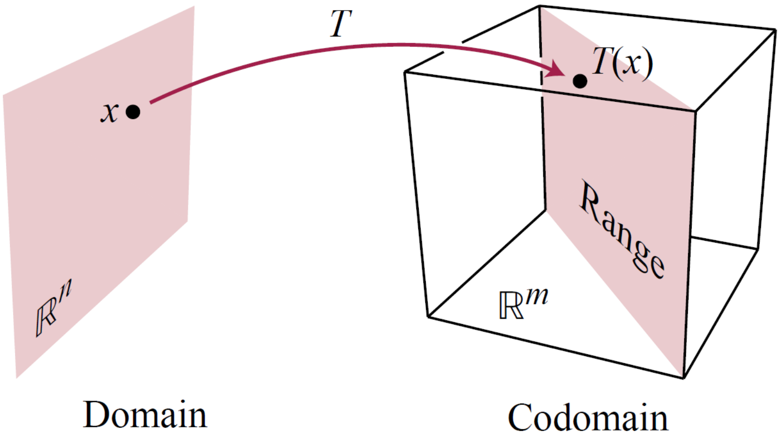
$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\text{multiply by } A} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Goal: think of the equation $A\mathbf{x} = \mathbf{b}$ in terms of the "multiplication by A" function: its input is $\mathbf{x}$ and its output is $\mathbf{b}$.

In this class, we are interested in functions that are linear (see p6 for the definition).
Key skills:

- i Determine whether a function is linear (p7-10);
  (This involves the important mathematical skill of "axiom checking", which also appears in other classes.)
- ii Find the standard matrix of a linear function (p13-14);
- iii Describe existence and uniqueness of solutions in terms of linear functions (p18-28).

Definition: A *function* f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $f(\mathbf{x})$ in $\mathbb{R}^m$. We write $f : \mathbb{R}^n \to \mathbb{R}^m$.



$\mathbb{R}^n$ is the *domain* of f.

$\mathbb{R}^m$ is the *codomain* of f.

$f(\mathbf{x})$ is the *image of* $\mathbf{x}$ *under* f.

The *range* is the set of all images. It is a subset of the codomain.

Example: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.
Its domain = codomain = $\mathbb{R}$, its range = $\{y \in \mathbb{R} \,|\, y \geq 0\}$.

Examples:

$f:\mathbb{R}^2\to\mathbb{R}^3$, defined by $f\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}x_2^3\\2x_1+x_2\\0\end{bmatrix}$.

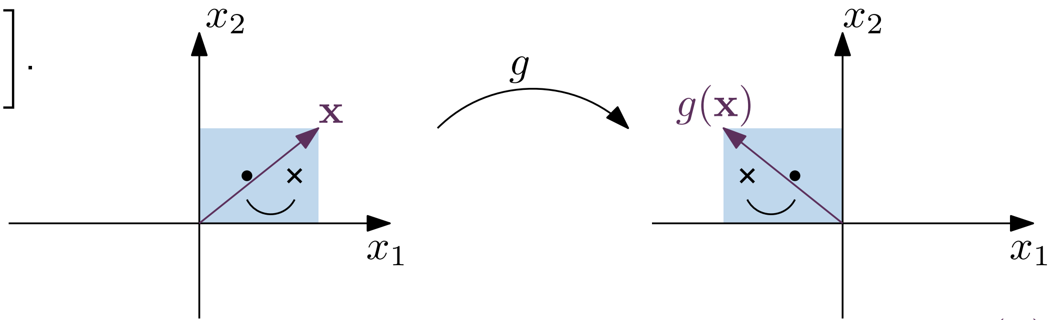
The range of f is the plane $z=0$ (it is obvious that the range must be a subset of the plane $z=0$, and with a bit of work (see p20), we can show that all points in $\mathbb{R}^3$ with $z=0$ is the image of some point in $\mathbb{R}^2$ under f).

$h:\mathbb{R}^3\to\mathbb{R}^2$, given by the matrix transformation $h(\mathbf{x})=\begin{bmatrix}8&3&-4\\5&1&2\end{bmatrix}\mathbf{x}$.

Geometric Examples:

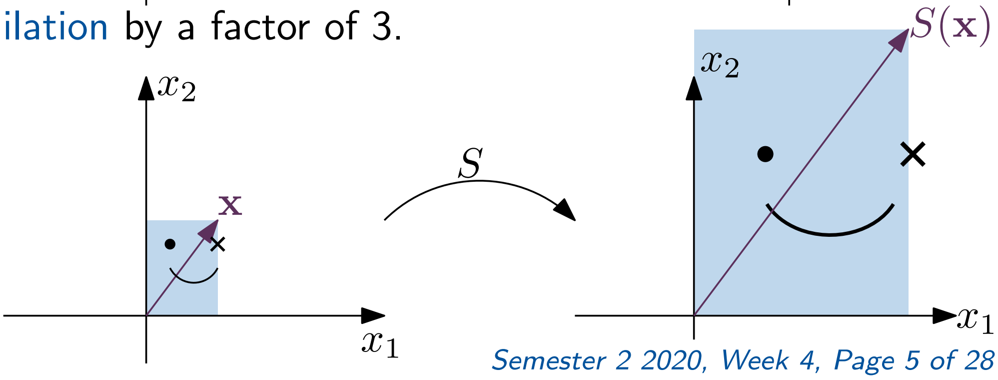
$g:\mathbb{R}^2\to\mathbb{R}^2$, given by reflection through the x_2-axis.

$g\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}-x_1\\x_2\end{bmatrix}$.



$S:\mathbb{R}^2\to\mathbb{R}^2$, given by dilation by a factor of 3.

$S(\mathbf{x})=3\mathbf{x}$.



In this class, we will concentrate on functions that are linear. (For historical reasons, people like to say "linear transformation" instead of "linear function".)

Definition: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in the domain of T.

For your intuition: the name "linear" is because these functions preserve lines:
A line through the point $\mathbf{p}$ in the direction $\mathbf{v}$ is the set $\{\mathbf{p} + s\mathbf{v} | s \in \mathbb{R}\}$.
If T is linear, then the image of this set is

$$T(\mathbf{p} + s\mathbf{v}) \overset{1}{=} T(\mathbf{p}) + T(s\mathbf{v}) \overset{2}{=} T(\mathbf{p}) + sT(\mathbf{v}),$$

the line through the point $T(\mathbf{p})$ in the direction $T(\mathbf{v})$.
(If $T(\mathbf{v}) = \mathbf{0}$, then the image is just the point $T(\mathbf{p})$.)

Fact: A linear transformation T must satisfy $T(\mathbf{0}) = \mathbf{0}$.

Proof: (sketch) Put $c = 0$ in condition 2.

Definition: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$, in the domain of T.

Example: $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2^3 \\ 2x_1 + x_2 \\ 0 \end{bmatrix}$ is not linear:

Take $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c = 2$:

$$f\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 6 \\ 0 \end{bmatrix}.$$

$$2f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 8 \\ 6 \\ 0 \end{bmatrix}.$$

So condition 2 is false for f.

Exercise: find a $\mathbf{u}$ and a $\mathbf{v}$ to show that condition 1 is also false.

DEFINITION: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$, in the domain of T.

EXAMPLE: $g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ (reflection through the x_2-axis) is linear:

1.

add the input vectors $$g\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = g\left(\begin{bmatrix} \quad \\ \quad \end{bmatrix}\right)$$

substitute $\mathbf{u} + \mathbf{v}$ for $\mathbf{x}$ in the formula for g $$=$$

separate the $\mathbf{u}$ terms and $\mathbf{v}$ terms $$=$$

check that this is $g(\mathbf{u}) + g(\mathbf{v})$ $$= g\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + g\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

2.

multiply the input vector $$g\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = g\left(\begin{bmatrix} \quad \\ \quad \end{bmatrix}\right)$$

substitute $c\mathbf{u}$ for $\mathbf{x}$ in the formula for g $$=$$

factor out c $$=$$

check that the remaining part is $g(\mathbf{u})$ $$= cg\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$$

Definition: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$, in the domain of T.

For simple functions, we can combine the two conditions at the same time, and check just one statement: $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$, for all scalars c, d and all vectors $\mathbf{u}, \mathbf{v}$. (Condition 1 is the case $c = d = 1$, condition 2 is the case $d = 0$. Exercise: show that if T satisfies conditions 1 and 2, then T satisfies the combined condition.)

Example: $S(\mathbf{x}) = 3\mathbf{x}$ (dilation by a factor of 3) is linear:

$$S(c\mathbf{u} + d\mathbf{v}) = 3(c\mathbf{u} + d\mathbf{v}) = 3c\mathbf{u} + 3d\mathbf{v} = cS(\mathbf{u}) + dS(\mathbf{v}).$$

Important Example: All matrix transformations $T(\mathbf{x}) = A\mathbf{x}$ are linear:

$$T(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u}) + A(d\mathbf{v}) = cA\mathbf{u} + dA\mathbf{v} = cT(\mathbf{u}) + dT(\mathbf{v}).$$

Definition: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$, in the domain of T.

Notice from the previous examples:
To show that a function is linear, check both conditions for general $\mathbf{u}, \mathbf{v}, c$ (i.e. use variables).
To show that a function is not linear, show that one of the conditions is not satisfied for a particular numerical values of $\mathbf{u}$ and $\mathbf{v}$ (for 1) or of c and $\mathbf{u}$ (for 2).

If you don't know whether a function is linear, work out the formulas for $T(c\mathbf{u})$ and $cT(\mathbf{u})$ separately (for general variables c and $\mathbf{u}$) and see if they are the same. If they're different, this should help you find numerical values for your counterexample (and similarly for $T(\mathbf{u}+\mathbf{v})$ and $T(\mathbf{u}) + T(\mathbf{v})$).

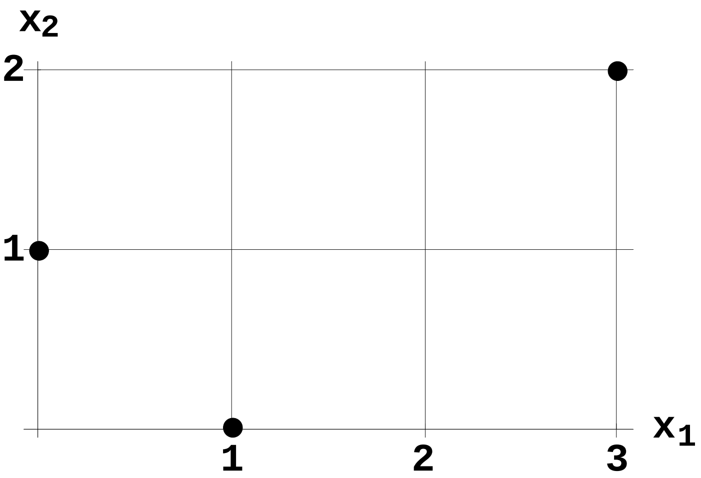
Some people find it easier to work with condition 2 first, before condition 1, because there are fewer vector variables.

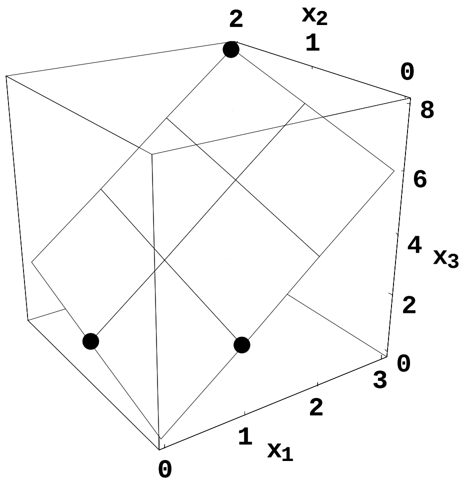
EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose $T : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ is a linear transformation with

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find the image of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Solution:





$$T(3\mathbf{e}_1 + 2\mathbf{e}_2) = 3T(\mathbf{e}_1) + 2T(\mathbf{e}_2)$$

In general:

Write $\mathbf{e}_i$ for the vector with 1 in row i and 0 in all other rows.
(So $\mathbf{e}_i$ means a different thing depending on which $\mathbb{R}^n$ we are working in.)

For example, in $\mathbb{R}^3$, we have $\mathbf{e_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{e_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

$\{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ span $\mathbb{R}^n$, and $\mathbf{x} = \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = x_1\mathbf{e_1} + \cdots + x_n\mathbf{e_n}$.

So, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$T(\mathbf{x}) = T(x_1\mathbf{e_1}+\cdots+x_n\mathbf{e_n}) = x_1T(\mathbf{e_1})+\cdots+x_nT(\mathbf{e_n}) = \begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & | & | \end{bmatrix} \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}$$

Theorem 10: The matrix of a linear transformation: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as a matrix transformation: $T(\mathbf{x}) = A\mathbf{x}$ where A is the *standard matrix for T*, the $m \times n$ matrix given by

$$A = \begin{bmatrix} | & | & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & | & | \end{bmatrix}.$$

We can think of the standard matrix as a compact way of storing the information about T.

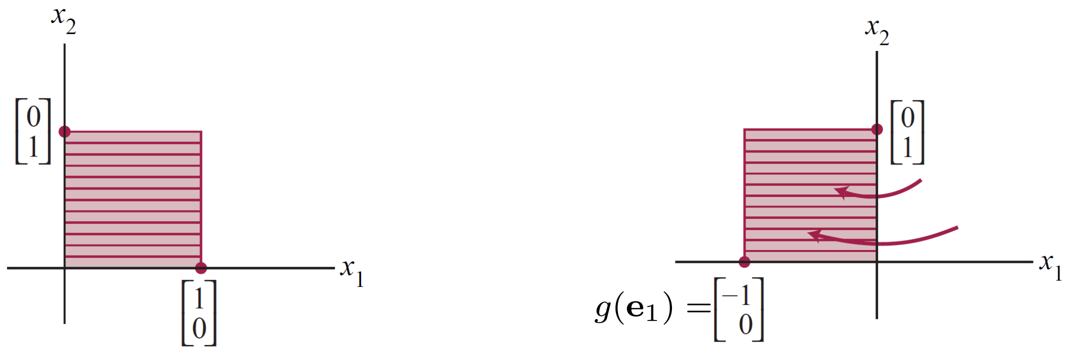
Other notations for the standard matrix for T (see §5.4, week 9) are $[T]$ and $[T]_{\mathcal{E}}$.

Example: $S : \mathbb{R}^2 \to \mathbb{R}^2$, given by dilation by a factor of 3, $S(\mathbf{x}) = 3\mathbf{x}$.

$$S(\mathbf{e_1}) = S\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 3\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\0\end{bmatrix}, \quad S(\mathbf{e_2}) = S\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 3\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\3\end{bmatrix}.$$

So the standard matrix of S is $\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$, i.e. $S(\mathbf{x}) = \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\mathbf{x}$.

Example: $g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ (reflection through the x_2-axis):

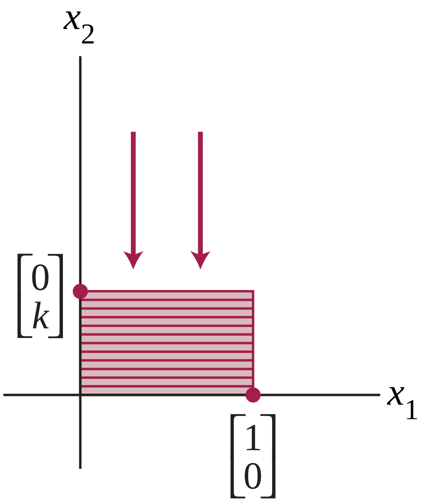
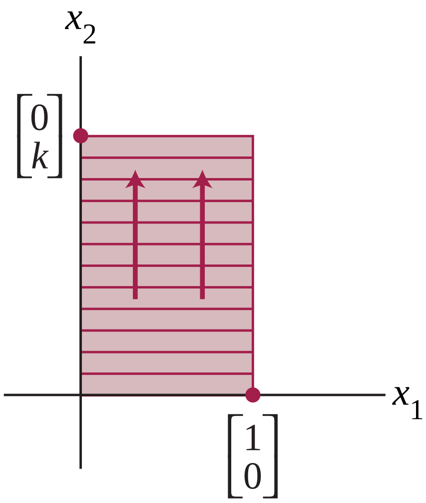


The standard matrix of g is $\begin{bmatrix} | & | \\ g(\mathbf{e}_1) & g(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

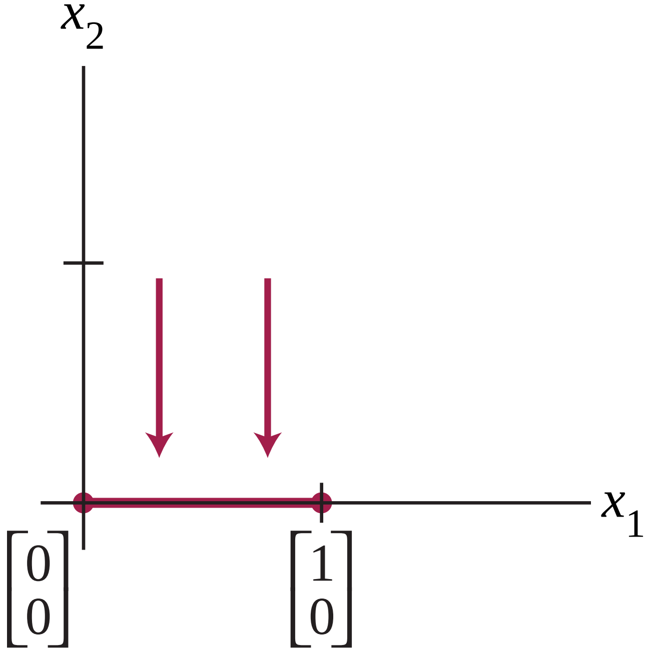
We can check that this gives the correct formula for g: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$.

Further examples of geometric linear transformations:

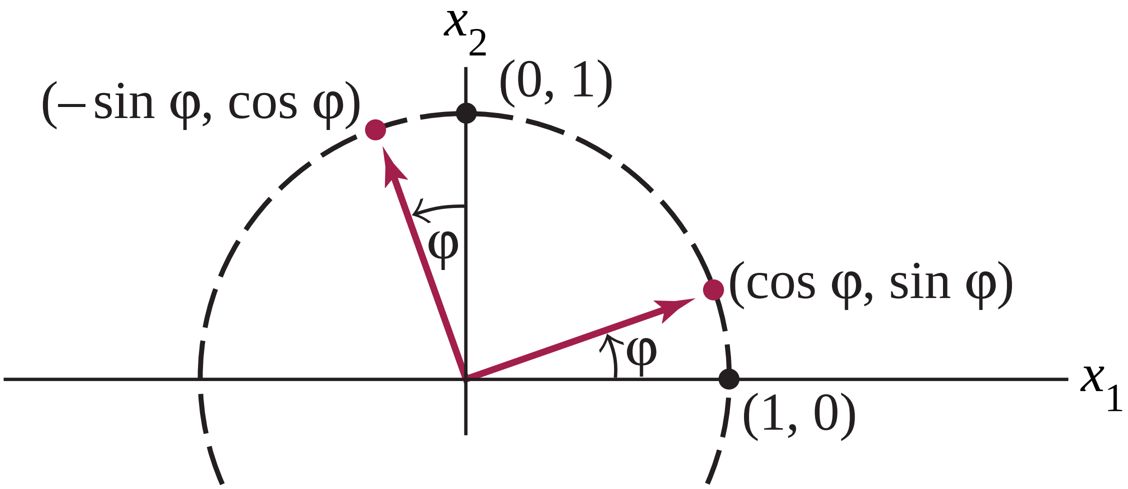
Vertical Contraction and Expansion

Image of the Unit Square		Standard Matrix
 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} & \\ & \end{bmatrix}$

Projection onto the x_1-axis

Image of the Unit Square	Standard Matrix
	$\begin{bmatrix} & \\ & \end{bmatrix}$

EXAMPLE: $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation counterclockwise about the origin through an angle φ:

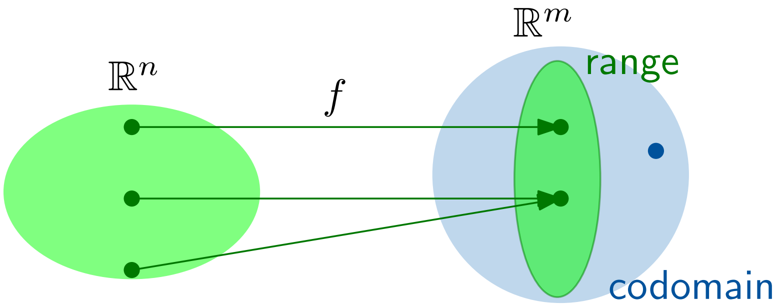


Now we rephrase our existence and uniqueness questions in terms of functions.

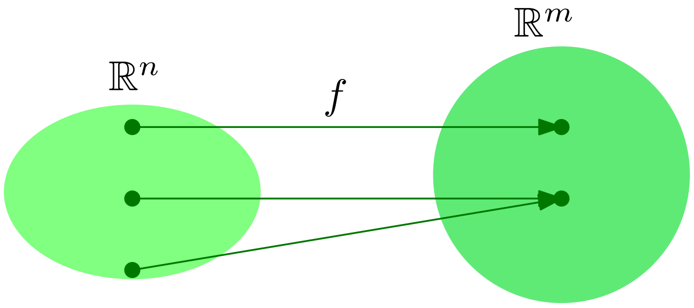
Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *onto* (surjective) if each $\mathbf{y}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

Other ways of saying this:

- The equation $f(\mathbf{x}) = \mathbf{y}$ has a solution for every $\mathbf{y}$ in $\mathbb{R}^m$,
- The range is all of the codomain $\mathbb{R}^m$.



f is not onto, because there are (blue) points in the codomain outside the range



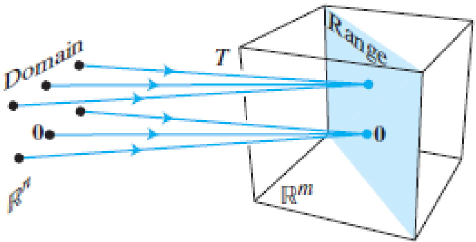
f is onto

Now we rephrase our existence and uniqueness questions in terms of functions.

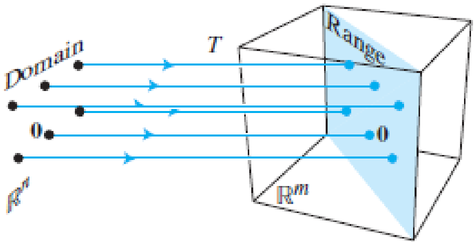
Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *one-to-one* (injective) if each $\mathbf{y}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Other ways of saying this:

- The equation $f(\mathbf{x}) = \mathbf{y}$ has no solutions or a unique solution,
- If $f(\mathbf{x}_1) = f(\mathbf{x}_2)$, then $\mathbf{x}_1 = \mathbf{x}_2$,
- ??? (A comparison of sets, but it only works for linear transformations, see p23).



T is *not* one-to-one



T is one-to-one

$f : \mathbb{R}^n \to \mathbb{R}^m$ is *onto* (surjective) if $f(\mathbf{x}) = \mathbf{y}$ has one or more solutions, for each $\mathbf{y}$ in $\mathbb{R}^m$.

$f : \mathbb{R}^n \to \mathbb{R}^m$ is *one-to-one* (injective) if $f(\mathbf{x}) = \mathbf{y}$ has zero or one solutions, for each $\mathbf{y}$ in $\mathbb{R}^m$.

Example: $f : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2^3 \\ 2x_1 + x_2 \\ 0 \end{bmatrix}$.

f is not onto, because $f(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ does not have a solution.

f is one-to-one:
if $y_3 \neq 0$, then $f(\mathbf{x}) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ does not have a solution,

if $y_3 = 0$, then the unique solution to $f(\mathbf{x}) = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}$ is $x_2 = \sqrt[3]{y_1}$,
$x_1 = \frac{1}{2}(y_2 - x_2) = \frac{1}{2}(y_2 - \sqrt[3]{y_1})$.

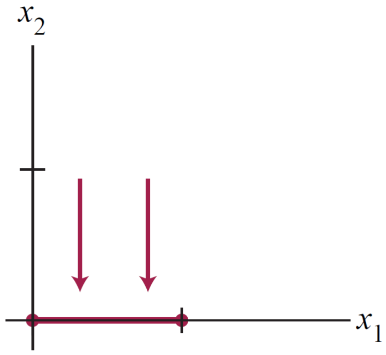
There is an easier way to check if a linear transformation is one-to-one:

Definition: The *kernel* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of solutions to $T(\mathbf{x}) = \mathbf{0}$.

Or, in set notation: $\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$.

Example: Let T be projection onto the x_1-axis, whose standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (i.e. $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x}$).

The kernel of T is the solution set of $T(\mathbf{x}) = \mathbf{0}$, i.e. the solution set of $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$. Using the usual algorithm, this solution set is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} t \middle| t \in \mathbb{R} \right\}$, which is the x_2-axis.



It is also clear from the geometric description of projection that the x_2-axis is mapped to the origin.

There is an easier way to check if a linear transformation is one-to-one:

Recall: given $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation, $\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$.

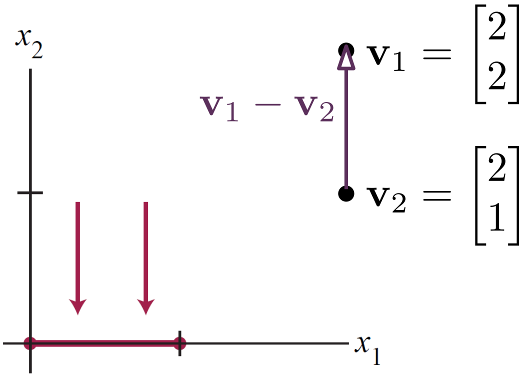
Fact: If $T(\mathbf{v_1}) = T(\mathbf{v_2})$, then $\mathbf{v_1} - \mathbf{v_2}$ is in the kernel of T.

Example: Let T be projection onto the x_1-axis.

The previous page showed that $\ker T$ is the x_2-axis.

Notice that $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and

$\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is in the kernel.



Proof of Fact: (We need to show $\mathbf{v}_1 - \mathbf{v}_2 \in \ker(T)$, i.e. $T(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$.)

$T(\mathbf{v_1} - \mathbf{v_2}) \underset{\uparrow}{=} T(\mathbf{v_1}) - T(\mathbf{v_2}) = \mathbf{0}$, so $\mathbf{v_1} - \mathbf{v_2} \in \ker T$.

$\because$ T is linear

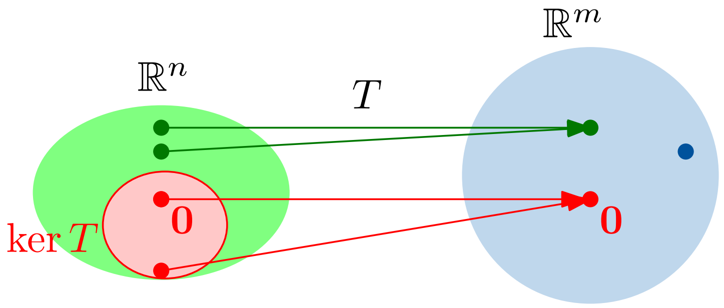
Tip: in any proof about linear transformations, use $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$

There is an easier way to check if a linear transformation is one-to-one:

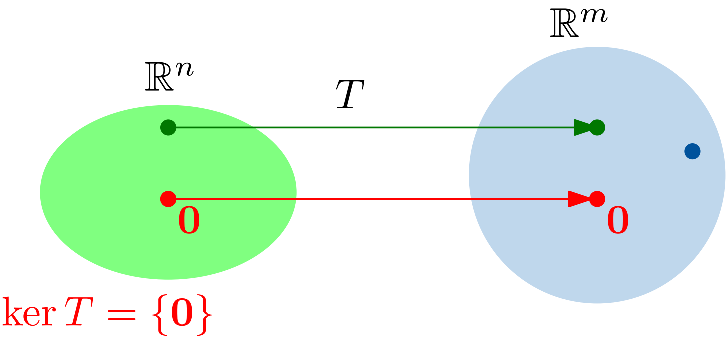
Recall: given $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation, $\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$.

Fact: If $T(\mathbf{v_1}) = T(\mathbf{v_2})$, then $\mathbf{v_1} - \mathbf{v_2}$ is in the kernel of T.

Theorem: A linear transformation is one-to-one if and only if its kernel is $\{\mathbf{0}\}$.



T is not one-to-one, because there are nonzero (red) points in the kernel, which T sends to $\mathbf{0}$.



T is one-to-one

There is an easier way to check if a linear transformation is one-to-one:

Recall: given $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation, $\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$.

Fact: If $T(\mathbf{v_1}) = T(\mathbf{v_2})$, then $\mathbf{v_1} - \mathbf{v_2}$ is in the kernel of T.

Theorem: A linear transformation is one-to-one if and only if its kernel is $\{\mathbf{0}\}$.

Warning: the theorem is only for linear transformations. For other functions, the solution sets of $f(\mathbf{x}) = \mathbf{y}$ and $f(\mathbf{x}) = \mathbf{0}$ are not related.

Proof:

Suppose T is one-to-one. Taking $\mathbf{y} = \mathbf{0}$ in the definition of one-to-one shows $T(\mathbf{x}) = \mathbf{0}$ has at most one solution. Since $\mathbf{0}$ is a solution (because T is linear), it must be the only one. So its kernel is $\{\mathbf{0}\}$.

Suppose the kernel of T is $\{\mathbf{0}\}$. We need to show T is one-to-one, (i.e. if $T(\mathbf{x}_1) = T(\mathbf{x}_2)$, then $\mathbf{x}_1 = \mathbf{x}_2$.)
If $T(\mathbf{x}_1) = T(\mathbf{x}_2)$, then by the Fact, $\mathbf{x}_1 - \mathbf{x}_2 \in \ker T = \{\mathbf{0}\}$, so $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$, so $\mathbf{x}_1 = \mathbf{x}_2$.

Theorem: A linear transformation is one-to-one if and only if its kernel is $\{\mathbf{0}\}$.

So a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one if and only if the set of solutions to $A\mathbf{x} = \mathbf{0}$ is $\{\mathbf{0}\}$. This is equivalent to many other things:

Theorem: Uniqueness of solutions to linear systems: For a matrix A, the following are equivalent:

a. $A\mathbf{x} = \mathbf{0}$ has no non-trivial solution (i.e. $\mathbf{x} = \mathbf{0}$ is the only solution).
b. If $A\mathbf{x} = \mathbf{b}$ is consistent, then it has a unique solution.
c. The columns of A are linearly independent.
d. A has a pivot position in every column (i.e. all variables are basic).
e. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
f. The kernel of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\{\mathbf{0}\}$.

Notice that e. is in terms of linear transformations, b. is in terms of matrices and linear equations, and they are the same thing.
f. is in terms of linear transformations, a. is in terms of matrices and linear equations, and they are the same thing.

Now let's think about onto and existence of solutions.

Recall that the range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of images, i.e. $\text{range}T = \{\mathbf{y} \in \mathbb{R}^m | \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\} = \{T(\mathbf{x}) | \mathbf{x} \in \mathbb{R}^n\}$.

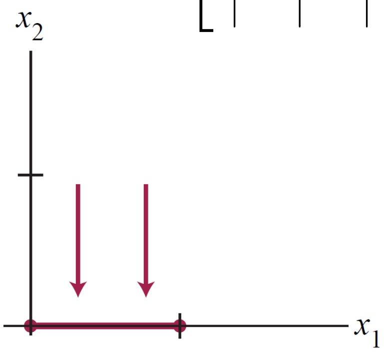
So, if $T(\mathbf{x}) = A\mathbf{x}$ (i.e. A is the standard matrix of T), then

$$\begin{aligned}\text{range}T &= \{\mathbf{y} \in \mathbb{R}^m | \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}\\ &= \{\mathbf{b} \in \mathbb{R}^m | \mathbf{b} = A\mathbf{x} \text{ has a solution}\}\\ &= \{\mathbf{b} \in \mathbb{R}^m | \mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n \text{ for some } x_i\} \quad \text{where } A = \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{bmatrix}.\\ &= \text{span of the columns of } A\end{aligned}$$

Example: Let T be projection onto the x_1-axis, whose standard matrix is $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Its range is the span of the columns of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, i.e.

Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, which is the x_1-axis.



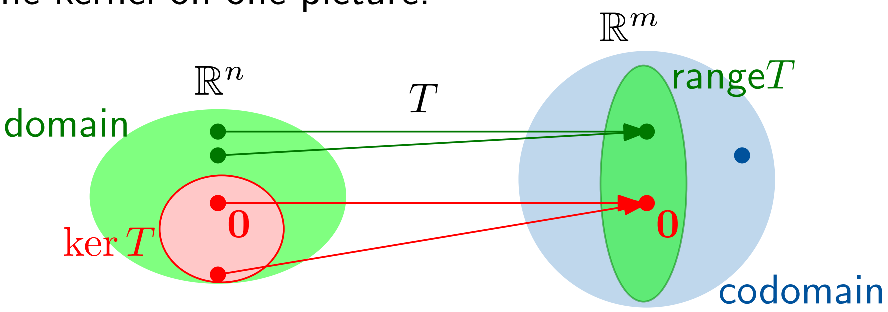
It is clear from the geometric description of projection that the set of images is the x_1-axis.

The range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.
And a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ is onto if and only if its range is all of $\mathbb{R}^m$.
Putting these together: $\mathbf{x} \mapsto A\mathbf{x}$ is onto if and only if $A\mathbf{x} = \mathbf{b}$ is always consistent, and this is equivalent to many things:

Theorem 4: Existence of solutions to linear systems: For an $m \times n$ matrix A, the following statements are logically equivalent (i.e. for any particular matrix A, they are all true or all false):

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.
e. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.
f. The range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.

The range and the kernel on one picture:



$\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$
defined by a condition

range$T = \{T(\mathbf{x}) | \mathbf{x} \in \mathbb{R}^n\}$
defined by a form

Remember from weeks 1-3 that existence and uniqueness are separate, unrelated concepts. Similarly, onto and one-to-one are unrelated:
Exercise 1: think of a linear transformation that is onto but not one-to-one, or both onto and one-to-one, or etc.

Exercise 2: consider the other linear transformations in this week's notes. Are they onto? Are they one-to-one?

§2.1: Matrix Operations

We have several ways to combine functions to make new functions:

- Addition: if f, g have the same domains and codomains, then we can set $(f+g)\mathbf{x} = f(\mathbf{x}) + g(\mathbf{x})$,
- Composition: if the codomain of f is the domain of g, then we can set $(g \circ f)\mathbf{x} = g(f(\mathbf{x}))$,
- Inverse (§2.2): if f is one-to-one and onto, then we can set $f^{-1}(\mathbf{y})$ to be the unique solution to $f(\mathbf{x}) = \mathbf{y}$.

It turns out that the sum, composition and inverse of linear transformations are also linear (exercise: prove it!), and we can ask how the standard matrix of the new function is related to the standard matrices of the old functions.

Notation:

The (i, j)-entry of a matrix A is the entry in row i, column j, and is written a_{ij} or $(A)_{ij}$.

The diagonal entries of A are the entries $a_{11}, a_{22}, \ldots$.

e.g. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$

A square matrix has the same number of rows as columns. The associated linear transformation has the same domain and codomain.

A diagonal matrix is a square matrix whose nondiagonal entries are 0.

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The identity matrix I_n is the $n \times n$ matrix whose diagonal entries are 1 and whose nondiagonal entries are 0.
It is the standard matrix for the identity transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = \mathbf{x}$.

e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Addition:

If A, B are the standard matrices for some linear transformations $S, T : \mathbb{R}^n \to \mathbb{R}^m$, then $(S+T)\mathbf{x} = S(\mathbf{x}) + T(\mathbf{x})$ is a linear transformation. What is its standard matrix $A+B$?

Proceed column by column:

First column of the standard matrix of $S+T$

$= (S+T)(\mathbf{e_1})$ definition of standard matrix of $S+T$

$= S(\mathbf{e_1}) + T(\mathbf{e_1})$ definition of $S+T$

$=$ first column of A + first column of B. definition of standard matrix of S and of T

i.e. $(i,1)$-entry of $A+B$=$a_{i1} + b_{i1}$.

The same is true of all the other columns, so $(A+B)_{ij}$=$a_{ij} + b_{ij}$.

Example: $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, \quad A+B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}.$

Scalar multiplication:

If A is the standard matrix for a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^m$, and c is a scalar, then $(cS)\mathbf{x} = c(S\mathbf{x})$ is a linear transformation. What is its standard matrix cA?

Proceed column by column:

First column of the standard matrix of cS

$= (cS)(\mathbf{e_1})$ definition of standard matrix of cS

$= c(S\mathbf{e_1})$ definition of cS

$=$ first column of A multiplied by c. definition of standard matrix of S

i.e. $(i,1)$-entry of cA=ca_{i1}.

The same is true of all the other columns, so $(cA)_{ij} = ca_{ij}$.

Example: $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \quad c = -3, \quad cA = \begin{bmatrix} -12 & 0 & -15 \\ 3 & -9 & -6 \end{bmatrix}.$

Addition and scalar multiplication satisfy some familiar rules of arithmetic:

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

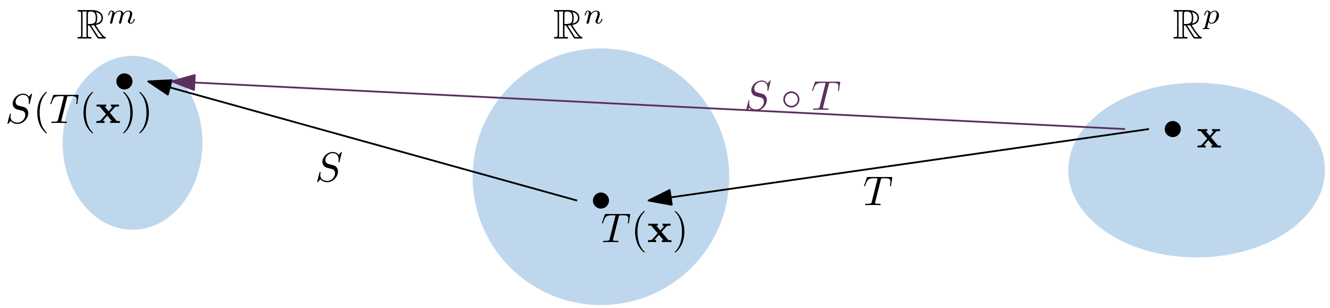
f. $r(sA) = (rs)A$

0 denotes the zero matrix:

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Composition:

If A is the standard matrix for a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^m$
and B is the standard matrix for a linear transformation $T : \mathbb{R}^p \to \mathbb{R}^n$
then the composition $S \circ T$ (T first, then S) is linear.
What is its standard matrix AB?



A is a $m \times n$ matrix,
B is a $n \times p$ matrix,
AB is a $m \times p$ matrix - so the (i, j)-entry of AB cannot simply be $a_{ij}b_{ij}$.

Composition:

Proceed column by column:
First column of the standard matrix of $S \circ T$
$= (S \circ T)(\mathbf{e_1})$ definition of standard matrix of $S \circ T$
$= S(T(\mathbf{e_1}))$ definition of $S \circ T$
$= S(\mathbf{b_1})$ definition of standard matrix of T (writing $\mathbf{b}_j$ for column j of B)
$= A\mathbf{b_1}$, and similarly for the other columns.

So
$$AB = A \begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_p \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\mathbf{b}_1 & \dots & A\mathbf{b}_p \\ | & & | \end{bmatrix}.$$

The jth column of AB is a linear combination of the columns of A using weights from the jth column of B.

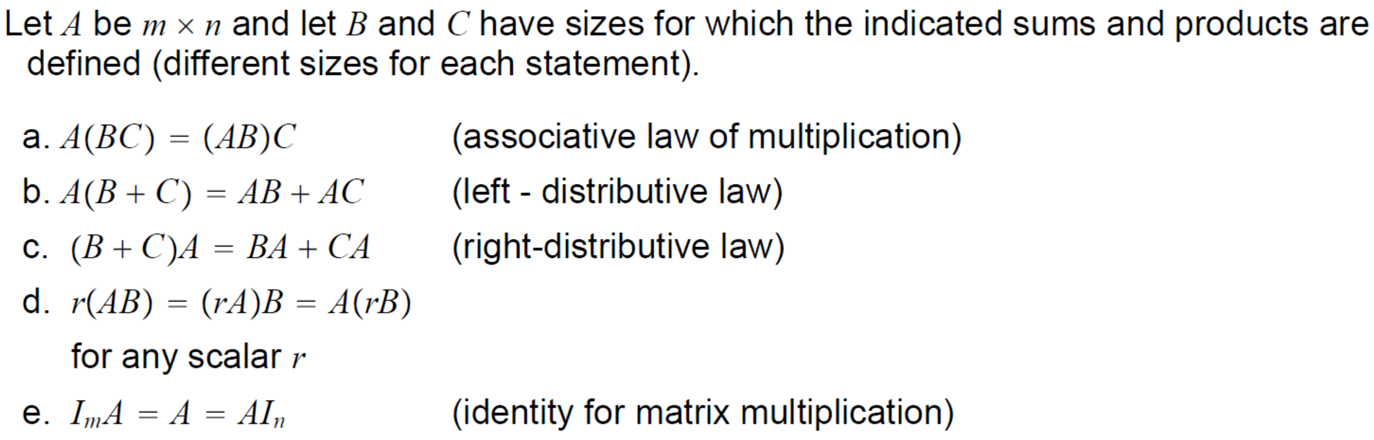
Another view is the row-column method: the (i, j)-entry of AB is
$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

$$AB = A\begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_p \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A\mathbf{b}_1 & \dots & A\mathbf{b}_p \\ | & | & | \end{bmatrix}.$$

The jth column of AB is a linear combination of the columns of A using weights from the jth column of B.

EXAMPLE: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Some familiar rules of arithmetic hold for matrix multiplication...



Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined (different sizes for each statement).

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left - distributive law)

c. $(B + C)A = BA + CA$ (right-distributive law)

d. $r(AB) = (rA)B = A(rB)$
for any scalar r

e. $I_mA = A = AI_n$ (identity for matrix multiplication)

... but not all of them:

- Usually, $AB \neq BA$ (because order matters for function composition: $S \circ T \neq T \circ S$);
- It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$ - so you cannot solve matrix equations by 'factorising'.

A fun application of matrix multiplication:

Consider rotations counterclockwise about the origin.
Rotation through $(\theta + \varphi)$ = (rotation through θ) $\circ$ (rotation through φ).

$$\begin{bmatrix} \cos(\theta + \varphi) & -\sin(\theta + \varphi) \\ \sin(\theta + \varphi) & \cos(\theta + \varphi) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\cos\varphi \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi \end{bmatrix}.$$

So, equating the entries in the first column:
$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$
$\sin(\theta + \varphi) = \cos\theta\sin\varphi + \sin\theta\cos\varphi$

Powers:

For a square matrix A, the kth power of A is $A^k = \underbrace{A \dots A}_{k \text{ times}}$.

If A is the standard matrix for a linear transformation T, then A^k is the standard matrix for T^k, the function that "applies T k times".

Examples:

$$\begin{bmatrix}1&2\\3&0\end{bmatrix}^3 = \begin{bmatrix}1&2\\3&0\end{bmatrix}\left(\begin{bmatrix}1&2\\3&0\end{bmatrix}\begin{bmatrix}1&2\\3&0\end{bmatrix}\right) = \begin{bmatrix}1&2\\3&0\end{bmatrix}\begin{bmatrix}7&2\\3&6\end{bmatrix} = \begin{bmatrix}13&14\\21&6\end{bmatrix}.$$

$$\begin{bmatrix}3&0\\0&-2\end{bmatrix}^3 = \begin{bmatrix}3&0\\0&-2\end{bmatrix}\left(\begin{bmatrix}3&0\\0&-2\end{bmatrix}\begin{bmatrix}3&0\\0&-2\end{bmatrix}\right) = \begin{bmatrix}3&0\\0&-2\end{bmatrix}\begin{bmatrix}9&0\\0&4\end{bmatrix} = \begin{bmatrix}27&0\\0&-8\end{bmatrix} = \begin{bmatrix}3^3&0\\0&(-2)^3\end{bmatrix}.$$

Exercise: show that $\begin{bmatrix}a_{11}&0\\0&a_{22}\end{bmatrix}^k = \begin{bmatrix}a_{11}^k&0\\0&a_{22}^k\end{bmatrix}$, and similarly for larger diagonal matrices.

We can consider polynomials involving square matrices:

Example: Let $p(x) = x^3 - 2x^2 + x - \boxed{2}$ and $A = \begin{bmatrix}1&2\\3&0\end{bmatrix}, D = \begin{bmatrix}3&0\\0&-2\end{bmatrix}$ as on the previous page. Then

use the identity matrix instead of constants

$$p(A) = A^3 - 2A^2 + A - \boxed{2I_2} = \begin{bmatrix}13&14\\21&6\end{bmatrix} - \begin{bmatrix}14&4\\6&12\end{bmatrix} + \begin{bmatrix}1&2\\3&0\end{bmatrix} - \begin{bmatrix}2&0\\0&2\end{bmatrix} = \begin{bmatrix}-2&12\\18&-8\end{bmatrix}.$$

$$p(D) = D^3 - 2D^2 + D - 2I_2 = \begin{bmatrix}27&0\\0&-8\end{bmatrix} - \begin{bmatrix}18&0\\0&8\end{bmatrix} + \begin{bmatrix}3&0\\0&-2\end{bmatrix} - \begin{bmatrix}2&0\\0&2\end{bmatrix} = \begin{bmatrix}10&0\\0&-20\end{bmatrix} = \begin{bmatrix}p(3)&0\\0&p(-2)\end{bmatrix}.$$

For a polynomial involving a single matrix, we can factorise and expand as usual:

Example: $x^3 - 2x^2 + x - 2 = (x^2+1)(x-2)$, and

$$(A^2 + I_2)(A - 2I_2) = \begin{bmatrix}8&2\\3&7\end{bmatrix}\begin{bmatrix}-1&2\\3&-2\end{bmatrix} = \begin{bmatrix}-2&12\\18&-8\end{bmatrix}.$$

But be careful with the order when there are two or more matrices:

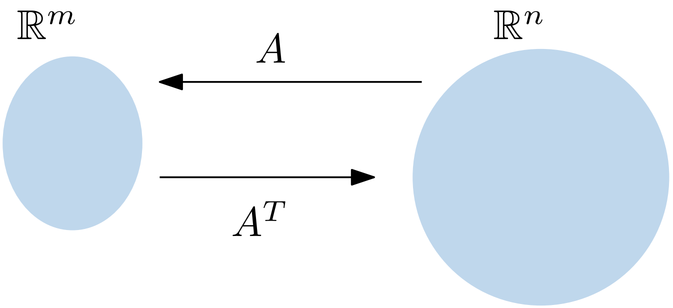
Example: $x^2 - y^2 = (x+y)(x-y)$, but

$(A+D)(A-D) = A^2 - AD + DA - D^2 \neq A^2 - D^2$.

Transpose:

The transpose of A is the matrix A^T whose (i, j)-entry is a_{ji}.
i.e. we obtain A^T by "flipping A through the main diagonal".

As a linear transformation, it "goes in the opposite direction", but it is NOT the inverse function.



Example: $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ 5 & 2 \end{bmatrix}$.

We will be interested in square matrices A such that
$A = A^T$ (symmetric matrix, self-adjoint linear transformation, §7.1), or
$A^{-1} = A^T$ (orthogonal matrix, or isometric linear transformation, §6.2).

Properties of the transpose:

Let A and B denote matrices whose sizes are appropriate for the following sums and products (different sizes for each statement).

a. $(A^T)^T = A$ (I.e., the transpose of A^T is A)

b. $(A + B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

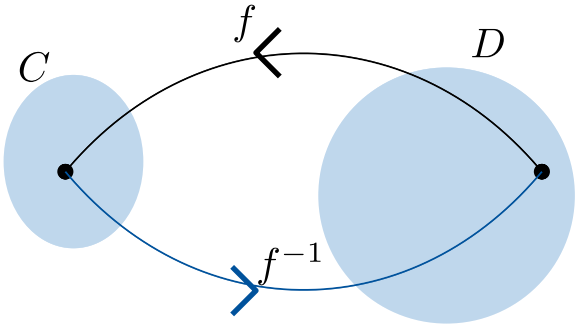
An example to explain d.
this is NOT a proof

$$\overset{A}{\begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}} = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Guess: $\cancel{\overset{A^T}{\begin{bmatrix} 4 & 3 & 0 \\ -2 & -5 & 1 \end{bmatrix}} \overset{B^T}{\begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix}}}$ Cannot multiply these matrices!

Guess again: $\overset{B^T}{\begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix}} \overset{A^T}{\begin{bmatrix} 4 & 3 & 0 \\ -2 & -5 & 1 \end{bmatrix}} = \begin{bmatrix} -4 & -24 & 6 \\ 2 & 26 & -7 \end{bmatrix}$

§2.2: The Inverse of a Matrix



Remember from calculus that the inverse of a function $f : D \to C$ is the function $f^{-1} : C \to D$ such that $f^{-1} \circ f =$ identity function on D and $f \circ f^{-1} =$ identity function on C.

Equivalently, $f^{-1}(y)$ is the unique solution to $f(x) = y$.
So f^{-1} exists if and only if f is one-to-one and onto. Then we say f is invertible.

Let T be a linear transformation whose standard matrix is A. From last week:

- T is one-to-one if and only if A has a pivot position in every column.
- T is onto if and only if A has a pivot position in every row.

So if T is invertible, then A must be a square matrix.

Warning: not all square matrices come from invertible linear transformations, e.g. $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

Remember from calculus that the inverse of a function $f : D \to C$ is the function $f^{-1} : C \to D$ such that $f^{-1} \circ f =$ identity function on D and $f \circ f^{-1} =$ identity function on C.

Definition: A $n \times n$ matrix A is *invertible* if there is a $n \times n$ matrix C satisfying $CA = AC = I_n$.

Fact: A matrix C with this property is unique:
if $BA = AC = I_n$, then $BAC = BI_n = B$ and $BAC = I_nC = C$ so $B = C$.

The matrix C is called the inverse of A, and is written A^{-1}. So

$$A^{-1}A = AA^{-1} = I_n.$$

A matrix that is not invertible is sometimes called singular.

Remember from calculus that the inverse of a function $f : D \to C$ is the function $f^{-1} : C \to D$ such that $f^{-1} \circ f =$ identity function on D and $f \circ f^{-1} =$ identity function on C.

Equivalently, $f^{-1}(y)$ is the unique solution to $f(x) = y$.

Theorem 5: Solving linear systems with the inverse: If A is an invertible $n \times n$ matrix, then, for each $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.

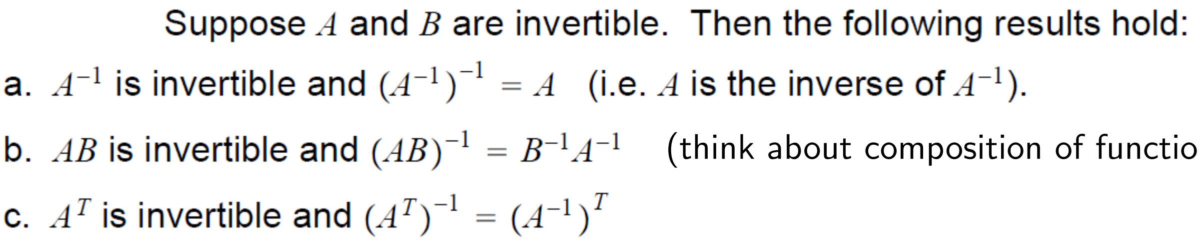
Proof:

1. We show $A^{-1}\mathbf{b}$ is a solution (i.e. $A(A^{-1}\mathbf{b}) = \mathbf{b}$).
$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b}$, so $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$:

2. We show this is the unique solution:
Let $\mathbf{u}$ be any solution to $A\mathbf{x} = \mathbf{b}$, so:
$$A\mathbf{u} = \mathbf{b}$$
Multiply both sides by A^{-1} on the left:
$$A^{-1}(A\mathbf{u}) = A^{-1}\mathbf{b}$$
$$\mathbf{u} = A^{-1}\mathbf{b}.$$

In particular, if A is an invertible $n \times n$ matrix, then rref$(A) = I_n$.

Properties of the inverse:

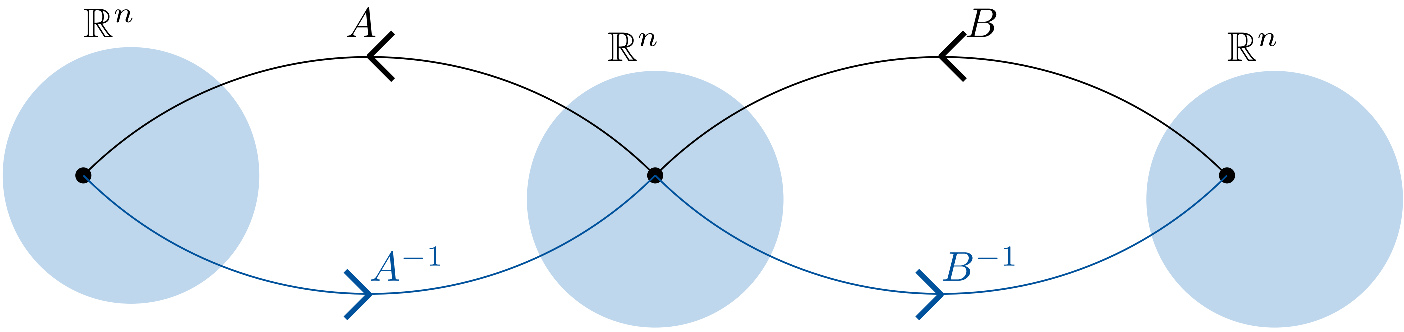


Suppose A and B are invertible. Then the following results hold:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).

b. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$ (think about composition of functions, see diagram below)

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$



Exercise: prove these properties.
(Hint: to show X is the inverse of Y, i.e. $Y^{-1} = X$, you should check $XY = YX = I$.)

Inverse of a 2×2 matrix:

Fact: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

i) if $ad - bc \neq 0$, then A is invertible and $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

ii) if $ad - bc = 0$, then A is not invertible.

Proof of i):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof of ii): next week.

Inverse of a 2×2 matrix:

Example: Let $A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$, the standard matrix of rotation about the origin through an angle φ counterclockwise.

$\cos\varphi\cos\varphi - (-\sin\varphi)\sin\varphi = \cos^2\varphi + \sin^2\varphi = 1 \neq 0$ so A is invertible, and $A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{bmatrix}$, the standard matrix of rotation about the origin through an angle φ clockwise.

Example: Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the standard matrix of projection to the x_1-axis.

$1 \cdot 0 - 0 \cdot 0 = 0$ so B is not invertible.

Exercise: choose a matrix C that is the standard matrix of a reflection, and check that C is invertible and $C^{-1} = C$.

Inverse of a $n \times n$ matrix:

Suppose A is an invertible $n \times n$ matrix.
Let $\mathbf{x}_i$ denote the ith column of A^{-1}.

So $\qquad A^{-1}\mathbf{e}_i = \mathbf{x}_i \qquad$ (to find the ith column of a matrix, multiply by $\mathbf{e}_i$)

$\qquad\qquad \mathbf{e}_i = A\mathbf{x}_i \qquad$ (left-multiply both sides by A)

So we can find $\mathbf{x}_i$ by row-reducing $[A|\mathbf{e_i}]$. Because rref$(A) = I_n$, the result should be $[I_n|\mathbf{x}_i]$.

We carry out this row-reduction for all columns at the same time, i.e. solve all n linear systems at the same time:

$$[A|I_n] = \left[A \left| \underbrace{\begin{matrix} | & | & | \\ \mathbf{e}_1 & \dots & \mathbf{e}_n \\ | & | & | \end{matrix}}_{\text{each column is the right hand side of a different linear system, which all have the same left hand side}} \right. \right] \xrightarrow{\text{row reduction}} \left[I_n \left| \begin{matrix} | & | & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & | & | \end{matrix} \right. \right] = [I_n|A^{-1}].$$

If A is an invertible matrix, then

$$[A|I_n] \xrightarrow{\text{row reduction}} [I_n|A^{-1}].$$

EXAMPLE: Find the inverse of $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 4 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ & & & & & \\ & & & & & \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{array}\right]$$

We showed that, if A is invertible, then $[A|I_n]$ row-reduces to $[I_n|A^{-1}]$.
In other words, if we already knew that A was invertible, then we can find its inverse by row-reducing $[A|I_n]$.
It would be useful if we could apply this without first knowing that A is invertible.

Indeed, we can:
Fact: If $[A|I_n]$ row-reduces to $[I_n|C]$, then A is invertible and $C = A^{-1}$.

Proof: (different from textbook, not too important)
If $[A|I_n]$ row-reduces to $[I_n|C]$, then $\mathbf{c}_i$ is the unique solution to $A\mathbf{x} = \mathbf{e}_i$, so $AC\mathbf{e}_i = A\mathbf{c}_i = \mathbf{e}_i$ for all i, so $AC = I_n$.
Also, by switching the left and right sides, and reading the process backwards, $[C|I_n]$ row-reduces to $[I_n|A]$. So $\mathbf{a}_i$ is the unique solution to $C\mathbf{x} = \mathbf{e}_i$, so $CA\mathbf{e}_i = C\mathbf{a}_i = \mathbf{e}_i$ for all i, so $CA = I_n$.

In particular: an $n \times n$ matrix A is invertible if and only if rref$(A) = I_n$.
Also equivalent: A has a pivot position in every row and column.
For a square matrix, having a pivot position in each row is the same as having a pivot position in each column.

§2.3: Characterisations of Invertible Matrices

As observed at the end of the previous page: for a square $n \times n$ matrix A, the following are equivalent:

- A is invertible.
- $\text{rref}(A) = I_n$.
- A has a pivot position in every row.
- A has a pivot position in every column.

This means that, in the very special case when A is a square matrix, all the statements in the Existence of Solutions Theorem ("green theorem") and all the statements in the Uniqueness of Solutions Thoerem ("red theorem") are all equivalent, so we can put the two lists together to make a giant list of equivalent statements, on the next page. (The third list, in blue, comes from combining the corresponding green and red statements.) (Re the last line: you proved on ex. sheet #9 Q2c,d that it implies the higher lines; exercise: prove that the higher lines imply it.)

Theorem 8: Invertible Matrix Theorem (IMT): For a square $n \times n$ matrix A, the following are equivalent:

A has a pivot position in every row.	A has a pivot position in every column.	$\text{rref}(A) = I_n$.
$A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.	$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	$A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
The columns of A span $\mathbb{R}^n$.	The columns of A are linearly independent.	
The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.	The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an invertible function.
There is a matrix D such that $AD = I_n$.	There is a matrix C such that $CA = I_n$.	A is invertible.

We will add more statements to the Invertible Matrix Theorem throughout the class.

Important consequences:

- line 3: A set of n vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if the set is linearly independent.
- line 4: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ (i.e. same domain and codomain) is one-to-one if and only if it is onto.

Students' main difficulty with IMT (or other theorems from later in the class) is when to use them, i.e. which theorems will help with which proof questions. Some tips:

- Each theorem connects two ideas, e.g. IMT connects existence and uniqueness. When the given information is about one idea, and the conclusion you want is about the other idea, then the theorem may be useful.
- If the situation of the question fits the conditions of the theorem, then that theorem may be useful. E.g. if you see a square matrix, consider IMT.

Theorem 8: Invertible Matrix Theorem continued: A is invertible if and only if A^T is invertible.

(Proof: from p18 $(A^T)^{-1} = (A^{-1})^T$.)

This means that the statements in the Invertible Matrix Theorem are equivalent to the corresponding statements with "row" instead of "column", for example:

- The columns of an $n \times n$ matrix are linearly independent if and only if its rows span $\mathbb{R}^n$. (This is in fact also true for rectangular matrices - transposing switches the green and red statements. Exercise: prove it.)
- If A is a square matrix and $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$, then the rows of A are linearly independent.

Advanced application (important for probability):

Let A be a square matrix. If the entries in each column of A sum to 1, then there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{v}$.

Hint: $(A - I)^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{0}$.

Recall from last week:

FACT: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

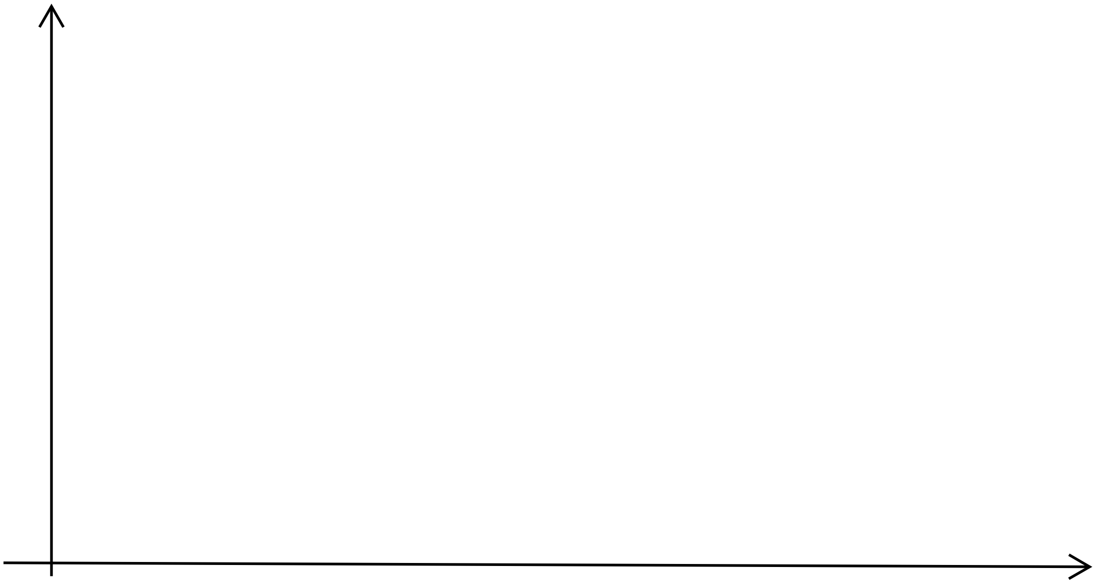
i) if $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, (proof in Week 5 p19)

ii) if $ad - bc = 0$, then A is not invertible. (proof below)

QUESTION: What is the mysterious quantity $ad - bc$?

It's easier to answer this using linear transformations.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation $T(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathbf{x}$. So $T(\mathbf{e_1}) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$ and $T(\mathbf{e_2}) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$.



So: if $ad - bc = 0$, then the image of the unit square under T has zero area, i.e. $T(\mathbf{e_1}), T(\mathbf{e_2})$ lie on the same line. So $T(\mathbf{e_1}), T(\mathbf{e_2})$ (i.e. the columns of A) is ________________________________,

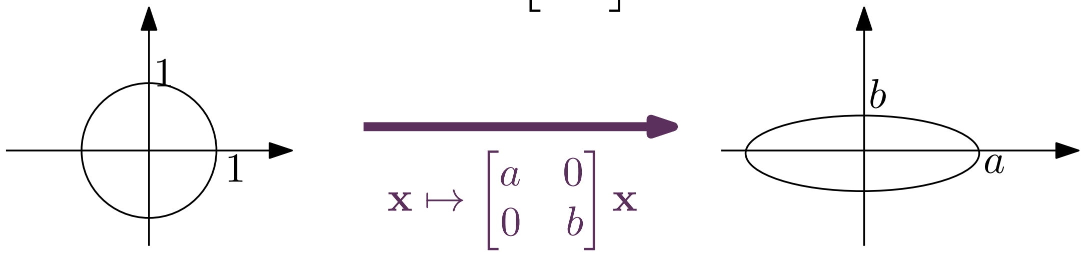
so A is ____________________.

§3.1-3.3: Determinants

Conceptually, the determinant $\det A$ of a square $n \times n$ matrix A is the signed area/volume scaling factor of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$, i.e.:

- For any region S in $\mathbb{R}^n$, the volume of its image $T(S)$ is $|\det A|$ multiplied by the original volume of S,
- If $\det A > 0$, then T does not change "orientation". If $\det A < 0$, then T changes "orientation".

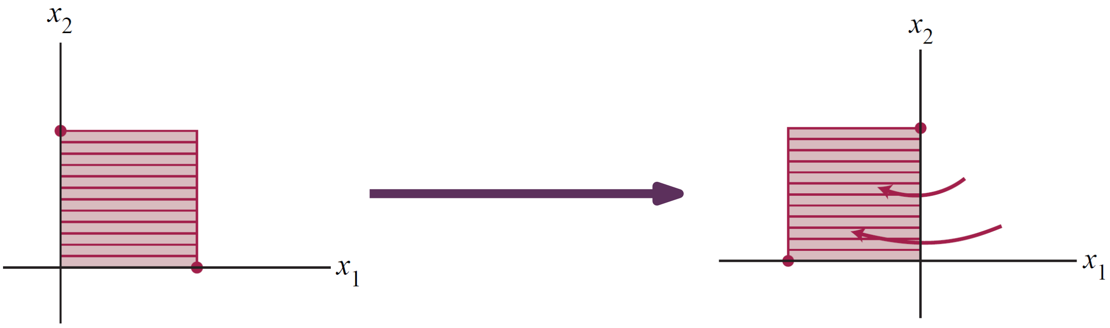
Example: Area of ellipse $= \det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \times$ area of unit circle $= ab\pi$.



This idea is useful in multivariate calculus.

Formula for 2×2 matrix: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

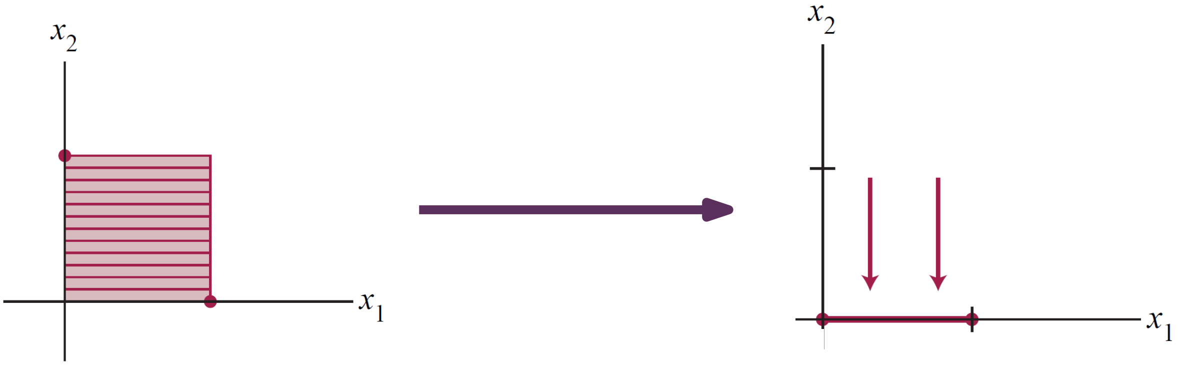
Example: The standard matrix for reflection through the x_2-axis is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Its determinant is -1 $\cdot 1 - 0 \cdot 0 = -1$: reflection does not change area, but changes orientation.



Exercise: Guess what the determinant of a rotation matrix is, and check your answer.

Formula for 2×2 matrix: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

Example: The standard matrix of projection onto the x_1-axis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Its determinant is $1 \cdot 0 - 0 \cdot 0 = 0$. Projection sends the unit square to a line, which has zero area.



Theorem: A is invertible if and only if $\det A \neq 0$.

Calculating Determinants

Notation: A_{ij} is the submatrix obtained from matrix A by deleting the ith row and jth column of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Recall that $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det[a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\begin{aligned} \det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n} a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\det A_{1j} \end{aligned}$$

EXAMPLE: Compute the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

THEOREM 1 The determinant of an $n \times n$ matrix A can be computed by expanding across any row or down any column:

$$\det A = (-1)^{i+1}a_{i1}\det A_{i1} + (-1)^{i+2}a_{i2}\det A_{i2} + \cdots + (-1)^{i+n}a_{in}\det A_{in}$$

$$= \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det A_{ij}$$ (expansion across row i)

$$\det A = (-1)^{1+j}a_{1j}\det A_{1j} + (-1)^{2+j}a_{2j}\det A_{2j} + \cdots + (-1)^{n+j}a_{nj}\det A_{nj}$$

$$= \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det A_{ij}$$ (expansion down column j)

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

EXAMPLE: An easier way to compute the determinant of

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

EXAMPLE:

$$\begin{vmatrix} 4 & 3 & 1 & 8 \\ 5 & 0 & 3 & -1 \\ 0 & 0 & -3 & 0 \\ 7 & 0 & 2 & 4 \end{vmatrix} =$$

It's easy to compute the determinant of a triangular matrix:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

EXAMPLE:

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} =$$

THEOREM 2: If A is a triangular matrix, then $\det A$ is the product of the diagonal entries of A.

How the determinant changes under row operations:

1. Replacement: add a multiple of one row to another row. $R_i \to R_i + cR_j$
   determinant does not change.
2. Interchange: interchange two rows. $R_i \to R_j,\ R_j \to R_i$
   determinant changes sign.
3. Scaling: multiply all entries in a row by a nonzero constant. $R_i \to cR_i,\ c \neq 0$
   determinant scales by a factor of c.

To help you remember:

original	after replacement	after interchange	after scaling
$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$	$\begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} = 1,$	$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1,$	$\begin{vmatrix} c & 0 \\ 0 & 1 \end{vmatrix} = c.$

Because we can compute the determinant by expanding down columns instead of across rows, the same changes hold for “column operations”.

---

1. Replacement: $R_i \to R_i + cR_j$ determinant does not change.
2. Interchange: $R_i \to R_j,\ R_j \to R_i$ determinant changes sign.
3. Scaling: $R_i \to cR_i,\ c \neq 0$ determinant scales by a factor of c.

Usually we compute determinants using a mixture of “expanding across a row or down a column with many zeroes” and “row reducing to a triangular matrix”.

Example:

$$\begin{vmatrix} 2 & 3 & 4 & 6 \\ 0 & 5 & 0 & 0 \\ 5 & 5 & 6 & 7 \\ 7 & 9 & 6 & 10 \end{vmatrix} = 5 \begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} \overset{\text{factor out 2 from } R_1}{=} 5 \cdot 2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} \overset{\substack{R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 7R_1}}{=} 5 \cdot 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$\overset{\text{factor out -4 from } R_2}{=} 5 \cdot 2 \cdot -4 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} \overset{R_3 \to R_3 + 8R_2}{=} 5 \cdot 2 \cdot -4 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix} = 5 \cdot 2 \cdot -4 \cdot 1 \cdot 1 \cdot 5 = -200.$$

1. Replacement: $R_i \to R_i + cR_j$ determinant does not change.
2. Interchange: $R_i \to R_j$, $R_j \to R_i$ determinant changes sign.
3. Scaling: $R_i \to cR_i$, $c \neq 0$ determinant scales by a factor of c.

Useful fact: If two rows of A are multiples of each other, then $\det A = 0$.

Proof: Use a replacement row operation to make one of the rows into a row of zeroes, then expand along that row.

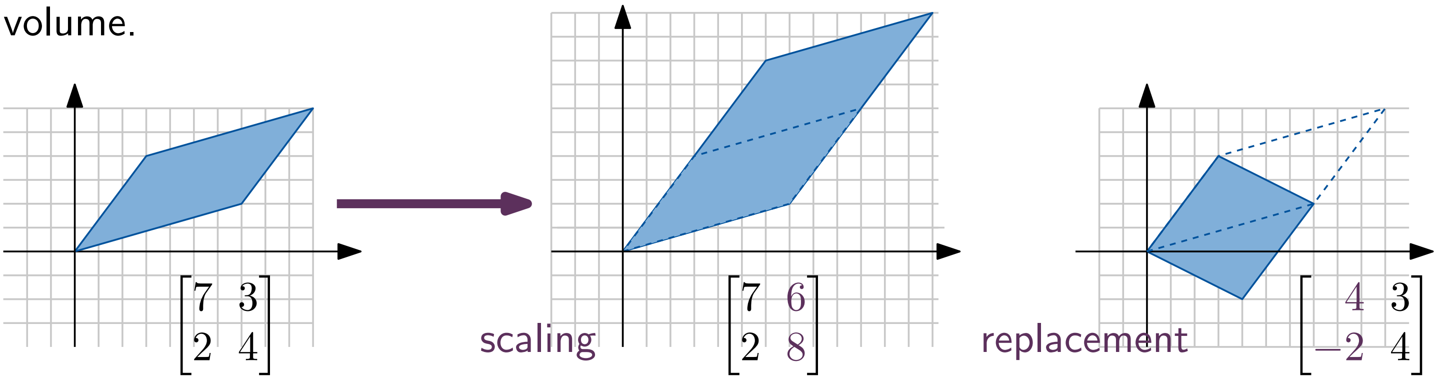
Example:

$$\begin{vmatrix} 1 & 3 & 4 \\ 5 & 9 & 3 \\ 2 & 6 & 8 \end{vmatrix} \overset{R_3 \to R_3 - 2R_1}{=} \begin{vmatrix} 1 & 3 & 4 \\ 5 & 9 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \begin{vmatrix} 3 & 4 \\ 9 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 4 \\ 5 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 \\ 5 & 9 \end{vmatrix} = 0.$$

Why does the determinant change like this under row and column operations? Two views:

Either: It is a consequence of the expansion formula in Theorem 1;

Or: By thinking about the signed volume of the image of the unit cube under the associated linear transformation:
2. Interchanging columns changes the orientation of the image of the unit cube.
3. Scaling a column applies an expansion to one side of the image of the unit cube.
1. Column replacement rearranges the image of the unit cube without changing its volume.



Properties of the determinant:

$\det(A^T) = \det A.$

Theorem 6: Determinants are multiplicative: For square matrices A and B:

$$\det(AB) = \det A \det B.$$

In particular:
(let $B = A^{-1}$)

$$\det(A^{-1}) = \frac{\det I_n}{\det A} = \frac{1}{\det A}, \quad \det(cA) = \det \begin{bmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{bmatrix} \det A = c^n \det A.$$

(where A is $n \times n$)

Properties of the determinant:

Theorem 4: Invertibility and determinants: A square matrix A is invertible if and only if $\det A \neq 0$.

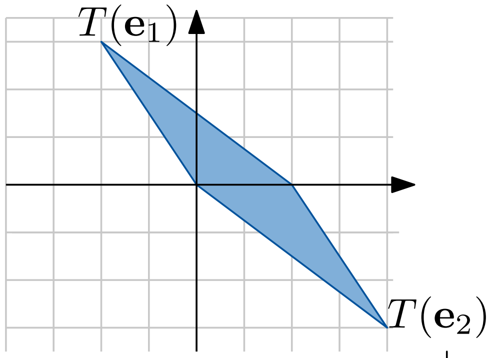
Proof 1: By the Invertible Matrix Theorem, A is invertible if and only if rref(A) has n pivots. Row operations multiply the determinant by nonzero numbers. So $\det A = 0$ if and only if $\det(\text{rref}(A)) = 0$, which happens precisely when rref(A) has fewer than n pivots.

Proof 2: By the Invertible Matrix Theorem, A is invertible if and only if its columns span $\mathbb{R}^n$. Since the image of the unit cube is a subset of the span of the columns, this image has zero volume if the columns do not span $\mathbb{R}^n$.

So we can use determinants to test whether $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^n$ is linearly independent, or if it spans $\mathbb{R}^n$: it does when $\det\left(\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & & | \end{bmatrix}\right) \neq 0.$

Application in MultiCal (MATH2205): determinants and volumes

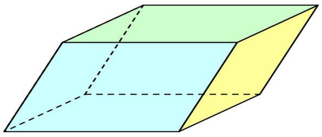
Example: Find the area of the parallelogram with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}, \begin{bmatrix}4\\-3\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}$.



Answer: This parallelogram is the image of the unit square under a linear transformation T with $T(\mathbf{e}_1) = \begin{bmatrix}-2\\3\end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix}4\\-3\end{bmatrix}$.

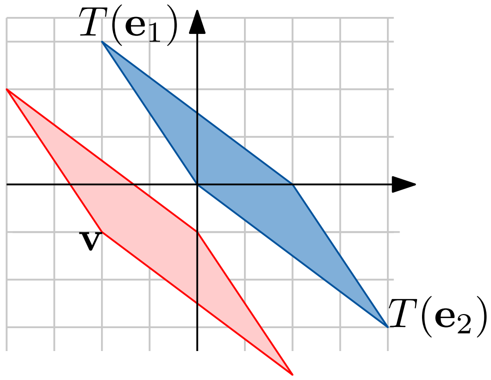
So area of parallelogram $= \left|\det \begin{bmatrix}-2 & 4\\3 & -3\end{bmatrix}\right| \times$ area of unit square $= |-6| \cdot 1 = 6$.

This works for any parallelogram where the origin is one of the vertices (and also in $\mathbb{R}^3$, for parallelopipeds).



Application in MultiCal (MATH2205): determinants and volumes

Example: Find the area of the parallelogram with vertices $\begin{bmatrix}-2\\-1\end{bmatrix}, \begin{bmatrix}-4\\2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}$.



Answer: Use a translation to move one of the vertices of the parallelogram to the origin - this does not change the area.

The formula for this translation function is $\mathbf{x} \mapsto \mathbf{x} - \mathbf{v}$, where $\mathbf{v}$ is one of the vertices of the parallelogram.

Here, the vertices of the translated parallelogram are $\begin{bmatrix}-2\\-1\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{bmatrix}-4\\2\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix}, \quad \begin{bmatrix}2\\-4\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}4\\-3\end{bmatrix}, \quad \begin{bmatrix}0\\-1\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}.$$

So, by the previous example, the area of the parallelogram is 6.

Application in ODE (MATH3405): determinants and linear systems

Cramer's rule: Let A be an invertible $n \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det \begin{bmatrix} | & & | & & | \\ \mathbf{a}_1 & \dots & \mathbf{b} & \dots & \mathbf{a}_n \\ | & & | & & | \end{bmatrix}}{\det A}.$$

put $\mathbf{b}$ in the ith column instead of $\mathbf{a}_i$

Proof:

$$A \begin{bmatrix} | & & | & & | \\ \mathbf{e}_1 & \dots & \mathbf{x} & \dots & \mathbf{e}_n \\ | & & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & & | \\ A\mathbf{e}_1 & \dots & A\mathbf{x} & \dots & A\mathbf{e}_n \\ | & & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & & | \\ \mathbf{a}_1 & \dots & \mathbf{b} & \dots & \mathbf{a}_n \\ | & & | & & | \end{bmatrix}.$$

So

$$\det A \boxed{\det \begin{bmatrix} | & & | & & | \\ \mathbf{e}_1 & \dots & \mathbf{x} & \dots & \mathbf{e}_n \\ | & & | & & | \end{bmatrix}} = \det \begin{bmatrix} | & & | & & | \\ \mathbf{a}_1 & \dots & \mathbf{b} & \dots & \mathbf{a}_n \\ | & & | & & | \end{bmatrix}.$$

To finish the proof, we need to show $\det \begin{bmatrix} | & & | & & | \\ \mathbf{e}_1 & \dots & \mathbf{x} & \dots & \mathbf{e}_n \\ | & & | & & | \end{bmatrix} = x_i$.

Note that the ith row of this matrix is $[0 \dots x_i \dots 0]$.
And expanding along this ith row gives $x_i \det(I_{n-1}) = x_i$.

Some examples:

$$n = 3, i = 1\text{: } \det \begin{bmatrix} | & | & | \\ \mathbf{x} & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix} = \det \begin{array}{r} \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = x_1 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$n = 3, i = 2\text{: } \det \begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{x} & \mathbf{e}_3 \\ | & | & | \end{bmatrix} = \det \begin{array}{r} \\ \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = x_2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying Cramer's rule to solve $A\mathbf{x} = \mathbf{e}_i$ gives a formula for the ith column of A^{-1} (see Theorem 8 in textbook; this formula is called the adjugate or classical adjoint).

Exercise: use this process to show the 2×2 formula: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Cramer's rule is much slower than row-reduction for linear systems with actual numbers, but is useful for obtaining theoretical results.

Example: If every entry of A is an integer and $\det A = 1$ or -1, then every entry of A^{-1} is an integer.

Proof: Cramer's rule tells us that every entry of A^{-1} is the determinant of an integer matrix divided by $\det A$. And the determinant of an integer matrix is an integer.

Exercise: using the fact $\det AB = \det A \det B$, prove the converse (if every entry of A and of A^{-1} is an integer, then $\det A = 1$ or -1).

Remember the addition and scalar multiplication of matrices:

$(A+B)_{ij} = a_{ij} + b_{ij}$, e.g $\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$.

$(cA)_{ij} = ca_{ij}$, e.g. $(-3)\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -12 & 0 & -15 \\ 3 & -9 & -6 \end{bmatrix}$.

Is this really different from $\mathbb{R}^6$?

$$\begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 6 \\ 2 \\ 8 \\ 9 \end{bmatrix}. \qquad (-3)\begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ -15 \\ 3 \\ -9 \\ -6 \end{bmatrix}.$$

Remember from calculus the addition and scalar multiplication of polynomials:

e.g $(2t^2+1) + (-t^2+3t+2) = t^2+3t+3$.

e.g $(-3)(-t^2+3t+2) = 3t^2-9t-6$.

Is this really different from $\mathbb{R}^3$?

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}. \qquad (-3)\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix}. \begin{matrix} \leftarrow \text{coefficient of } 1 \\ \leftarrow \text{coefficient of } t \\ \leftarrow \text{coefficient of } t^2 \end{matrix}$$

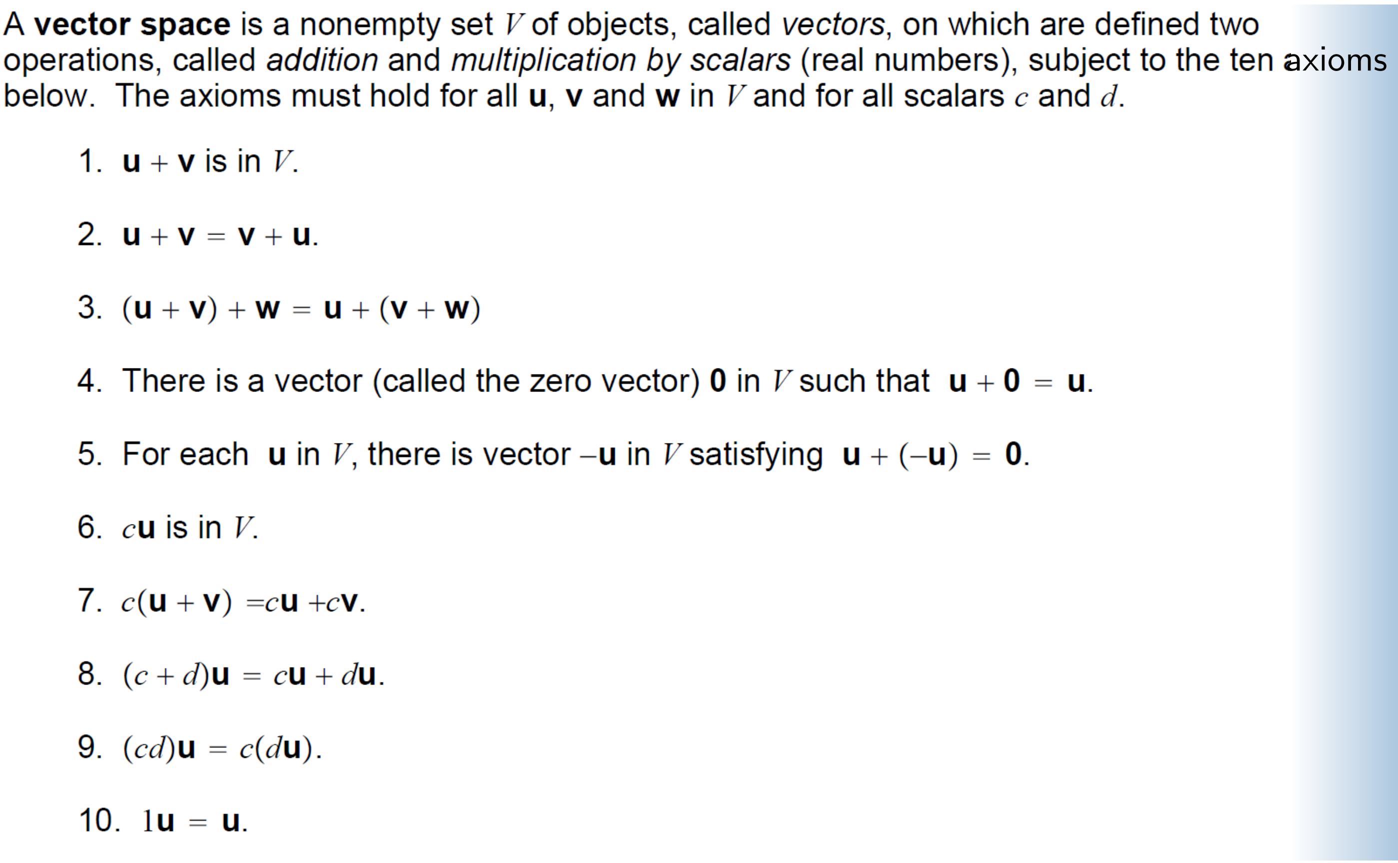
§4.1, pp217-218: Abstract Vector Spaces

As the examples above showed, there are many objects in mathematics that "looks" and "feels" like $\mathbb{R}^n$. We will also call these vectors.

The real power of linear algebra is that everything we learned in Chapters 1-3 can be applied to all these abstract vectors, not just to column vectors.

You should think of abstract vectors as objects which can be added and multiplied by scalars - i.e. where the concept of "linear combination" makes sense. This addition and scalar multiplication must obey some "sensible rules" called axioms (see next page).

The axioms guarantee that the proof of every result and theorem from Chapters 1-3 will work for our new definition of vectors.



A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $(cd)\mathbf{u}=c(d\mathbf{u})$.
10. $1\mathbf{u}=\mathbf{u}$.

Examples of vector spaces:

$M_{2\times 3}$, the set of 2×3 matrices.

Let's check axiom 4. There is a vector (called the zero vector) **0** in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

The zero vector for $M_{2\times 3}$ is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

You can check the other 9 axioms by using the properties of matrix addition and scalar multiplication (page 5 of week 5 slides, theorem 2.1 in textbook).

Similarly, $M_{m\times n}$, the set of all $m \times n$ matrices, is a vector space.

Is the set of all matrices (of all sizes) a vector space?

No, because we cannot add two matrices of different sizes, so axiom 1 does not hold.

Examples of vector spaces:

$\mathbb{P}_n$, the set of polynomials of degree at most n.

Each of these polynomials has the form

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n,$$

for some numbers $a_0, a_1, \ldots, a_n$.

Let's check axiom 4. There is a vector (called the zero vector) **0** in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

The zero vector for $\mathbb{P}_n$ is $0 + 0t + 0t^2 + \cdots + 0t^n$.

Let's check axiom 1. $\mathbf{u} + \mathbf{v}$ is in V.

$(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) + (b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n)$
$= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n$, which also has degree at most n.

Exercise: convince yourself that the other axioms are true.

Examples of vector spaces:

Warning: the set of polynomials of degree exactly n is not a vector space, because axiom 1 does not hold:

$$\underbrace{(t^3+t^2)}_{\text{degree 3}}+\underbrace{(-t^3+t^2)}_{\text{degree 3}}=\underbrace{2t^2}_{\text{degree2}}$$

$\mathbb{P}$, the set of all polynomials (no restriction on the degree) is a vector space.

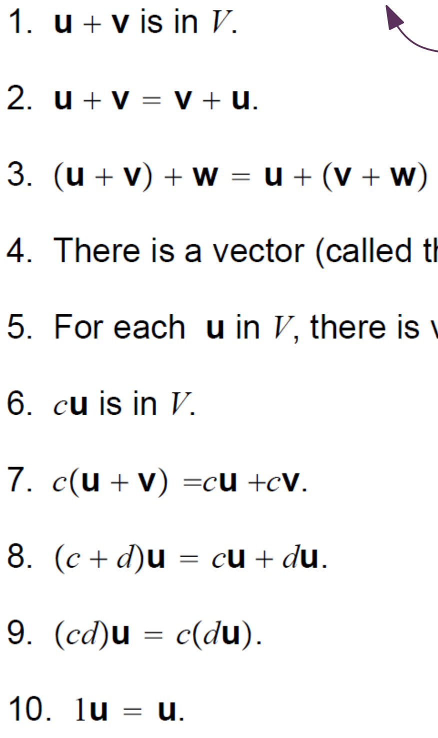
$C(\mathbb{R})$, the set of all continuous functions is a vector space (because the sum of two continuous functions is continuous, the zero function is continuous, etc.)

These last two examples are a bit different from $M_{m\times n}$ and $\mathbb{P}_n$ because they are infinite-dimensional (more later, see week 8.5 §4.5).

(You do not have to remember the notation $M_{m\times n}, \mathbb{P}_n$, etc. for the vector spaces.)

Let W be the set of symmetric 2×2 matrices. Is W a vector space?

$A = A^T$, i.e. $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ for some a, b, d



1. **u** + **v** is in V.
2. **u** + **v** = **v** + **u**.
3. (**u** + **v**) + **w** = **u** + (**v** + **w**)
4. There is a vector (called the zero vector) **0** in V such that **u** + **0** = **u**.
5. For each **u** in V, there is vector –**u** in V satisfying **u** + (–**u**) = **0**.
6. c**u** is in V.
7. c(**u** + **v**) = c**u** + c**v**.
8. $(c + d)$**u** = c**u** + d**u**.
9. (cd)**u** = $c(d$**u**$)$.
10. 1**u** = **u**.

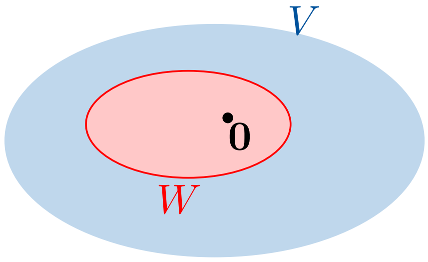
Note that the axioms come in two types:

- Axioms 2, 3, 5, 7, 8, 9, 10 are about the interactions of vectors with each other, they do not mention the space V. Since they hold for $M_{2\times 2}$, they also hold for W.
- So we only need to check axioms 1, 4, 6, that mention both the vectors and the space V.

Definition: A subset W of a vector space V is a *subspace* of V if the closure axioms 1,4,6 hold:

4. The zero vector is in W.
1. If $\mathbf{u}, \mathbf{v}$ are in W, then their sum $\mathbf{u}+\mathbf{v}$ is in W. (closed under addition)
6. If $\mathbf{u}$ is in W and c is any scalar, the scalar multiple $c\mathbf{u}$ is in W. (closed under scalar multiplication)

Fact: W is itself a vector space (with the same addition and scalar multiplication as V) if and only if W is a subspace of V.



To show that W is a subspace, check all three axioms directly, for all $\mathbf{u}, \mathbf{v}, c$ (i.e. use variables). (You may find it easier to check 6. before 1.)

To show that W is not a subspace, show that one of the axioms is false, for a particular value of $\mathbf{u}, \mathbf{v}, c$.

Definition: A subset W of a vector space V is a *subspace* of V if:

4. The zero vector is in W.
1. If $\mathbf{u}, \mathbf{v}$ are in W, then their sum $\mathbf{u}+\mathbf{v}$ is in W.
6. If $\mathbf{u}$ is in W and c is any scalar, the scalar multiple $c\mathbf{u}$ is in W.

Tip: to show that a vector is in a set defined by "$\{*|\dagger\}$" notation, you show that it has the form in $*$, satisfying the conditions in $\dagger$.

Example: Let $W=\left\{\begin{bmatrix}a\\0\\b\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$, i.e. the x_1x_3-plane. We show W is a subspace of $\mathbb{R}^3$:

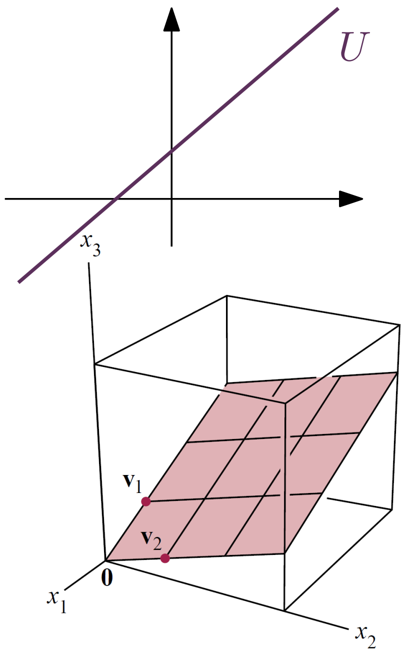
4. The zero vector is in W because it is the vector with $a=0$, $b=0$.
1. Take two arbitrary vectors in W: $\begin{bmatrix}a\\0\\b\end{bmatrix}$ and $\begin{bmatrix}x\\0\\y\end{bmatrix}$. Then $\begin{bmatrix}a\\0\\b\end{bmatrix}+\begin{bmatrix}x\\0\\y\end{bmatrix}=\begin{bmatrix}a+x\\0\\b+y\end{bmatrix}\in W$.
6. Take an arbitrary vector in W: $\begin{bmatrix}a\\0\\b\end{bmatrix}$, and any $c\in\mathbb{R}$. Then $c\begin{bmatrix}a\\0\\b\end{bmatrix}=\begin{bmatrix}ca\\0\\cb\end{bmatrix}\in W$.

Example: Let $U = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} \middle| x \in \mathbb{R} \right\}$. To show that U is not a subspace of $\mathbb{R}^2$, we need to find one counterexample to one of the axioms, e.g.

4. The zero vector is not in U, because there is no value of x with $\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

An alternative answer:

1. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are in U, but $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not of the form $\begin{bmatrix} x \\ x+1 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not in U. So U is not closed under addition.



Best examples of a subspace: lines and planes containing the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example: Let $Q = \{\mathbf{p} \in \mathbb{P}_3 | \mathbf{p}(t) = at + 3a$ for some $a \in \mathbb{R}\}$, i.e. $Q = \{at + 3a | a \in \mathbb{R}\}$. We show that Q is a subspace of $\mathbb{P}_3$:

4. The zero polynomial $(0 + 0t + 0t^2 + 0t^3)$ is in Q because it is $at + 3a$ when $a = 0$.
1. Take two arbitrary polynomials in Q: $at + 3a$ and $bt + 3b$. Then $(at + 3a) + (bt + 3b) = (a+b)t + 3(a+b) \in Q$.
6. Take an arbitrary polynomial in Q: $at + 3a$, and any $c \in \mathbb{R}$. Then $c(at + 3a) = (ca)t + 3(ca) \in Q$.

Every vector space V contains two subspaces (its smallest and biggest ones):

- The set $\{\mathbf{0}\}$ containing only the zero vector is the zero subspace:
    4. $\mathbf{0}$ is clearly in the subspace.
    1. $\mathbf{0} + \mathbf{0} = \mathbf{0}$ (use axiom 4: $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V).
    6. $c\mathbf{0} = \mathbf{0}$ (use axiom 7: $c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$; and left hand side is $c\mathbf{0}$.)
- The whole space V is a subspace of V.

The first of two shortcuts to show that a set is a subspace:

Theorem 1: Spans are subspaces: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

"separate" the "free variables" like how we write a solution in parametric form (week 2 p31)

Redo Example: (p10) Let $W = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$.

We can rewrite W as $\left\{ a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

So W is a subspace of $\mathbb{R}^3$.

The first of two shortcuts to show that a set is a subspace:

Theorem 1: Spans are subspaces: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Redo Example: (p8) Let $Sym_{2\times 2}$ be the set of symmetric 2×2 matrices. Then

$$\begin{aligned} Sym_{2\times 2} &= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in M_{2\times 2} \middle| a, b, d \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| a, b, d \in \mathbb{R} \right\} \\ &= \text{Span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \end{aligned}$$

so $Sym_{2\times 2}$ is a subspace of $M_{2\times 2}$.

Warning: Theorem 1 does not help us show that a set is not a subspace.

THEOREM 1: Spans are subspaces

If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are vectors in a vector space V, then Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

Proof: We check axioms 4, 1 and 6 in the definition of a subspace.

4. **0** is in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ since

$$\mathbf{0} = \underline{\hspace{2em}}\mathbf{v}_1 + \underline{\hspace{2em}}\mathbf{v}_2 + \cdots + \underline{\hspace{2em}}\mathbf{v}_p$$

1. To show that Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is closed under addition, we choose two arbitrary vectors in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= \underline{\hspace{2em}}\;\underline{\hspace{2em}}\mathbf{v}_1 + \underline{\hspace{2em}}\;\underline{\hspace{2em}}\mathbf{v}_2 + \cdots + \underline{\hspace{2em}}\;\underline{\hspace{2em}}\mathbf{v}_p$$

So $\mathbf{u} + \mathbf{v}$ is in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

6. To show that Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$:

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$c\mathbf{v} = c(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

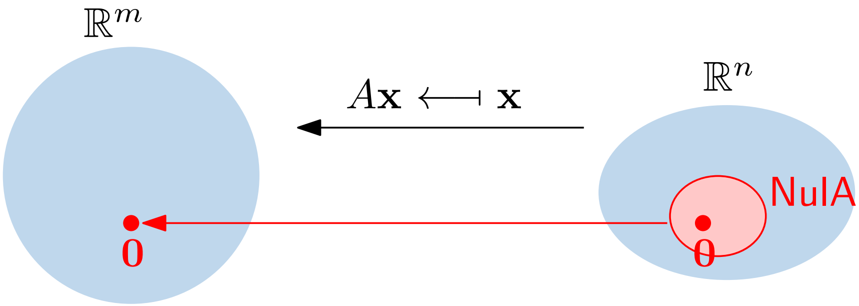
$$= \underline{\hspace{3em}}\mathbf{v}_1 + \underline{\hspace{3em}}\mathbf{v}_2 + \cdots + \underline{\hspace{3em}}\mathbf{v}_p$$

So $c\mathbf{v}$ is in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

Since 4,1,6 hold, Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

The second of two shortcuts to show that a set is a subspace:

Definition: The null space of a $m \times n$ matrix A, written Nul$A$, is the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Theorem 2: Null Spaces are Subspaces: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

This theorem is useful for showing that a set defined by conditions is a subspace.

Warning: If $\mathbf{b} \neq \mathbf{0}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is not a subspace, because it does not contain $\mathbf{0}$.

The second of two shortcuts to show that a set is a subspace:

Definition: The null space of a $m \times n$ matrix A, written Nul$A$, is the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem 2: Null Spaces are Subspaces: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Example: Show that the line $L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \middle| y = 2x \right\}$ is a subspace of $\mathbb{R}^2$.

Here, we do not have "$x, y \in \mathbb{R}$": instead, x and y are related by the condition $y = 2x$. In these situations, it's often easier to show that the given set is a null space.

Answer: $y = 2x$ is the same as $2x - y = 0$, which in matrix form is $\begin{bmatrix} 2 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$. So L is the solution set to $\begin{bmatrix} 2 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$, which is the null space of the matrix $\begin{bmatrix} 2 & -1 \end{bmatrix}$. Because null spaces are subspaces, L is a subspace.

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

THEOREM 2 The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. We check axioms 4,1,6 in the definition of a subspace.

4. **0** is in Nul A because

1. Axiom 1 says:

So we need to show (the conclusion in formulas):

6. Axiom 6 says:

So we need to show:

Since axioms 4,1,6 hold, Nul A is a subspace of $\mathbf{R}^n$.

Summary:

Axioms for a subspace: Warning: no functions are involved!

4. The zero vector is in W.
1. If $\mathbf{u}, \mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W. (closed under addition)
6. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W. (closed under scalar multiplication)

Ways to show that a set W is a subspace:

- $\{\quad * \quad | s, t \in \mathbb{R}\} \xrightarrow{\text{choose } \mathbf{v}, \mathbf{w}} \{s\mathbf{v} + t\mathbf{w} | s, t \in \mathbb{R}\} = \text{Span}\,\{\mathbf{v}, \mathbf{w}\}$.
- $\{\mathbf{x} \in \mathbb{R}^n | \quad \dagger \quad \} \xrightarrow{\text{choose } A} \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\} = \text{Nul}A$.
- Show that W is the kernel or range of a linear transformation (later, p41-42).
- Check all three axioms directly, for all $\mathbf{u}, \mathbf{v}, c$.

To show that a set is not a subspace:

- Show that one of the axioms is false, for a particular value of $\mathbf{u}, \mathbf{v}, c$.

Best examples of a subspace: lines and planes containing the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$.

One example of the power of abstract vector spaces - solving differential equations:

Question: What are all the polynomials $\mathbf{p}$ of degree at most 5 that satisfy

$$\frac{d^2}{dt^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t) = 3t - 1?$$

Answer: The differentiation function $D : \mathbb{P}_5 \to \mathbb{P}_5$ given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$ is a linear transformation (later, p39).

The function $T : \mathbb{P}_5 \to \mathbb{P}_5$ given by $T(\mathbf{p}) = \frac{d^2}{dt^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t)$ is a sum of compositions of linear transformations, so T is also linear.

We can check that the polynomial $t + 1$ is a solution.

So, by the Solutions and Homogeneous Equations Theorem, the solution set to the above differential equation is all polynomials of the form $t + 1 + \mathbf{q}(t)$ where $T(\mathbf{q}) = 0$.

Extra: $\mathbb{P}_5$ is both the domain and codomain of T, so the Invertible Matrix Theorem applies. So, if the above equation has more than one solution, then there is a polynomial $\mathbf{g}$ such that $\frac{d^2}{dx^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t) = \mathbf{g}(t)$ has no solutions.

§4.2, pp229-230, pp249-250: Subspaces and Matrices

Each linear transformation has two vector subspaces associated to it.

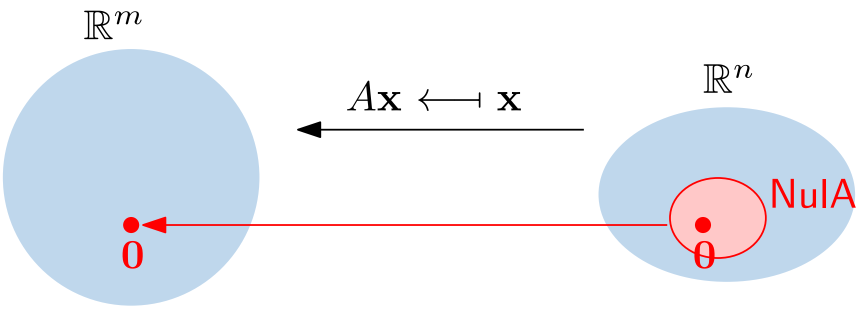
For each of these two subspaces, we are interested in two problems:

a. given a vector $\mathbf{v}$, is it in the subspace?
b. can we write this subspace as Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$? The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is then called a spanning set of the subspace.
b*. can we write this subspace as Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$? The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is then called a basis of the subspace.

Problem b is important because it means every vector in the subspace can be written as $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$. This allows us to calculate with and prove statements about arbitrary vectors in the subspace.
Problem b* is important because it means every vector in the subspace can be written uniquely as $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ (proof next week, §4.4).
We turn a spanning set into a basis by removing some vectors - this is the Spanning Set Theorem / casting-out algorithm (p28, also week 8 p10).

Remember from p16:

Definition: The null space of a $m \times n$ matrix A, written Nul$A$, is the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Nul$A$ is implicitly defined (i.e. defined by conditions) - problem a is easy, problem b takes more work.

Example: Let $A = \begin{bmatrix} 1 & -3 & 4 & -3 \\ 3 & -7 & 8 & -5 \end{bmatrix}$. a. Is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in Nul$A$?

b. Find vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ which span Nul$A$.

Answer:

a. $A\mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \mathbf{0}$, so $\mathbf{v}$ is not in Nul$A$.

b. $[A|\mathbf{0}] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x_1 = 2x_3 - 3x_4 \\ x_2 = 2x_3 - 2x_4 \\ x_3 = \quad x_3 \\ x_4 = \qquad\quad x_4 \end{array}$

So the solution set is $\left\{ s\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$. So Nul$A$ = Span $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

linearly independent

In general: the solution to $A\mathbf{x} = \mathbf{0}$ in parametric form looks like $\{s_i\mathbf{w}_i + s_j\mathbf{w}_j + \dots | s_i, s_j, \dots \in \mathbb{R}\}$, where $x_i, x_j, \dots$ are the free variables (one vector for each free variable).

To determine if the $\mathbf{w}$s are linearly independent, solve $c_i\mathbf{w}_i + c_j\mathbf{w}_j + \dots = \mathbf{0}$ for the cs.

Look in the ith row: the ith row of $\mathbf{w_i}$ is 1; the ith row of any other $\mathbf{w}_j$ is 0. So $c_i = 0$.

The same argument shows that all cs are zero, so the $\mathbf{w}$s are linearly independent.

b. $[A|\mathbf{0}] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x_1 = 2x_3 - 3x_4 \\ x_2 = 2x_3 - 2x_4 \\ x_3 = \quad x_3 \\ x_4 = \qquad\quad x_4 \end{array}$

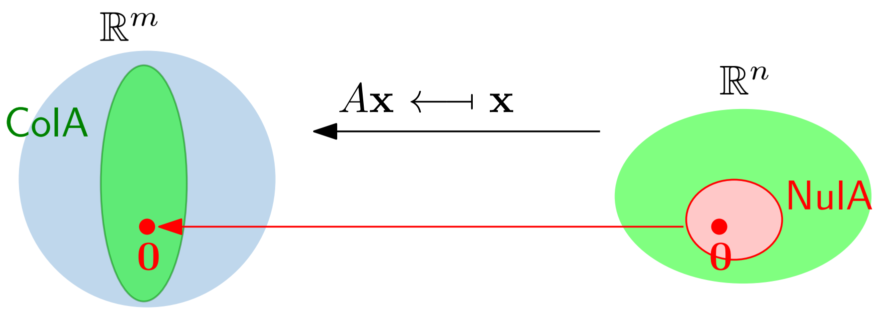
So the solution set is $\left\{ s\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$. So Nul$A$ = Span $\left\{ \underset{\substack{\uparrow \\ \mathbf{w_3}}}{\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}}, \underset{\substack{\uparrow \\ \mathbf{w_4}}}{\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}} \right\}$.

linearly independent

Definition: The column space of a $m \times n$ matrix A, written Col$A$, is the span of the columns of A.

Because spans are subspaces, it is obvious that Col$A$ is a subspace of $\mathbb{R}^m$.

It follows from §1.3-1.4 that Col$A$ is the set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has solutions.



Col$A$ is explicitly defined - problem a takes work, problem b is easy.

Example: Let $A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$. a. Is $\mathbf{v} = \begin{bmatrix} -5 \\ 9 \\ 5 \end{bmatrix}$ in Col$A$?

b. Find vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ which span Col$A$.

Answer:

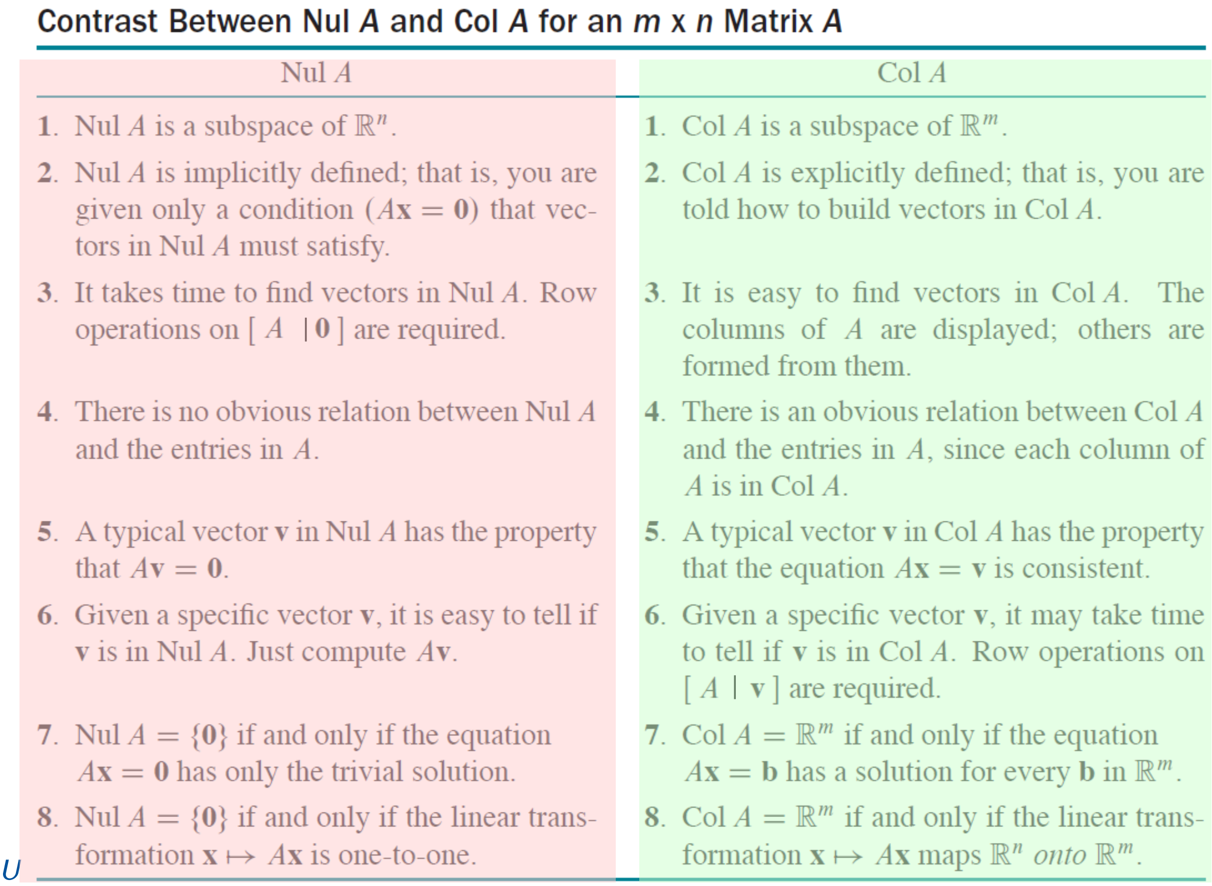
a. $$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{array}\right] \xrightarrow[\text{to echelon form}]{\text{row reduction}} \left[\begin{array}{ccccc|c} 1 & -3 & 4 & 3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

There is no row $[0 \dots 0|\blacksquare]$, so $\mathbf{v}$ is in Col$A$.

b. By definition, Col$A$ is the span of the columns of A, so

$$\text{Col}A = \text{Span}\left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}.$$

Note that this spanning set is not linearly independent (more than 3 vectors in $\mathbb{R}^3$).



Contrast Between Nul A and Col A for an m x n Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\ A \ \ \mathbf{0}\ ]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\ A \ \ \mathbf{v}\ ]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

p.222 of textbook

← problem b

← problem a

As we saw on p26, it is easy to obtain a spanning set for Col$A$ (just take all the columns of A), but usually this spanning set is not linearly independent.

To obtain a linearly independent set that spans Col$A$, take the pivot columns of A - this is called the casting-out algorithm.

Example: Let $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$.

Find a linearly independent set that spans Col$A$.

Answer: $$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to echelon form}]{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & 3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivot columns are 1,2 and 5, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\} = \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}$ is one answer.

(The answer from the casting-out algorithm is not the only answer - see p35.)

Casting-out algorithm: the pivot columns of A is a linearly independent set that spans Col$A$.

Why does the casting-out algorithm work part 1: why the pivot columns are linearly independent:

Example:

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to echelon form}]{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & 3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So $\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_5 \\ | & | & | \end{bmatrix}$ is row-equivalent to $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, which has no free variables.

So $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is linearly independent.

Casting-out algorithm: the pivot columns of A is a linearly independent set that spans Col$A$.

Why does the casting-out algorithm work part 2: why the pivot columns span Col$A$:

To explain this, we need to look at the solutions to $A\mathbf{x}=\mathbf{0}$:

Example:

$$A=\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix}=\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So the solution set to $A\mathbf{x}=\mathbf{0}$ is

$$\begin{matrix} x_3=1 \\ x_4=0 \end{matrix} \quad s\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}+t\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{matrix} x_3=0 \\ x_4=1 \end{matrix}$$

where s,t can take any value.

These correspond respectively to the linear dependence relations $2\mathbf{a}_1+2\mathbf{a}_2+\mathbf{a}_3=\mathbf{0}$ and $-3\mathbf{a}_1-2\mathbf{a}_2+\mathbf{a}_4=\mathbf{0}$.

Rearranging: $\mathbf{a}_3=-2\mathbf{a}_1-2\mathbf{a}_2$ and $\mathbf{a}_4=3\mathbf{a}_1+2\mathbf{a}_2$.

$$A(2,2,1,0,0)=\mathbf{0} \longrightarrow 2\mathbf{a}_1+2\mathbf{a}_2+\mathbf{a}_3=\mathbf{0} \longrightarrow \mathbf{a}_3=-2\mathbf{a}_1-2\mathbf{a}_2.$$

$$A(-3,-2,0,1,0)=\mathbf{0} \longrightarrow -3\mathbf{a}_1-2\mathbf{a}_2+\mathbf{a}_4=\mathbf{0} \longrightarrow \mathbf{a}_4=3\mathbf{a}_1+2\mathbf{a}_2.$$

In other words: consider the solution to $A\mathbf{x}=\mathbf{0}$ where one free variable x_i is 1, and all other free variables are 0. This corresponds to a linear dependence relation among the columns of A, which can be rearranged to express the column $\mathbf{a}_i$ as a linear combination of the pivot columns.

Why this is useful: any vector $\mathbf{v}$ in Col$A$ has the form

$$\mathbf{v}=c_1\mathbf{a}_1+c_2\mathbf{a}_2+c_3\mathbf{a}_3+c_4\mathbf{a}_4+c_5\mathbf{a}_5,$$

which we can rewrite as

$$\begin{aligned} & c_1\mathbf{a}_1+c_2\mathbf{a}_2+c_3(-2\mathbf{a}_1-2\mathbf{a}_2)+c_4(3\mathbf{a}_1+2\mathbf{a}_2)+c_5\mathbf{a}_5 \\ =&(c_1-2c_3+3c_4)\mathbf{a}_1+(c_2-2c_3+2c_4)\mathbf{a}_2+c_5\mathbf{a}_5, \end{aligned}$$

a linear combination of the pivot columns $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_5$. So $\mathbf{v}$ is in Span $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_5\}$, and so Col$A=$ Span $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_5\}$.

Another view: the casting-out algorithm as a greedy algorithm:

Example:

$$\begin{bmatrix} \vert & \vert & \vert & \vert & \vert \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ \vert & \vert & \vert & \vert & \vert \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref}\left(\begin{bmatrix} \vert \\ \mathbf{a}_1 \\ \vert \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ has a pivot in every column, so $\{\mathbf{a}_1\}$ is linearly independent, so we keep $\mathbf{a}_1$.

$$\text{rref}\left(\begin{bmatrix} \vert & \vert \\ \mathbf{a}_1 & \mathbf{a}_2 \\ \vert & \vert \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$ has a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent, so we keep $\mathbf{a}_2$.

$$\text{rref}\left(\begin{bmatrix} \vert & \vert & \vert \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ \vert & \vert & \vert \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$ does not have a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent, so we remove $\mathbf{a}_3$.

Another view: the casting-out algorithm as a greedy algorithm (continued):

Example:

$$\begin{bmatrix} \vert & \vert & \vert & \vert & \vert \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ \vert & \vert & \vert & \vert & \vert \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref}\left(\begin{bmatrix} \vert & \vert & \vert \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 \\ \vert & \vert & \vert \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ does not have a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly dependent, so we remove $\mathbf{a}_4$.

$$\text{rref}\left(\begin{bmatrix} \vert & \vert & \vert \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_5 \\ \vert & \vert & \vert \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ has a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is linearly independent, so we keep $\mathbf{a}_5$.

So the casting-out algorithm is a greedy algorithm in that it prefers vectors that are earlier in the set.

Example: Let $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$.

Find a linearly independent set containing $\mathbf{a}_3$ that spans Col$A$.

Answer: To ensure that the set contains $\mathbf{a}_3$, we should make it the leftmost column - e.g. we row-reduce $\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_3 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix}$ and take the pivot columns.

Warning: the example on the previous two pages is a little misleading: a subset of the columns of rref(A) is not always the reduced echelon form of those columns of A, e.g.rref $\left(\begin{bmatrix} | & | \\ \mathbf{a}_2 & \mathbf{a}_3 \\ | & | \end{bmatrix} \right) \neq \begin{bmatrix} 0 & -2 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$ (because this isn't in reduced echelon form).

The correct statement is that a subset of the columns of rref(A) is row equivalent to those columns of A.

Definition: The row space of a $m \times n$ matrix A, written Row$A$, is the span of the rows of A. It is a subspace of $\mathbb{R}^n$.

Example: $A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 1 & -3 & 2 \end{bmatrix}$ rref$(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{Row}A = \text{Span}\left\{(0,1,0,4),(0,2,0,8),(1,1,-3,2)\right\}.$$

Row$A$ is explicitly defined - indeed, it is equivalent to ColA^T.
So, to see if a vector $\mathbf{v}$ is in Row$A$, row-reduce $[A^T|\mathbf{v}^T]$.
To find a linear independent set that spans Row$A$, take the pivot columns of A^T, or..

Theorem 13: Row operations do not change the row space. In particular, the nonzero rows of rref(A) is a linearly independent set whose span is Row$A$.
E.g. for the above example, Row$A = \text{Span}\left\{(1,0,-3,-2),(0,1,0,4)\right\}$.

Warning: the "pivot rows" of A do not usually span Row$A$:
e.g. here $(1,1,-3,2)$ is in Row$A$ but not in Span$\left\{(0,1,0,4),(0,2,0,8)\right\}$.

Theorem 13: Row operations do not change the row space. In particular, the nonzero rows of rref(A) is a linearly independent set whose span is Row$A$.

An example to explain why row operations do not change the row space:

$$A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 1 & -3 & 2 \end{bmatrix} \quad \begin{array}{r} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & -2 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{array}{l} R_1' \\ R_2' \\ R_3' \end{array}$$

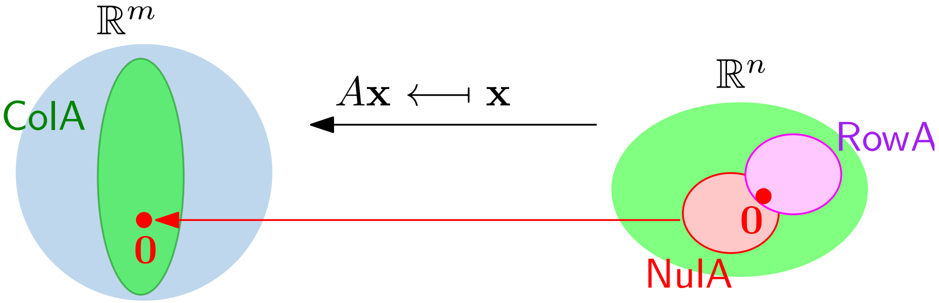
Take any vector in the row space, i.e. any linear combination of R_1, R_2, R_3,
e.g. $R_2 + R_3 = (1,3,-3,10)$.
We can rewrite it as a linear combination of the rows R_1', R_2', R_3' of rref$A$:
e.g. $(1,3,-3,10) = R_2 + R_3 = (R_2 - 2R_1) + (R_3 - R_1) + 3R_1 = R_3' + R_1' + 3R_2'$.

Proof of the second sentence in Theorem 13:
From the first sentence, Row(A) = Row(rref(A)) = Span of the nonzero rows of rref(A). Because each nonzero row has a 1 in one pivot column (different column for each row) and 0s in all other pivot columns, these rows are linearly independent.

Summary:

A basis for W is a linearly independent set that spans W (more next week).

- Nul$A$=solutions to $A\mathbf{x}=\mathbf{0}$, basis for Nul$A$: solve $A\mathbf{x}=\mathbf{0}$ via the rref.
- Col$A$=span of columns of A, basis for Col$A$: pivot columns of A.
- Row$A$=span of rows of A. basis for Row$A$: nonzero rows of rref(A).



Col$A$ is in $\mathbb{R}^m$. Nul$A$, Row$A$ are in $\mathbb{R}^n$.

In general, Col$A \neq$ Col(rref(A)). Nul$A$ = Nul(rref(A)), Row$A$ = Row(rref(A)).

(think about $A=\begin{bmatrix}1&1\\2&2\end{bmatrix}$.)

PP222-223: Linear Transformations for Vector Spaces

Recall (week 4 §1.8) the definition of a linear transformation:

Definition: A function $T:\mathbb{R}^n\to\mathbb{R}^m$ is a *linear transformation* if:

1. $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ for all $\mathbf{u},\mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u})=cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in the domain of T.

Now consider a function $T:V\to W$, where V,W are abstract vector spaces.
Because we can add and scalar-multiply in V, the left hand sides of the equations in 1,2 make sense.
Because we can add and scalar-multiply in W, the right hand sides of the equations in 1,2 make sense.
So we can ask if functions between abstract vector spaces are linear:

Definition: A function $T:V\to W$ is a *linear transformation* if:

1. $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ for all $\mathbf{u},\mathbf{v}$ in V;
2. $T(c\mathbf{u})=cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in V.

Hard exercise: show that the set of all linear transformations $V\to W$ is a vector space.

Definition: A function $T : V \to W$ is a *linear transformation* if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in V.

Example: The differentiation function $D : \mathbb{P}_n \to \mathbb{P}_{n-1}$ given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$,

$$D(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1},$$

is linear.

If you've taken a calculus class, then you already know this:

When you calculate $\frac{d}{dt}(3t + 2t^2) = 3 \quad + 2 \cdot 2t$

you're really thinking $3\frac{d}{dt}t + 2\frac{d}{dt}t^2$

Method A to show that D is linear:

$$D(\mathbf{p}+\mathbf{q}) = \frac{d}{dt}(\mathbf{p}+\mathbf{q}) = \frac{d}{dt}\mathbf{p} + \frac{d}{dt}\mathbf{q} = D(\mathbf{p}) + D(\mathbf{q}); \text{ and}$$

$$D(c\mathbf{p}) = \frac{d}{dt}(c\mathbf{p}) \qquad = c\frac{d}{dt}\mathbf{p} \qquad = cD(\mathbf{p})$$

Definition: A function $T : V \to W$ is a *linear transformation* if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in V.

Example: The differentiation function $D : \mathbb{P}_n \to \mathbb{P}_{n-1}$ given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$,

$$D(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1},$$

is linear.

Method B to show that D is linear - use the formula:

$$\begin{aligned}
&D((a_0+b_0) + (a_1+b_1)t + (a_2+b_2)t^2 + \cdots + (a_n+b_n)t^n)\\
&= (a_1+b_1) + 2(a_2+b_2)t + \cdots + n(a_n+b_n)t^{n-1}\\
&= a_1 + 2a_2 t + \cdots + n a_n t^{n-1} + b_1 + 2b_2 t + \cdots + n b_n t^{n-1}\\
&= D(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) + D(b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n); \text{ and}
\end{aligned}$$

$$\begin{aligned}
D((ca_0) + (ca_1)t + (ca_2)t^2 + \cdots + (ca_n)t^n) &= (ca_1) + 2(ca_2)t + \cdots + n(ca_n)t^{n-1}\\
&= c(a_1 + 2a_2 t + \cdots + n a_n t^{n-1})\\
&= cD(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n).
\end{aligned}$$

Example: The "multiplication by t" function $M : \mathbb{P}_n \to \mathbb{P}_{n+1}$ given by $M(\mathbf{p}(t)) = t\mathbf{p}(t)$,

$$M(a_0 + a_1 t + \cdots + a_n t^n) = t(a_0 + a_1 t + \cdots + a_n t^n),$$

is linear:

Method A: $M(\mathbf{p}+\mathbf{q}) = t[(\mathbf{p}+\mathbf{q})(t)] = t\mathbf{p}(t) + t\mathbf{q}(t) = M(\mathbf{p}) + M(\mathbf{q})$; and

$$M(c\mathbf{p}) = t[(c\mathbf{p})(t)] \qquad = c[t(\mathbf{p}(t)] \qquad = cM(\mathbf{p})$$

Method B:

$$\begin{aligned}&M((a_0+b_0) + (a_1+b_1)t + \cdots + (a_n+b_n)t^n)\\ &= t((a_0+b_0) + (a_1+b_1)t + \cdots + (a_n+b_n)t^n))\\ &= t(a_0 + a_1 t + \cdots + a_n t^n) + t(b_0 + b_1 t + \cdots + b_n t^n)\\ &= M(a_0 + a_1 t + \cdots + a_n t^n) + M(b_0 + b_1 t + \cdots + b_n t^n);\ \text{and}\end{aligned}$$

$$\begin{aligned}M((ca_0) + (ca_1)t + \cdots + (ca_n)t^n) &= t((ca_0) + (ca_1)t + \cdots + (ca_n)t^n)\\ &= ct(a_0 + a_1 t + \cdots + a_n t^n)\\ &= cM(a_0 + a_1 t + \cdots + a_n t^n).\end{aligned}$$

The concepts of kernel and range (week 4, §1.9) make sense for linear transformations between abstract vector spaces:

Definition: The *kernel* of T is $\ker T = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}\}$.

Definition: The *range* of T is range$T = \{\mathbf{w} \in W | \mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V\}$.

Example: Consider the differentiation function $D : \mathbb{P}_n \to \mathbb{P}_{n-1}$, given by $D(\mathbf{p}) = \dfrac{d}{dt}\mathbf{p}$.

$\ker D = \left\{\mathbf{p} \in \mathbb{P}_n \,|\, \dfrac{d}{dt}\mathbf{p} = 0\right\}$ = the set of constant polynomials $\{a_0 | a_0 \in \mathbb{R}\}$.

range$D = \left\{\mathbf{q} \in \mathbb{P}_{n-1} \,|\mathbf{q} = \dfrac{d}{dt}\mathbf{p} \text{ for some } \mathbf{p} \in \mathbb{P}_n\right\}$. For any $\mathbf{q} \in \mathbb{P}_{n-1}$, letting $\mathbf{p} = \int \mathbf{q}\, dt$ solves $\mathbf{q} = \dfrac{d}{dt}\mathbf{p}$, so range$D$ is all of $\mathbb{P}_{n-1}$ (i.e. D is onto).

Exercise: what is the kernel and range of the multiplication by t function $M(\mathbf{p}) = t\mathbf{p}(t)$?

Our proof that null spaces are subspaces (p18) can be modified to show that the kernel of a linear transformation is a subspace.

Exercise: show that the range of a linear transformation is a subspace.

Recall from p17: to prove that a subset of $\mathbb{R}^n$ defined by conditions is a subspace, we can try to show it's a null space:

$$\{\mathbf{x} \in \mathbb{R}^n | \quad \dagger \quad \} \xrightarrow{\text{choose } A} \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\} = \text{Nul}A.$$

If our subset defined by conditions is in a different vector space from $\mathbb{R}^n$, then we can similarly try to show it's a kernel.

$$\{\mathbf{x} \in V | \quad \dagger \quad \} \xrightarrow{\text{choose } T: V \to ?} \{\mathbf{x} \in V | T(\mathbf{x}) = \mathbf{0}\} = \ker T.$$

You will need to show that the T you choose is linear.

The second answer (p46) to this example uses this new shortcut.

Example: Let $K = \{\mathbf{p} \in \mathbb{P}_3 | \mathbf{p}(2) = 0\}$, i.e. the polynomials $\mathbf{p}(t)$ of degree at most 3, which output 0 when we set $t = 2$. Show that K is a subspace of $\mathbb{P}_3$.

Before we answer the question, let's make sure we understand what K is:
e.g. $t - 1$ is not in K because $2 - 1 = 1 \neq 0$.
e.g. $-t^2 + 3t + 2$ is in K because $-2^2 + 3 \cdot 2 + 2 = 0$.

Example: Let $K = \{\mathbf{p} \in \mathbb{P}_3 | \mathbf{p}(2) = 0\}$. Show that K is a subspace of $\mathbb{P}_3$.

Answer 1: Checking the axioms directly.

4. The zero polynomial $(0 + 0t + 0t^2 + 0t^3)$ is in K because $\mathbf{0}(2) = 0 + 0 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 = 0$.

1. We need to show that, if $\mathbf{p}, \mathbf{q}$ are in K, then $\mathbf{p} + \mathbf{q}$ is in K.
   Translation: $\mathbf{p}(2) = 0, \mathbf{q}(2) = 0 \qquad (\mathbf{p} + \mathbf{q})(2) = 0.$

   Method A: $(\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = 0 + 0 = 0.$

   Method B: Suppose $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ so $a_0 + a_1 2 + a_2 2^2 + a_3 2^3 = 0$.
   Suppose $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$ so $b_0 + b_1 2 + b_2 2^2 + b_3 2^3 = 0$.
   $(\mathbf{p} + \mathbf{q})(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + (a_3 + b_3)t^3$.
   So
   $$\begin{aligned}(\mathbf{p} + \mathbf{q})(2) &= (a_0 + b_0) + (a_1 + b_1)2 + (a_2 + b_2)2^2 + (a_3 + b_3)2^3 \\ &= (a_0 + a_1 2 + a_2 2^2 + a_3 2^3) + (b_0 + b_1 2 + b_2 2^2 + b_3 2^3) \\ &= 0 + 0 = 0\end{aligned}$$

Example: Let $K = \{\mathbf{p} \in \mathbb{P}_3 | \mathbf{p}(2) = 0\}$. Show that K is a subspace of $\mathbb{P}_3$.

Answer 1: (continued): Checking the axioms directly.

6. Method A:
   For $\mathbf{p}$ in K and any scalar c, we have $(c\mathbf{p})(2) = c(\mathbf{p}(2)) = c0 = 0$, so $c\mathbf{p}$ is in K.

   Method B:
   Take $\mathbf{p} = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ in K, so $a_0 + a_1 2 + a_2 2^2 + a_3 2^3 = 0$. Then $c\mathbf{p}(2) = (ca_0) + (ca_1)2 + (ca_2)2^2 + (ca_3)2^3 = c(a_0 + a_1 2 + a_2 2^2 + a_3 2^3) = c0 = 0,$ so $c\mathbf{p}$ is in K.

Example: Let $K = \{\mathbf{p} \in \mathbb{P}_3 | \mathbf{p}(2) = 0\}$. Show that K is a subspace of $\mathbb{P}_3$.

Answer 2: Showing that K is a kernel.
Consider the evaluation-at-2 function $E_2 : \mathbb{P}_3 \to \mathbb{R}$ given by $E_2(\mathbf{p}) = \mathbf{p}(2)$,

$$E_2(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_0 + a_1 2 + a_2 2^2 + a_3 2^3$$

E_2 is a linear transformation because

1. For $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_3$, we have
   $E_2(\mathbf{p} + \mathbf{q}) = (\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = E_2(\mathbf{p}) + E_2(\mathbf{q})$.
2. For $\mathbf{p}$ in $\mathbb{P}_3$ and any scalar c, we have $E_2(c\mathbf{p}) = (c\mathbf{p})(2) = c(\mathbf{p}(2)) = cE_2(\mathbf{p})$.

So E_2 is a linear transformation. K is the kernel of E_2, so K is a subspace.

Can we write K as Span $\{\mathbf{p}_1, \ldots, \mathbf{p}_p\}$ for some linearly independent polynomials $\mathbf{p}_1, \ldots, \mathbf{p}_p$?

One idea: associate a matrix A to E_2 and take a basis of Nul$A$ using the rref.
To do computations like this, we need coordinates.

From the beginning of last week:
Remember from calculus the addition and scalar multiplication of polynomials:

e.g $(2t^2 + 1) + (-t^2 + 3t + 2) = t^2 + 3t + 3.$

e.g $(-3)(-t^2 + 3t + 2) = 3t^2 - 9t - 6.$

We want to represent abstract vectors as column vectors so we can do calculations (e.g. row-reduction) to study linear systems (e.g. week 7 p21) and linear transformations (e.g. week 7 p44).

Is this really different from $\mathbb{R}^3$?

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}. \qquad (-3)\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix}. \begin{array}{l} \leftarrow \text{coefficient of } 1 \\ \leftarrow \text{coefficient of } t \\ \leftarrow \text{coefficient of } t^2 \end{array}$$

We want to represent abstract vectors as column vectors so we can do calculations (e.g. row-reduction) to study linear systems and linear transformations.

In $\mathbb{R}^n$, $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.$

$\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ must span V

We can copy this idea: in V, pick a special set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, write each $\mathbf{x}$ in V uniquely as $c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ and represent $\mathbf{x}$ by the column vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

$\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ must be linearly independent

Example: In $\mathbb{P}_2$, let $\mathbf{b}_1 = 1, \mathbf{b}_2 = t, \mathbf{b}_3 = t^2$.

Then we represent $a_0 + a_1 t + a_2 t^2$ by $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ (see previous page).

§4.3: Bases

Definition: Let W be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a *basis for W* if

i $\mathcal{B}$ is a linearly independent set, and

ii Span $\{\mathbf{b}_1, \dots, \mathbf{b}_p\} = W$.

The order matters: $\{\mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathbf{b}_2, \mathbf{b}_1\}$ are different bases.

i means: The only solution to $x_1\mathbf{b}_1 + \dots + x_p\mathbf{b}_p = \mathbf{0}$ is $x_1 = \dots = x_p = 0$.

ii means: W is the set of vectors of the form $c_1\mathbf{b}_1 + \dots + c_p\mathbf{b}_p$ where $c_1, \dots, c_p$ can take any value.

Condition ii implies that $\mathbf{b}_1, \dots, \mathbf{b}_p$ must be in W, because Span $\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ contains each of $\mathbf{b}_1, \dots, \mathbf{b}_p$.

Every vector space V is a subspace of itself, so we can take $W = V$ in the definition and talk about bases for V.

Definition: Let W be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a *basis for W* if

i $\mathcal{B}$ is a linearly independent set, and

ii Span $\{\mathbf{b}_1, \dots, \mathbf{b}_p\} = W$.

Example: The standard basis for $\mathbb{R}^3$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

To check that this is a basis: $\begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is in reduced echelon form.

The matrix has a pivot in every column, so its columns are linearly independent.
The matrix has a pivot in every row, so its columns span $\mathbb{R}^3$.

Definition: Let W be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a *basis for W* if

i $\mathcal{B}$ is a linearly independent set, and
ii Span $\{\mathbf{b}_1, \dots, \mathbf{b}_p\} = W$.

A basis for W is not unique: (different bases are useful in different situations, see week 9).
Let's look for a different basis for $\mathbb{R}^3$.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^3$?

Answer: No, because two vectors cannot span $\mathbb{R}^3$: $\begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$ cannot have a pivot in every row.

Definition: Let W be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a *basis for W* if

i $\mathcal{B}$ is a linearly independent set, and
ii Span $\{\mathbf{b}_1, \dots, \mathbf{b}_p\} = W$.

A basis for W is not unique: (different bases are useful in different situations, see week 9).
Let's look for a different basis for $\mathbb{R}^3$.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for $\mathbb{R}^3$?

Answer: Form the matrix $A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Because $\det A = 1 \neq 0$, the matrix A is invertible, so (by Invertible Matrix Theorem) its columns are linearly independent and its columns span $\mathbb{R}^3$.

Definition: Let W be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a *basis for W* if

i $\mathcal{B}$ is a linearly independent set, and

ii Span $\{\mathbf{b}_1, \dots, \mathbf{b}_p\} = W$.

A basis for W is not unique: (different bases are useful in different situations, see week 9).

Let's look for a different basis for $\mathbb{R}^3$.

Example: Let $\mathbf{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ a basis for $\mathbb{R}^3$?

Answer: No, because four vectors in $\mathbb{R}^3$ must be linearly dependent:

$\begin{bmatrix}| & | & | & |\\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\\ | & | & | & |\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & 0\\ 2 & 1 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$ cannot have a pivot in every column.

By the same logic as in the above examples:

Fact: $\{\mathbf{v}_1, \dots \mathbf{v}_p\}$ is a basis for $\mathbb{R}^n$ if and only if:

- $p = n$ (i.e. the set has exactly n vectors), and
- $\det \begin{bmatrix}| & & |\\ \mathbf{v}_1 & \dots & \mathbf{v}_n\\ | & & |\end{bmatrix} \neq 0$.

Fewer than n vectors: not enough vectors, can't span $\mathbb{R}^n$.
More than n vectors: too many vectors, linearly dependent.

Example: The standard basis for $\mathbb{P}_n$ is $\mathcal{B} = \left\{1, t, t^2, \ldots, t^n\right\}$.

To check that this is a basis:

ii By definition of $\mathbb{P}_n$, every element of $\mathbb{P}_n$ has the form $a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$, so $\mathcal{B}$ spans $\mathbb{P}_n$.

i To see that $\mathcal{B}$ is linearly independent, we show that $c_0 = c_1 = \cdots = c_n = 0$ is the only solution to

$$c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n = 0. \text{ (the zero function)}$$

Substitute $t = 0$: we find $c_0 = 0$.

Differentiate, then substitute $t = 0$: we find $c_1 = 0$.

Differentiate again, then substitute $t = 0$: we find $c_2 = 0$.

Repeating many times, we find $c_0 = c_1 = \cdots = c_n = 0$.

Once we have the standard basis of $\mathbb{P}_n$, it will be easier to check if other sets are bases of $\mathbb{P}_n$, using coordinates (later, p14).

Advanced exercise: what do you think is the standard basis for $M_{m\times n}$?

One way to make a basis for V is to start with a set that spans V.

Theorem 5: Spanning Set Theorem: If $V = \text{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then some subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis for V.

Proof: basically, the idea of the casting-out algorithm (week 7 p29-35) works in abstract vector spaces too.

- If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, it is a basis for V.
- If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent, then one of the $\mathbf{v}_i$s is a linear combination of the others. Removing this $\mathbf{v}_i$ from the set still gives a set that spans V. Continue removing vectors in this way until the remaining vectors are linearly independent.

Example: $\mathbb{P}_2 = \text{Span}\left\{5, 3+t, 1+2t^2, 4+2t-4t^2\right\}$, but this set is not linearly independent because $4+2t-4t^2$ is a linear combination of the other polynomials: $4+2t-4t^2 = 2(3+t) - 2(1+2t^2)$. So remove $4+2t-4t^2$ to get the set $\left\{5, 3+t, 1+2t^2\right\}$, which is in fact a basis (we can show this with coordinates, p14-15).

PP 234, 238-240 (§4.4), 307-308 (§5.4): Coordinates

Recall (p2) that our motivation for finding a basis is because we want to write each vector $\mathbf{x}$ as $c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$ in a unique way. Let's show that this is indeed possible

Theorem 7: Unique Representation Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Proof:

Since $\mathcal{B}$ spans V, there exists scalars $c_1, \ldots, c_n$ such that the above equation holds.

Suppose $\mathbf{x}$ has another representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n.$$

for some scalars $d_1, \ldots, d_n$. Then

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_n - d_n)\mathbf{b}_n.$$

Because $\mathcal{B}$ is linearly independent, all the weights in this equation must be zero, i.e. $(c_1 - d_1) = \cdots = (c_n - d_n) = 0$. So $c_1 = d_1, \ldots, c_n = d_n$.

Because of the Unique Representation Theorem, we can make the following definition:

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V. Then, for any $\mathbf{x}$ in V, the *coordinates of $\mathbf{x}$ relative to $\mathcal{B}$*, or the *$\mathcal{B}$-coordinates of $\mathbf{x}$*, are the unique weights $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

The vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the *coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$*, or the *$\mathcal{B}$-coordinate vector of $\mathbf{x}$*.

Example: Let $\mathcal{B} = \left\{1, t, t^2, t^3\right\}$ be the standard basis for $\mathbb{P}_3$. Then the coordinate vector of an arbitrary polynomial is $[a_0 + a_1t + a_2t^2 + a_3t^3]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

Why are coordinate vectors useful?

Because of the Unique Representation Theorem, the function V to $\mathbb{R}^n$ given by $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ (e.g. $a_0 + a_1t + a_2t^2 + a_3t^3 \mapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$) is linear, one-to-one and onto.

Definition: A linear transformation $T : V \to W$ that is both one-to-one and onto is called an *isomorphism*. We say V and W are isomorphic.

This means that, although the notation and terminology for V and W are different, the two spaces behave the same as vector spaces. Every vector space calculation in V is accurately reproduced in W, and vice versa.

Important consequence: if V has a basis of n vectors, then V and $\mathbb{R}^n$ are isomorphic, so we can solve problems about V (e.g. span, linear independence) by working in $\mathbb{R}^n$.

If V has a basis of n vectors, then V and $\mathbb{R}^n$ are isomorphic, so we can solve problems about V (e.g. span, linear independence) by working in $\mathbb{R}^n$.

Example: Is the set of polynomials $\left\{1, 2-t, (2-t)^2, (2-t)^3\right\}$ linearly independent?

Answer: The coordinates of these polynomials relative to the standard basis of $\mathbb{P}_3$ are

$$[1]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \qquad [(2-t)^2]_{\mathcal{B}} = [4-4t+t^2]_{\mathcal{B}} = \begin{bmatrix}4\\-4\\1\\0\end{bmatrix},$$

$$[2-t]_{\mathcal{B}} = \begin{bmatrix}2\\-1\\0\\0\end{bmatrix}, \quad [(2-t)^3]_{\mathcal{B}} = [(8-12t+6t^2-t^3]_{\mathcal{B}} = \begin{bmatrix}8\\-12\\6\\-1\end{bmatrix}$$

The set of polynomials is linearly independent if and only if their coordinate vectors are linearly independent (continued on next page).

Example: Is the set of polynomials $\left\{1, 2-t, (2-t)^2, (2-t)^3\right\}$ linearly independent?

Answer: (continued). The matrix $\begin{bmatrix}1 & 2 & 4 & 8\\0 & -1 & -4 & -12\\0 & 0 & 1 & 6\\0 & 0 & 0 & -1\end{bmatrix}$

is in echelon form and has a pivot in every column, so its columns are linearly independent in $\mathbb{R}^4$. So the polynomials are linearly independent.

(Alternative: this matrix is upper triangular so its determinant is the product of the diagonal entries $1 \cdot -1 \cdot 1 \cdot -1 = 1 \neq 0$, so the matrix is invertible, and by the Invertible Matrix Theorem, its columns are linearly independent in $\mathbb{R}^4$.)

In fact the polynomials form a basis: IMT says that the columns of the above matrix also span $\mathbb{R}^4$, so the polynomials span $\mathbb{P}_3$.

Advanced exercise: if $\mathbf{p}_i$ has degree exactly i, then $\{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n\}$ is a basis for $\mathbb{P}_n$. (This idea is how I usually prove that a set is a basis in my research work.)

If $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$, then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n, \quad \text{so } [\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}.$$

Harder example: (in preparation for week 9, change of coordinates) Let $\mathcal{F} = \left\{1, 2-t, (2-t)^2, (2-t)^3\right\}$. We just showed that $\mathcal{F}$ is a basis. So if the $\mathcal{F}$-coordinates of a polynomial $\mathbf{p}$ is $[\mathbf{p}]_{\mathcal{F}} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$, then what is $\mathbf{p}$?

Answer:

$[\mathbf{p}]_{\mathcal{F}} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$ means $\mathbf{p} = 2 + 4(2-t) + 0(2-t)^2 - 1(2-t)^3 = 2 + 8t - 6t^2 + t^3$.

If V has a basis of n vectors, then V and $\mathbb{R}^n$ are isomorphic, so we can solve problems about V (e.g. span, linear independence) by working in $\mathbb{R}^n$.

What about problems concerning linear transformations $T : V \to W$?

Remember from week 4 §1.9: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$, where

apply T to ith basis vector, put the coordinates of the result into column i

$$A = \begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & & | \end{bmatrix} \quad \text{(standard matrix of } T\text{).}$$

A reminder why $T(\mathbf{x}) = A\mathbf{x}$:

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \dots + x_nT(\mathbf{e}_n) = \begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

We do a similar calculation in an abstract vector space V, with basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. Suppose $T : V \to V$ is a linear transformation, and $\mathbf{v} \in V$ is $\mathbf{v} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$:

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n) = c_1T(\mathbf{b}_1) + \dots + c_nT(\mathbf{b}_n) \\ [T(\mathbf{v})]_{\mathcal{B}} &= [T(c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n)]_{\mathcal{B}} = c_1[T(\mathbf{b}_1)]_{\mathcal{B}} + \dots + c_n[T(\mathbf{b}_n)]_{\mathcal{B}} \\ &= \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} [\mathbf{v}]_{\mathcal{B}}. \end{aligned}$$

Definition: If V is a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $T : V \to V$ is a linear transformation, then the *matrix for T relative to $\mathcal{B}$* is

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}.$$

(so the standard matrix of T is $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}$, where $\mathcal{E}$ is the standard basis of $\mathbb{R}^n$.)

DEFINITION:If V is a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $T : V \to V$ is a linear transformation, then the matrix for T relative to $\mathcal{B}$ is

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = \begin{bmatrix} | & | & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}.$$

EXAMPLE:Let $D : \mathbb{P}_2 \to \mathbb{P}_2$ be the differentiation function

$$D(a_0 + a_1 t + a_2 t^2) = \frac{d}{dt}(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t.$$

Work in the standard basis of $\mathbb{P}_2$: $\mathbf{b}_1 = 1, \mathbf{b}_2 = t, \mathbf{b}_3 = t^2$.

$D(\mathbf{b_1}) =$ $\qquad$ $D(\mathbf{b_2}) =$ $\qquad$ $D(\mathbf{b_3}) =$

$$[D(\mathbf{b_1})]_{\mathcal{B}} = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} \qquad [D(\mathbf{b_2})]_{\mathcal{B}} = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} \qquad [D(\mathbf{b_3})]_{\mathcal{B}} = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}$$

So

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[D]} =$$

The matrix $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$ is useful because
$$[T(\mathbf{x})]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\,[\mathbf{x}]_{\mathcal{B}}, \qquad (*)$$
so we can solve $T(\mathbf{x}) = \mathbf{y}$ by row-reducing $\left[\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\middle|[\mathbf{y}]_{\mathcal{B}}\right]$.

Example: Let $D : \mathbb{P}_2 \to \mathbb{P}_2$ be the differentiation function $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$ as on the previous page. Here is an example of equation $(*)$ for $\mathbf{x} = 2 + 3t - t^2$.

$$D(2 + 3t - t^2) = \frac{d}{dt}(2 + 3t - t^2) \qquad = 3 - 2t$$
$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[D]}\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}3\\-2\\0\end{bmatrix}.$$

Some other things about D that we can learn from the matrix $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[D]}$:

- We can solve the differential equation $\frac{d}{dt}\mathbf{p} = 1 - 3t$ by row-reducing $\left[\begin{array}{ccc|c}0&1&0&1\\0&0&2&-3\\0&0&0&0\end{array}\right]$.
- $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[D]}$ is in echelon form, and it does not have a pivot in every column, so D is not one-to-one (which you know from calculus - this is why indefinite integrals have $+C$).

As you may have guessed from the notation $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$, it is possible to define similar matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{[T]}$ using different "input" and "output" bases $\mathcal{B}$ and $\mathcal{C}$. This is useful when $T : V \to W$ has different domain and codomain. But we will not consider this more general case.

Note that the textbook writes $[T]_{\mathcal{B}}$ for $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$. You need to understand this notation in homework and exams.

Warning: if $\mathbf{x}$ is a vector , then $[\mathbf{x}]_{\mathcal{B}}$ is a column vector.
if T is a linear transformation, then $[T]_{\mathcal{B}}$ is a matrix.
i.e. the notation $[\ ]_{\mathcal{B}}$ means a different thing depending on what is inside the bracket.

Basis and coordinates for subspaces:

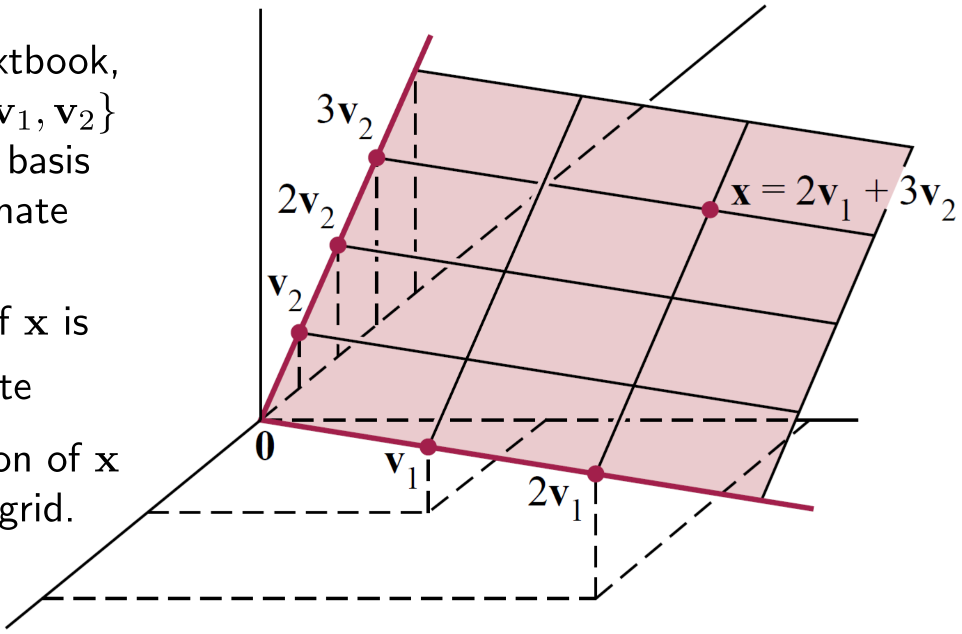
Example: Let $W = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$. We showed (week 7 p14) that W is a subspace of $\mathbb{R}^3$ because $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Since $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is furthermore linearly independent, it is a basis for W.

Because $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, the coordinate vector of $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, relative to the basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, is $\begin{bmatrix} a \\ b \end{bmatrix}$. So $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \end{bmatrix}$ is an isomorphism from W to $\mathbb{R}^2$.

Coordinates for subspaces (e.g. planes in $\mathbb{R}^3$) are useful as they allow us to represent points in the subspace with fewer numbers (e.g. with 2 numbers instead of 3 numbers).

In this picture (p239 of textbook, example 7 in §4.4), $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for a plane. The basis allows us to draw a coordinate grid on the plane.

The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. This coordinate vector describes the location of $\mathbf{x}$ relative to this coordinate grid.



An “abstract” example of coordinates:

EXAMPLE: Let $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ be a basis for V.

1. What is the $\mathcal{B}$-coordinate vector of $\mathbf{b_1} + \mathbf{b_2}$?

Suppose $T : V \to V$ is a linear transformation satisfying

$$T(\mathbf{b_1}) = \mathbf{b_1} + \mathbf{b_2}, \quad T(\mathbf{b_2}) = \mathbf{b_1} - 2\mathbf{b_3}, \quad T(\mathbf{b_3}) = \mathbf{b_3}.$$

2. Find the matrix $\underset{\mathcal{B} \leftarrow \mathcal{B}}{[T]}$ for T relative to $\mathcal{B}$.

3. Find $T(\mathbf{b_1} + \mathbf{b_2})$.

§4.5: Dimension

From last week:

- Given a vector space V, a basis for V is a linearly independent set that spans V.
- If $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V, then the $\mathcal{B}$-coordinates of $\mathbf{x}$ are the weights c_i in the linear combination $\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_p\mathbf{b}_p$.
- Coordinate vectors allow us to test for spanning / linear independence, to solve linear systems, and to test for one-to-one / onto by working in $\mathbb{R}^n$.

Another example of this idea:

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V.

i Any set in V containing more than n vectors must be linearly dependent (theorem 9 in textbook).

ii Any set in V containing fewer than n vectors cannot span V.

We prove this (next page) using coordinate vectors, and the fact that we already know it is true for $V = \mathbb{R}^n$.

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V.

i Any set in V containing more than n vectors must be linearly dependent.

ii Any set in V containing fewer than n vectors cannot span V.

Proof: Let our set of vectors in V be $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$, and consider the matrix

$$A = \begin{bmatrix} | & | & | \\ [\mathbf{u}_1]_{\mathcal{B}} & \dots & [\mathbf{u}_p]_{\mathcal{B}} \\ | & | & | \end{bmatrix},$$

which has p columns and n rows.

i If $p > n$, then rref(A) cannot have a pivot in every column, so $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_p]_{\mathcal{B}}\}$ is linearly dependent in $\mathbb{R}^n$, so $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is linearly dependent in V.

ii If $p < n$, then rref(A) cannot have a pivot in every row, so the set of coordinate vectors $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_p]_{\mathcal{B}}\}$ cannot span $\mathbb{R}^n$, so $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ cannot span V.

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V.

i Any set in V containing more than n vectors must be linearly dependent.

ii Any set in V containing fewer than n vectors cannot span V.

As a consequence:

Theorem 10: Every basis has the same size: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

So the following definition makes sense:

Definition: Let V be a vector space.

- If V is spanned by a finite set, then V is *finite-dimensional*.
  The *dimension* of V, written $\dim V$, is the number of vectors in a basis for V. (This number is finite because of the spanning set theorem.)
- If V is not spanned by a finite set, then V is *infinite-dimensional*.

Note that the definition does not involve "infinite sets".

Definition: (or convention) The dimension of the zero vector space $\{\mathbf{0}\}$ is 0.

Definition: The *dimension* of V is the number of vectors in a basis for V.

Examples:

- The standard basis for $\mathbb{R}^n$ is $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, so $\dim \mathbb{R}^n = n$.
- The standard basis for $\mathbb{P}_n$ is $\{1, t, \dots, t^n\}$, so $\dim \mathbb{P}_n = n + 1$.
- Exercise: Show that $\dim M_{m\times n} = mn$.

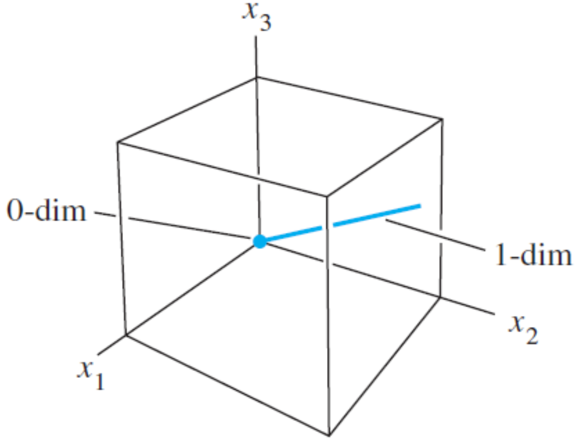
Example: Let $W = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \middle| a, b, \in \mathbb{R} \right\}$. We showed (week 8 p20) that a basis for W is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. So $\dim W$=2.

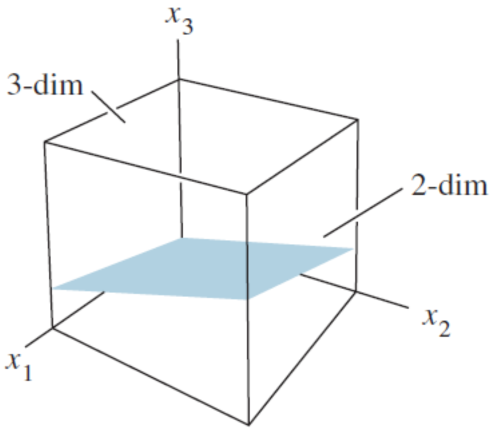
Why is it useful to know the dimension of W? One example: From the theorem on p2, we know that any set of 3 vectors in W must be linearly dependent, because $3 > \dim W$.

Example: We classify the subspaces of $\mathbb{R}^3$ by dimension:

- 0-dimensional: only the zero subspace $\{\mathbf{0}\}$.
- 1-dimensional, i.e. Span $\{\mathbf{v}\}$: lines through the origin.
- 2-dimensional, i.e. Span $\{\mathbf{u}, \mathbf{v}\}$ where $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent: planes through the origin.
- 3-dimensional: by Invertible Matrix Theorem, 3 linearly independent vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$, so the only 3-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

The theorem on the next page shows that other dimensions are not possible.





Here is a counterpart to the spanning set theorem (week 8 p10):

Theorem 11: Linearly Independent Set Theorem: Let W be a subspace of a finite-dimensional vector space V. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly independent set in W, we can find $\mathbf{v}_{p+1}, \ldots, \mathbf{v}_n$ so that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for W.

Proof:

- If Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = W$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis for W.
- Otherwise $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ does not span W, so there is a vector $\mathbf{v}_{p+1}$ in W that is not in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Adding $\mathbf{v}_{p+1}$ to the set still gives a linearly independent set. Continue adding vectors in this way until the set spans W. This process must stop after at most $\dim V - p$ additions, because a set of more than $\dim V$ elements must be linearly dependent.

The above logic proves something stronger:

Theorem 11 part 2: Subspaces of Finite-Dimensional Spaces: If W is a subspace of a finite-dimensional vector space V, then W is also finite-dimensional and $\dim W \leq \dim V$.

Because of the spanning set theorem and linearly independent set theorem:

Theorem 12: Basis Theorem: If V is a p-dimensional vector space, then

- i Any linearly independent set of exactly p elements in V is a basis for V.
- ii Any set of exactly p elements that span V is a basis for V.

In other words, to prove that $\mathcal{B}$ is a basis of a p-dimensional vector space V, we only need to show two of the following three things (the third will be automatic):

- $\mathcal{B}$ contains exactly p vectors in V;
- $\mathcal{B}$ is linearly independent;
- Span$\mathcal{B} = V$.

If V is a subspace of U, these two statements are usually easier to check because we can work in the big space U (see p9, p14, ex#18).

Proof:

- i By the linearly independent set theorem, we can add elements to any linearly independent set in V to obtain a basis for V. But that larger set must contain exactly $\dim V = p$ elements. So our starting set must already be a basis.
- ii By the spanning set theorem, we can remove elements from any set that spans V to obtain a basis for V. But that smaller set must contain exactly $\dim V = p$ elements. So our starting set must already be a basis.

Summary:

- If V is spanned by a finite set, then V is finite-dimensional and $\dim V$ is the number of vectors in any basis for V.
- If V is not spanned by a finite set, then V is infinite-dimensional.
- If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans V, then some subset is a basis for V (week 8 p10).
- If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent and V is finite-dimensional, then it can be expanded to a basis for V (p4).

If $\dim V = p$ (so V and $\mathbb{R}^p$ are isomorphic):

- Any set of more than p vectors in V is linearly dependent (p2).
- Any set of fewer than p vectors in V cannot span V (p2).
- Any linearly independent set of exactly p elements in V is a basis for V (p7).
- Any set of exactly p elements that span V is a basis for V (p7).

To prove that $\mathcal{B}$ is a basis of V, show two of the following three things:

- $\mathcal{B}$ contains exactly p vectors in V;
- $\mathcal{B}$ is linearly independent;
- Span$\mathcal{B} = V$.

The basis theorem is useful for finding bases of subspaces:

Example:

Let $W = \text{Span}\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$. Is $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\0\\3\end{bmatrix}, \begin{bmatrix}2\\0\\0\\5\end{bmatrix}, \begin{bmatrix}3\\0\\1\\2\end{bmatrix} \right\}$ a basis for W?

Answer: We are given that $W = \text{Span}\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\}$ and $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\}$ is a linearly independent set, so $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\}$ is a basis for W, and so $\dim W = 3$.
The vectors in $\mathcal{B}$ are all in W, and $\mathcal{B}$ consists of exactly 3 vectors, so it's enough to check whether $\mathcal{B}$ is linearly independent.

Row reduction: $\begin{bmatrix}1&2&3\\0&0&0\\0&0&1\\3&5&2\end{bmatrix} \xrightarrow[R_4-3R_1]{} \begin{bmatrix}1&2&3\\0&0&0\\0&0&1\\0&-1&-7\end{bmatrix} \xrightarrow[R]{R_4} \begin{bmatrix}1&2&3\\0&-1&-7\\0&0&1\\0&0&0\end{bmatrix}$ has a pivot in each column, so $\mathcal{B}$ is linearly independent, and is therefore a basis.
Note that we never had to work in W, only in $\mathbb{R}^4$.

§4.6: Rank

Next we look at how the idea of dimension can help us answer questions about existence and uniqueness of solutions to linear systems.

Definition: The *rank* of a matrix A is the dimension of its column space.
The *nullity* of a matrix A is the dimension of its null space.

Example: Let $A = \begin{bmatrix}5&-3&10\\7&2&14\end{bmatrix}$, rref$(A) = \begin{bmatrix}1&0&1/2\\0&1&0\end{bmatrix}$.

A basis for Col$A$ is $\left\{ \begin{bmatrix}5\\7\end{bmatrix}, \begin{bmatrix}-3\\2\end{bmatrix} \right\}$ ⟵ one vector per pivot

A basis for Nul$A$ is $\left\{ \begin{bmatrix}-1/2\\0\\1\end{bmatrix} \right\}$. ⟵ one vector per free variable

A basis for Row$A$ is $\{(1,0,1/2),(0,1,0)\}$. ⟵ one vector per pivot

So rank$A = 2$, nullity$A = 1$.

So rank$A$ + nullity$A$ $=?$

Theorem 14:

Rank Theorem: rank$A = \dim \text{Col}A = \dim \text{Row}A =$ number of pivots in rref(A).

Rank-Nullity Theorem: For an $m \times n$ matrix A,

$$\text{rank}A + \text{nullity}A = n \text{ , the number of columns.}$$

Proof: From our algorithms for bases of Col$A$ and Nul$A$ (see week 7 slides):
rank$A =$ number of pivots in rref(A) $=$ number of basic variables,
nullity$A =$ number of free variables.
Each variable is either basic or free, and the total number of variables is n, the number of columns.

An application of the Rank-Nullity theorem:

Example: Suppose a homogeneous system of 10 equations in 12 variables has a solution set that is spanned by two linearly independent vectors (i.e. 2 free variables). Then the nullity of this system is 2, so the rank is $12 - 2 = 10$. So this system has 10 pivots. Since there are ten equations, there must be a pivot in every row, so any nonhomogeneous system with the same coefficients always has a solution.

Using our new ideas of dimension, we can add more statements to the Existence theorem, the Uniqueness theorem, and the Invertible Matrix Theorem:

Theorem 8: Invertible Matrix Theorem (IMT): For a square $n \times n$ matrix A, the following are equivalent:

A has a pivot position in every row.

$A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

The columns of A span $\mathbb{R}^n$.

The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

There is a matrix D such that $AD = I_n$.

Col$A = \mathbb{R}^n$.

rank$A = n$.

A has a pivot position in every column.

$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

The columns of A are linearly independent.

The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

There is a matrix C such that $CA = I_n$.

Nul$A = \{\mathbf{0}\}$.

nullity$A = 0$.

$\det A \neq 0$.

rref(A) $= I_n$.

$A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

The columns of A form a basis for $\mathbb{R}^n$.

The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an invertible function.

A is an invertible matrix.

Advanced application of the Rank-Nullity Theorem and the Basis Theorem:

Redo Example: (p10) Let $A = \begin{bmatrix} 5 & -3 & 10 \\ 7 & 2 & 14 \end{bmatrix}$. Find a basis for Nul$A$ and Col$A$.

Answer: (a clever trick without any row-reduction)

- Observe that $2\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$. So nullity$A \geq 1$.
- The first two columns of A are linearly independent (not multiples of each other), so $\left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ is a linearly independent set in Col$A$, so rank$A \geq 2$.
- But rank$A$ + nullity$A = 3$, so in fact rank$A = 2$ and nullity$A = 1$, and, by the Basis Theorem, the linearly independent sets we found above are bases: so $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis for Nul$A$, $\left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ is a basis for Col$A$.

So for a general $m \times n$ matrix, it's enough to find k linearly independent vectors in Nul$A$ and $n - k$ linearly independent vectors in Col$A$.

The Rank-Nullity theorem also holds for linear transformations $T : V \to W$ whenever V is finite-dimensional (to prove it yourself, work through optional q8 of homework 5):

$$\underbrace{\dim \text{range of } T}_{\text{rank}T} + \underbrace{\dim \text{kernel of } T}_{\text{nullity}T} = \dim \text{domain of } T.$$

Advanced application:

Example: Find a basis for $K = \{\mathbf{p} \in \mathbb{P}_3 | \mathbf{p}(2) = 0\}$, i.e. polynomials $\mathbf{p}(t)$ of degree at most 3 with $\mathbf{p}(2) = 0$.

Answer: Remember (week 7 p46) that K is the kernel of the evaluation-at-2 function $E_2 : \mathbb{P}_3 \to \mathbb{R}$ given by $E_2(\mathbf{p}) = \mathbf{p}(2)$,

$$E_2(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_0 + a_1 2 + a_2 2^2 + a_3 2^3.$$

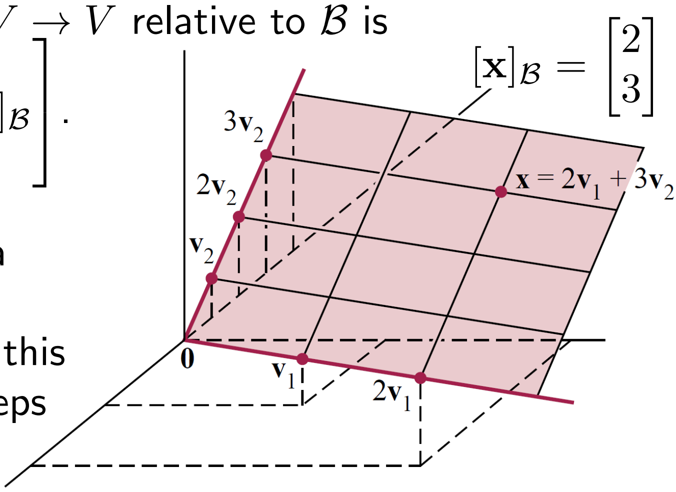
range$(E_2) = \mathbb{R}$, because, for each $c \in \mathbb{R}$, the polynomial $\mathbf{p} = c$ satisfies $E_2(\mathbf{p}) = c$.
So $\dim K = \dim(\text{domain}E_2) - \dim(\text{range}E_2) = \dim \mathbb{P}_3 - \dim \mathbb{R} = 4 - 1 = 3$.
Now $\mathcal{B} = \left\{(2-t), (2-t)^2, (2-t)^3\right\}$ is a subset of K, and is linearly independent (check with coordinate vectors relative to the standard basis of $\mathbb{P}_3$, or because these three polynomials have different degrees - see week 8 p14-15). Since $\mathcal{B}$ contains exactly $3 = \dim K$ vectors, it is a basis for K.

§4.4, 4.7, 5.4: Change of Basis

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V. Remember:

- The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ where $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.
- The matrix for a linear transformation $T : V \to V$ relative to $\mathcal{B}$ is

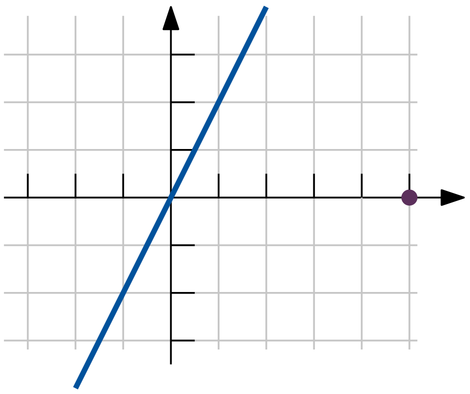
$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} \text{ (or } [T]_\mathcal{B}) = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_\mathcal{B} & \dots & [T(\mathbf{b}_n)]_\mathcal{B} \\ | & & | \end{bmatrix}.$$



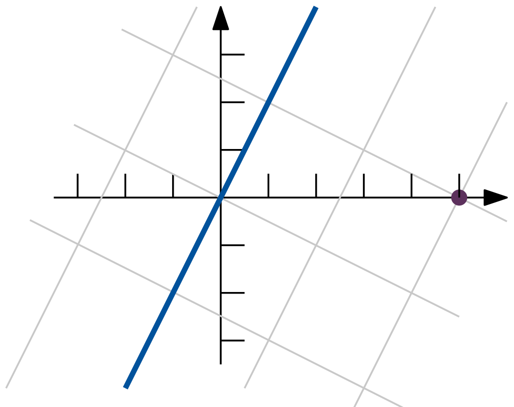
A basis for this plane in $\mathbb{R}^3$ allows us to draw a coordinate grid on the plane. The coordinate vectors then describe the location of points on this plane relative to this coordinate grid (e.g. 2 steps in $\mathbf{v}_1$ direction, 3 steps in $\mathbf{v}_2$ direction.)

Although we already have the standard coordinate grid on $\mathbb{R}^n$, some computations are much faster and more accurate in a different basis i.e. using a different coordinate grid (see also p18-20):

Example: Find the image of the point $(5, 0)$ under reflection about the line $y = 2x$.

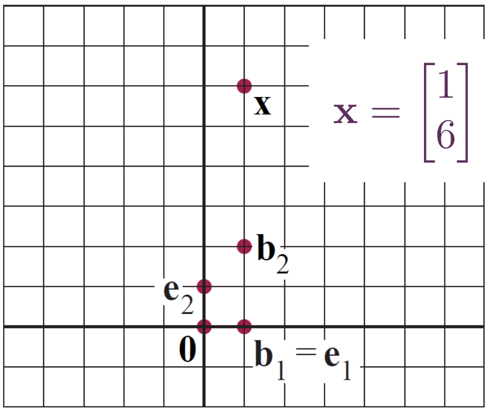


Horizontal and vertical grid lines are not useful for this problem because $y = 2x$ is not horizontal nor vertical.

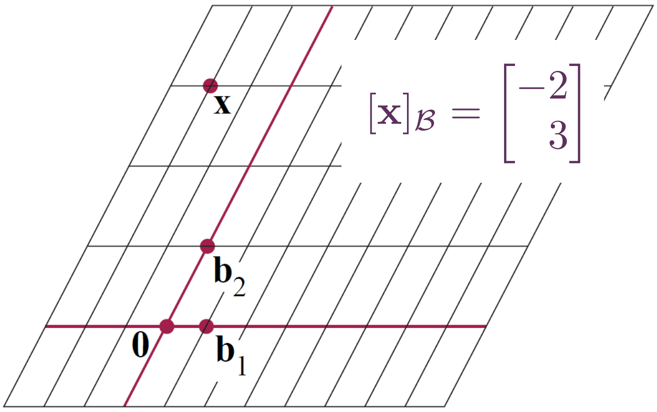


It is more useful to work with lines parallel and perpendicular to $y = 2x$.

Another example of two coordinate grids (note that the lines don't have to be perpendicular):



standard coordinate grid



$\mathcal{B}$-coordinate grid

Important questions:

i how are $\mathbf{x}$ and $[\mathbf{x}]_{\mathcal{B}}$ related (p4-7, §4.4 in textbook);

ii how are $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{F}}$ related for two bases $\mathcal{B}$ and $\mathcal{F}$ (p8-11, §4.7);

iii how are the standard matrix of T and the matrix $[T]_{\mathcal{B}}$ (or $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$) related (p12-16, §5.4)

Changing from any basis to the standard basis of $\mathbb{R}^n$

EXAMPLE: (see the picture on p3) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of $\mathbb{R}^2$.

a. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, then what is $\mathbf{x}$?

b. If $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then what is $\mathbf{v}$?

Solution: (a) Use the definition of coordinates:

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ means that $\mathbf{x} =$ ______ $\mathbf{b}_1 +$ ______ $\mathbf{b}_2 =$

(b) Use the definition of coordinates:

$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ means that $\mathbf{v} =$

In general, if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$, and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, then

$\mathbf{x} =$

i.e.

where $\underset{\mathcal{E} \leftarrow \mathcal{B}}{\mathcal{P}}$ is called the **change-of-coordinates matrix from $\mathcal{B}$ to the standard basis** ($\mathcal{P}_{\mathcal{B}}$ in textbook).

In the opposite direction

Changing from the standard basis to any other basis of $\mathbb{R}^n$

EXAMPLE: (see the picture on p3) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of $\mathbb{R}^2$.

a. If $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, then what are its $\mathcal{B}$-coordinates $[\mathbf{x}]_\mathcal{B}$?

b. If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then what are its $\mathcal{B}$-coordinates $[\mathbf{v}]_\mathcal{B}$?

Solution: (a) Suppose $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. This means that

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = \mathbf{x} =$$

So (c_1, c_2) is the solution to the linear system $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 6 \end{array}\right]$.

Row reduction: $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right]$

So $[\mathbf{x}]_\mathcal{B} =$

(b) The $\mathcal{B}$-coordinate vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ of $\mathbf{v}$ satisfies $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

So $[\mathbf{v}]_\mathcal{B}$ is the solution to

In general, if $\mathcal{B}=\{\mathbf{b}_1,\dots,\mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$, and $\mathbf{v}$ is any vector in $\mathbb{R}^n$, then $[\mathbf{v}]_{\mathcal{B}}$ is a solution to $\begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix}\mathbf{x}=\mathbf{v}$.

This matrix is $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}$

Because $\mathcal{B}$ is a basis, the columns of $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}$ are linearly independent, so by the Invertible Matrix Theorem, $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}$ is invertible, and the unique solution to $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}\mathbf{x}=\mathbf{v}$ is

$$[\mathbf{v}]_{\mathcal{B}}=\begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix}^{-1}\mathbf{v}.$$

So we can write $[\mathbf{v}]_{\mathcal{B}}=\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}\mathbf{v}$, where $\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}$, the change-of-coordinates matrix from the standard basis to $\mathcal{B}$, satisfies $\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}=\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}$.

Check with previous example: $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}\mathbf{x}=\begin{bmatrix}1&1\\0&2\end{bmatrix}^{-1}\begin{bmatrix}1\\6\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2&-1\\-0&1\end{bmatrix}\begin{bmatrix}1\\6\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix}$.

A very common mistake is to get the direction wrong:

Does multiplication by $\begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix}$ change from standard coordinates to $\mathcal{B}$-coordinates, or from $\mathcal{B}$-coordinates to standard coordinates?

Don't memorise the formulas. Instead, remember the definition of coordinates:

$$[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix} \text{ means } \quad \mathbf{x}=c_1\mathbf{b}_1+\dots+c_n\mathbf{b}_n=\begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix}[\mathbf{x}]_{\mathcal{B}}$$

and you won't go wrong.

ii: Changing between two non-standard bases:

Example: As before, $\mathbf{b}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

Another basis: $\mathbf{f}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{f}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$.

If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$, then what are its $\mathcal{F}$-coordinates $[\mathbf{x}]_{\mathcal{F}}$?

Answer 1: $\mathcal{B}$ to standard to $\mathcal{F}$ - works only in $\mathbb{R}^n$, in general easiest to calculate.

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$ means $\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2 = -2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$.

So if $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix}d_1\\d_2\end{bmatrix}$, then $d_1\begin{bmatrix}1\\1\end{bmatrix} + d_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$.

Row-reducing $\left[\begin{array}{cc|c}1&0&1\\1&1&6\end{array}\right]$ shows $d_1 = 1, d_2 = 5$ so $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix}1\\5\end{bmatrix}$.

In other words, $\mathbf{x} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{F}} = \underset{\mathcal{F}\leftarrow\mathcal{E}}{\mathcal{P}}\mathbf{x}$, so $[\mathbf{x}]_{\mathcal{F}} = \underset{\mathcal{F}\leftarrow\mathcal{E}}{\mathcal{P}}\,\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[\mathbf{x}]_{\mathcal{B}}\ (= \mathcal{P}_{\mathcal{F}}^{-1}\mathcal{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}})$

Answer 2: A different view that works for abstract vector spaces (without reference to a standard basis) - important theoretically, but may be hard to calculate for general examples in $\mathbb{R}^n$.

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$ means $\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2$.

So $[\mathbf{x}]_{\mathcal{F}} = [-2\mathbf{b}_1 + 3\mathbf{b}_2]_{\mathcal{F}} \underset{\curvearrowright}{=} -2[\mathbf{b}_1]_{\mathcal{F}} + 3[\mathbf{b}_2]_{\mathcal{F}} = \begin{bmatrix}| & | \\ [\mathbf{b}_1]_{\mathcal{F}} & [\mathbf{b}_2]_{\mathcal{F}} \\ | & | \end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}$.

because $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{F}}$ is an isomorphism, so every vector space calculation is accurately reproduced using coordinates.

$$\left.\begin{aligned}\mathbf{b}_1 &= \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \mathbf{f}_1 - \mathbf{f}_2 \text{ so } [\mathbf{b}_1]_{\mathcal{F}} = \begin{bmatrix}1\\-1\end{bmatrix}.\\ \mathbf{b}_2 &= \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \mathbf{f}_1 + \mathbf{f}_2 \text{ so } [\mathbf{b}_2]_{\mathcal{F}} = \begin{bmatrix}1\\1\end{bmatrix}.\end{aligned}\right\}$$

This step can be hard to calculate if the $\mathbf{b}_i$ are not "easy" linear combinations of the $\mathbf{f}_i$. But if you need to change bases in a practical application, the bases are probably "nicely" related.

So $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$.

Theorem 15: Change of Basis: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be two bases of a vector space V. Then, for all $\mathbf{x}$ in V,

$$[\mathbf{x}]_\mathcal{F} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_\mathcal{F} & \dots & [\mathbf{b}_n]_\mathcal{F} \\ | & & | \end{bmatrix} [\mathbf{x}]_\mathcal{B}.$$

Another way to say this: $[\mathbf{x}]_\mathcal{F} = \underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}}[\mathbf{x}]_\mathcal{B}$ where $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_\mathcal{F} & \dots & [\mathbf{b}_n]_\mathcal{F} \\ | & & | \end{bmatrix}$ is the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{F}$.

A tip to get the direction correct:

$$[\mathbf{x}]_\mathcal{F} = \underbrace{\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}}[\mathbf{x}]_\mathcal{B}}$$

A $\mathcal{F}$-coordinate vector

a linear combination of columns of $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}}$, so these columns should be $\mathcal{F}$-coordinate vectors

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Theorem 15: Change of Basis: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be two bases of a vector space V. Then, for all $\mathbf{x}$ in V,

$$[\mathbf{x}]_\mathcal{F} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_\mathcal{F} & \dots & [\mathbf{b}_n]_\mathcal{F} \\ | & & | \end{bmatrix} [\mathbf{x}]_\mathcal{B}.$$

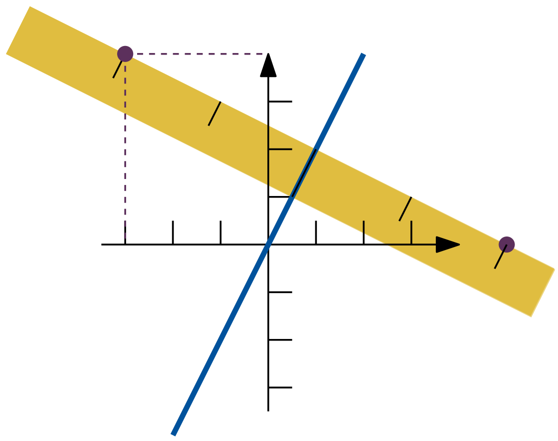
Properties of the change-of-coordinates matrix $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_\mathcal{F} & \dots & [\mathbf{b}_n]_\mathcal{F} \\ | & & | \end{bmatrix}$:

- $\underset{\mathcal{B}\leftarrow\mathcal{F}}{\mathcal{P}} = \underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}$.
- In the special case that V is $\mathbb{R}^n$ and $\mathcal{F}$ is the standard basis $\mathcal{E} = \{\mathbf{e}_1, \dots \mathbf{e}_n\}$, then the above formula says $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_\mathcal{E} & \dots & [\mathbf{b}_n]_\mathcal{E} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix}$, because $[\mathbf{b}_i]_\mathcal{E} = \mathbf{b}_i$. So this agrees with what we found earlier (part i, p4).
- If V is $\mathbb{R}^n$, then $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}} = \underset{\mathcal{F}\leftarrow\mathcal{E}}{\mathcal{P}}\,\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_\mathcal{F}^{-1}\mathcal{P}_\mathcal{B}$ (see p8).

iii: Change of coordinates and linear transformations:

Recall our problem from the start of this week's notes:

Example: Find the image of the point $(5,0)$ under reflection about the line $y=2x$.



An efficient solution:

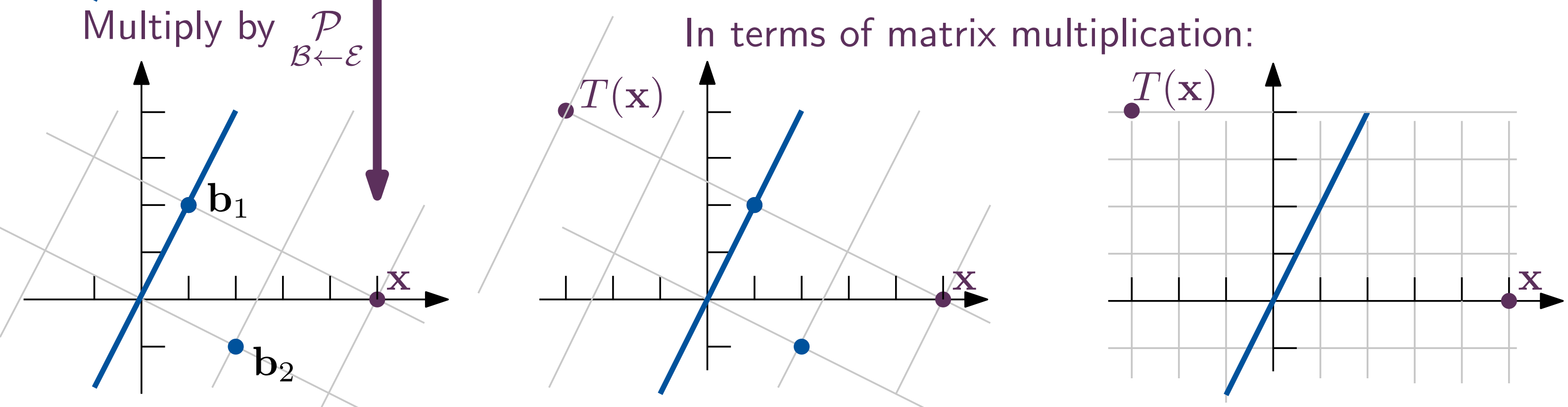
1. Measure the perpendicular distance from $(5,0)$ to the line;
2. The image of $(5,0)$ is the point that is the same distance away on the other side of the line;
3. Read off the coordinates of this point: $(-3,4)$.

The previous solution in the language of coordinates:

Let $\mathbf{b}_1=\begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{b}_2=\begin{bmatrix}2\\-1\end{bmatrix}$ and work in the basis $\mathcal{B}=\{\mathbf{b}_1,\mathbf{b}_2\}$.

Let T be reflection about the line $y=2x$, and $\mathbf{x}=\begin{bmatrix}5\\0\end{bmatrix}$.

So we want $T(\mathbf{x})$.



Multiply by $\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}$

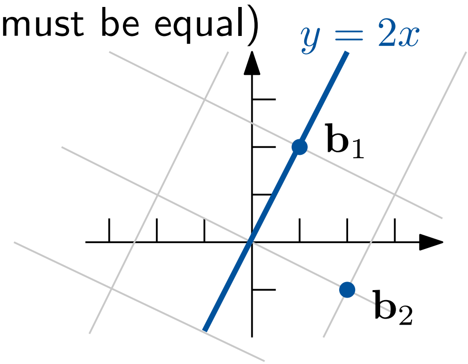
In terms of matrix multiplication:

$$1.\ [\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}1\\2\end{bmatrix}\xrightarrow[\text{Multiply by }\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}]{}2.\ [T(\mathbf{x})]_{\mathcal{B}}=\begin{bmatrix}1\\-2\end{bmatrix}\xrightarrow[\text{Multiply by }\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}]{}3.\ T(\mathbf{x})=\begin{bmatrix}-3\\4\end{bmatrix}$$

The 3-step solution above shows that $T(\mathbf{x}) = \underset{\mathcal{E}\leftarrow\enclose{circle}{\mathcal{B}}}{\mathcal{P}} \underset{\enclose{circle}{\mathcal{B}}\leftarrow\enclose{circle}{\mathcal{B}}}{[T]} \underset{\enclose{circle}{\mathcal{B}}\leftarrow\mathcal{E}}{\mathcal{P}} \mathbf{x}$.

Write $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}$ for the standard matrix of T. Then $T(\mathbf{x}) = \underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}\,\mathbf{x}$, so the equation $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}\,\mathbf{x} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} \underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} \mathbf{x}$ is true for all $\mathbf{x}$. So the matrices on the two sides must be equal (e.g. letting $\mathbf{x} = \mathbf{e}_i$ shows that each column of the matrices must be equal)

$$\boxed{\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} \underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}.}$$



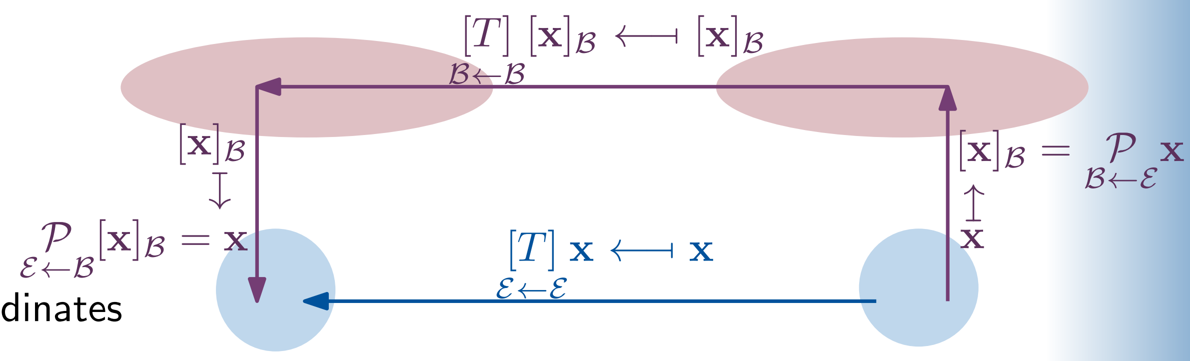
This equation is useful because, for geometric linear transformations T, it is often easier to find $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$ for some "natural" basis $\mathcal{B}$ than to find the standard matrix $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}$.

E.g. in our example of reflection in $y = 2x$:

$\begin{bmatrix}1\\2\end{bmatrix}$ is on the line $y = 2x$, so it is unchanged by the reflection: $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$.

$\begin{bmatrix}2\\-1\end{bmatrix}$ is perpendicular to $y = 2x$, so its image is its negative: $T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) = \begin{bmatrix}-2\\1\end{bmatrix}$.

A different picture to understand $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} \underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}$:



$\mathcal{B}$-coordinates

$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\,[\mathbf{x}]_{\mathcal{B}} \longleftarrow [\mathbf{x}]_{\mathcal{B}}$

$[\mathbf{x}]_{\mathcal{B}} \downarrow \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$

$[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}\mathbf{x} \uparrow \mathbf{x}$

$\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}\,\mathbf{x} \longleftarrow \mathbf{x}$

Standard coordinates

Because $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_{\mathcal{B}}$ and $\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \mathcal{P}_{\mathcal{B}}^{-1}$: $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} = \mathcal{P}_{\mathcal{B}} \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} \mathcal{P}_{\mathcal{B}}^{-1}$

Multiply both sides by $\mathcal{P}_{\mathcal{B}}^{-1}$ on the left and by $\mathcal{P}_{\mathcal{B}}$ on the right: $\mathcal{P}_{\mathcal{B}}^{-1} \underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} \mathcal{P}_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$

These two equations are hard to remember ("where does the inverse go?"). Instead, remember $\boxed{\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} \underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}}$ (which works for all vector spaces, not just $\mathbb{R}^n$).

EXAMPLE: Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of $\mathbb{R}^2$.

Suppose T is a linear transformation satisfying $T(\mathbf{b}_1) = \mathbf{b}_1$ and $T(\mathbf{b}_2) = -\mathbf{b}_2$.
Find $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}$, the standard matrix of T.

Solution: We will use change of coordinates:

$$\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} =$$

Now we need to find the matrices on the right hand side.

- Find $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$ from the information $\begin{aligned} T(\mathbf{b}_1) &= \mathbf{b}_1 \\ T(\mathbf{b}_2) &= -\mathbf{b}_2 \end{aligned}$

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} =$$

- Find the change-of-coordinate matrices, using the definition of coordinates:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ means } \mathbf{x} = \qquad\qquad = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

Putting it all together:

$$\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} =$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \quad \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Check that our answer satisfies the conditions given in the question:

$$T(\mathbf{b}_1) = \underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}\,\mathbf{b}_1 =$$

$$T(\mathbf{b}_2) = \underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}\,\mathbf{b}_2 =$$

Remember

$$\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}.$$

This motivates the following definition:

Definition: Two square matrices A and D are *similar* if there is an invertible matrix P such that $A = PDP^{-1}$.

Similar matrices represent the same linear transformation in different bases.

Similar matrices have the same determinant and the same rank, because the signed volume scaling factor and the dimension of the image are coordinate-independent properties of the linear transformation. (Exercise: prove that $\det D = \det(PDP^{-1})$ using the multiplicative property of determinants.)

Why is change of basis important?

Example: If x, y are the prices of two stocks on a particular day, then their prices the next day are respectively $\frac{1}{2}y$ and $-x + \frac{3}{2}y$. How are the prices after many days related to the prices today?

Answer: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function representing the changes in stock prices from one day to the next, i.e. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}y \\ -x + \frac{3}{2}y \end{bmatrix}$. We are interested in T^k for large k. (You will NOT be required to do this step.)

T is a linear transformation; its standard matrix is $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$.

Calculating $\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}^k$ by direct matrix multiplication will take a long time.

Answer: (continued) Let $\mathbf{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

$$T(\mathbf{b}_1) = \begin{bmatrix}0 & \frac{1}{2}\\-1 & \frac{3}{2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \tfrac{1}{2}\mathbf{b}_1, \quad T(\mathbf{b}_2) = \begin{bmatrix}0 & \frac{1}{2}\\-1 & \frac{3}{2}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} = \mathbf{b}_2,$$

so $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = \begin{bmatrix}| & |\\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}}\\ | & |\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix}$. Use $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}$:

$$\begin{aligned}
\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}^k &= \left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}\right)^k\\
&= \left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\cancel{\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}}\right)\left(\cancel{\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\cancel{\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}}\right)\cancel{\cdots\cdots}\left(\cancel{\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}\right)\\
&= \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}^k\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}\\
&= \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix}^k\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}^{-1} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{2}^k & 0\\0 & 1^k\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}^{-1} = \begin{bmatrix}-1+\frac{1}{2}^{k-1} & 1-\frac{1}{2}^k\\-2+\frac{1}{2}^{k-1} & 2-\frac{1}{2}^k\end{bmatrix}.
\end{aligned}$$

So $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}^k = \begin{bmatrix}-1+\frac{1}{2}^{k-1} & 1-\frac{1}{2}^k\\-2+\frac{1}{2}^{k-1} & 2-\frac{1}{2}^k\end{bmatrix}$. When k is very large, this is very close to $\begin{bmatrix}-1 & 1\\-2 & 2\end{bmatrix}$.
So essentially the stock prices after many days is $-x+y$ and $-2x+2y$, where x, y are the prices today. (In particular, the prices stabilise, which was not clear from $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}$.)

The important points in this example:

- We have a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ and we want to find T^k for large k.
- We find a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $T(\mathbf{b}_1) = \lambda_1\mathbf{b}_1$ and $T(\mathbf{b}_2) = \lambda_2\mathbf{b}_2$ for some scalars λ_1, λ_2. (In the example, $\lambda_1 = \frac{1}{2}, \lambda_2 = 1$.)
- Relative to the basis $\mathcal{B}$, the matrix for T is a diagonal matrix $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}$.
- It is easy to compute with $[T]_{\mathcal{B}}$, and we can then use change of coordinates to transfer the result to the standard matrix $\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}$.

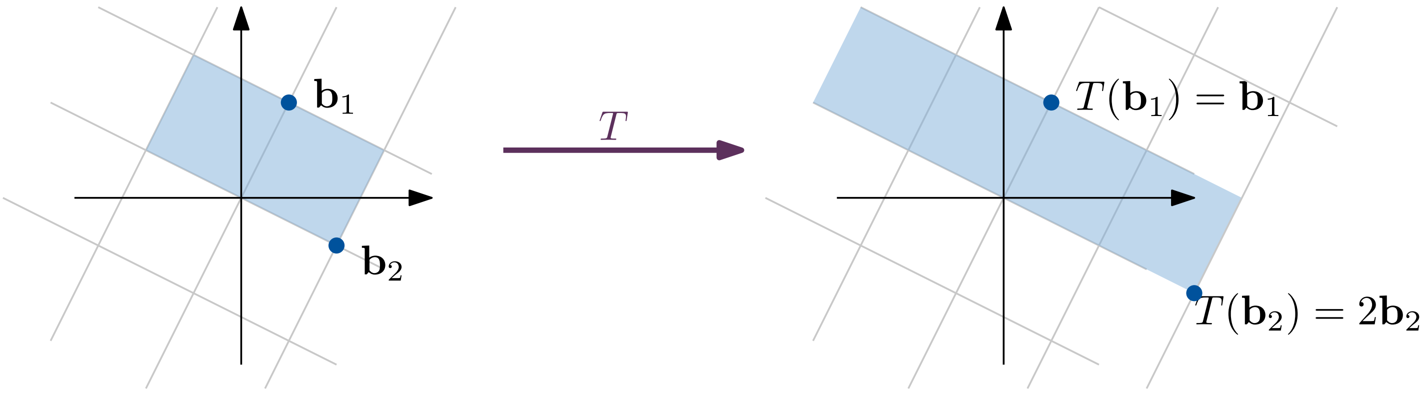
Next week (§5): does a "magic" basis like this always exist, and how to find it?
(Don't worry: you can do many of the computations in §5 without fully understanding change of coordinates.)

Remember from last week (week 9 p20):
Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, the "right" basis to work in is $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ where $T(\mathbf{b}_i) = \lambda_i \mathbf{b}_i$ for some scalars λ_i. Then the matrix for T relative to $\mathcal{B}$ is a diagonal matrix:

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = \begin{bmatrix} | & | & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

Computers are much faster and more accurate when they work with diagonal matrices, because many entries are 0.

Also, it's much easier to understand the linear transformation T from a diagonal matrix, e.g. if $T(\mathbf{b}_1) = \mathbf{b}_1$ and $T(\mathbf{b}_2) = 2\mathbf{b}_2$, so $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then T is an expansion by a factor of 2 in the $\mathbf{b}_2$ direction.



So it is important to study the equation $T(\mathbf{x}) = \lambda\mathbf{x}$.

(It's also very useful in ODEs - see MATH3405.)

§5.1-5.2: Eigenvectors and Eigenvalues

Definition: Let A be a square matrix.
An *eigenvector* of A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.
Then we call $\mathbf{x}$ an *eigenvector corresponding to λ* (or a λ-eigenvector).
An *eigenvalue* of A is a scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$.

If $\mathbf{x}$ is an eigenvector of A, then $\mathbf{x}$ and its image $A\mathbf{x}$ are in the same (or opposite, if $\lambda < 0$) direction. Multiplication by A stretches $\mathbf{x}$ by a factor of λ.
If $\mathbf{x}$ is not an eigenvector, then $\mathbf{x}$ and $A\mathbf{x}$ are not geometrically related in any obvious way.

Warning: eigenvalues and eigenvectors exist for square matrices only. If A is not a square matrix, then $\mathbf{x}$ and $A\mathbf{x}$ are in different vector spaces (they are column vectors with a different number of rows), so it doesn't make sense to ask whether $A\mathbf{x}$ is a multiple of $\mathbf{x}$.

$$A\mathbf{x} = \lambda\mathbf{x}$$

eigenvector: cannot be $\mathbf{0}$.
$A\mathbf{0} = \lambda\mathbf{0}$ is always true, so it holds no information about A.

eigenvalue: can be 0.
$A\mathbf{x} = 0\mathbf{x}$ for a nonzero vector $\mathbf{x}$ does hold information about A - it tells you that A is not invertible. In fact, A is invertible if and only if 0 is not an eigenvalue (add to IMT!).

Important computations:

i. given an eigenvalue, how to find the corresponding eigenvectors (p5-9, §5.1);
ii. how to find the eigenvalues (p10-13, §5.2);
iii. how to determine if there is a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$ where each $\mathbf{b}_i$ is an eigenvector (p15-31, §5.3).

Warm up:

Example: Let $A = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix}$. Determine whether $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are eigenvectors of A.

Answer:

$\begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (because its entries are not equal), so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector of A.

$\begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 6.

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i: Given the eigenvalues, find the corresponding eigenvectors:

i.e. we know λ, and we want to solve $A\mathbf{x} = \lambda\mathbf{x}$.
This equation is equivalent to $A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$,
which is equivalent to $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

So the eigenvectors of A corresponding to the eigenvalue λ are the nonzero solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$, which we can find by row-reducing $A - \lambda I$.

The eigenvectors of A corresponding to the eigenvalue λ are the nonzero solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

EXAMPLE: Let $A = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix}$. Find the eigenvectors of A corresponding to the eigenvalue 2.

Solution:

To solve $(A - 2I_2)(\mathbf{x}) = \mathbf{0}$, we need to find $A - 2I_2$ first:

$$A - 2I_2 = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} 8 - \underline{\quad} & 4 \\ -3 & \quad \end{bmatrix} =$$

Now we solve $(A - 2I_2)(\mathbf{x}) = \mathbf{0}$:

$$\left[\begin{array}{cc|c} 6 & 4 & 0 \\ -3 & -2 & 0 \end{array}\right] \longrightarrow$$

so the eigenvectors are $\left\{ \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} s \;\middle|\; s\underline{\quad} \right\}$.

A nicer-looking answer: $\left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix} s \;\middle|\; s\underline{\quad} \right\}$.

Check our answer:

Because it is sometimes convenient to talk about the eigenvectors and $\mathbf{0}$ together:

Definition: The *eigenspace* of A corresponding to the eigenvalue λ (or the λ-eigenspace of A, sometimes written $E_\lambda(A)$) is the solution set to $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Because λ-eigenspace of A is the null space of $A - \lambda I$, eigenspaces are subspaces. In the previous example, the eigenspace is a line, but there can also be two-dimensional eigenspaces:

Example: Let $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$. Find a basis for the eigenspace corresponding to the eigenvalue -3.

Answer:

$$B - (-3)I_3 = \begin{bmatrix} -3+3 & 0 & 0 \\ -1 & -2+3 & 1 \\ -1 & 1 & -2+3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row-reduction}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So solutions are $x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, for all values of x_2, x_3. So a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Be careful how you write your answer, depending on what the question asks for:

The eigenvectors: $\left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \middle| \boxed{s, t \text{ not both zero}} \right\}$.

DON'T write $s, t \neq 0$, because that's confusing: do you mean $s \neq 0$ AND $t \neq 0$?

The eigenspace: $\left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$.

A basis for the eigenspace: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Tip: if you found that $(B - \lambda I)\mathbf{x} = \mathbf{0}$ has no nonzero solutions, then you've made an arithmetic error. Please do not write that the eigenvector is $\mathbf{0}$!

ii: Given a matrix, find its eigenvalues:

λ is an eigenvalue of A if $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has non-trivial solutions.
By the Invertible Matrix Theorem, this happens precisely when $A - \lambda I$ is not invertible.
So we must have $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is the characteristic polynomial of A (sometimes written χ_A). If A is $n \times n$, then this is a polynomial of degree n. So A has at most n different eigenvalues.
$\det(A - \lambda I) = 0$ is the characteristic equation of A.
We find the eigenvalues by solving the characteristic equation.

We find the eigenvalues by solving the characteristic equation $\det(A - \lambda I) = 0$.

EXAMPLE: Find the eigenvalues of $A = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix}$.

Solution:

$\det(A - \lambda I) =$

So the eigenvalues are the solutions to

Factorise:

We find the eigenvalues by solving the characteristic equation $\det(A - \lambda I) = 0$.

Example: Find the eigenvalues of $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$.

Answer:

(expand along top row)

$$\det(B - \lambda I) = \begin{vmatrix} -3-\lambda & 0 & 0 \\ -1 & -2-\lambda & 1 \\ -1 & 1 & -2-\lambda \end{vmatrix} = (-3-\lambda) \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix}$$

Tip: if you already have a factor, don't expand it

$$\begin{aligned} &= (-3-\lambda)[(-2-\lambda)(-2-\lambda) - 1] \\ &= (-3-\lambda)[\lambda^2 + 4\lambda + 3] \\ &= (-3-\lambda)(\lambda+3)(\lambda+1). \end{aligned}$$

So the eigenvalues are the solutions to $(-3-\lambda)(\lambda+3)(\lambda+1) = 0$,

which are $-3, \quad -3, \quad -1$.

Tips:

- Because of the variable λ, it is easier to find $\det(A - \lambda I)$ by expanding across rows or down columns than by using row operations.
- If you already have a factor, do not expand it (e.g. previous page)
- Do not "cancel" λ in the characteristic equation: remember that $\lambda = 0$ can be an eigenvalue (see below).
- The eigenvalues of A are usually not related to the eigenvalues of rref(A).

Example: Find the eigenvalues of $C = \begin{bmatrix} 3 & 6 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \end{bmatrix}$.

Answer: $C - \lambda I = \begin{bmatrix} 3-\lambda & 6 & -2 \\ 0 & -\lambda & 2 \\ 0 & 0 & 6-\lambda \end{bmatrix}$ is upper triangular, so its determinant is the product of its diagonal entries: $\det(C - \lambda I) = (3-\lambda)(-\lambda)(6-\lambda)$, whose solutions are 3, 0, 6.

By a similar argument (for upper or lower triangular matrices):

Fact: The eigenvalues of a triangular matrix are the diagonal entries.

Summary: To find the eigenvalues and eigenvectors of a square matrix A:
Step 1 Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues;
Step 2 For each eigenvalue λ, solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find the eigenvectors.

Thinking about eigenvectors conceptually:

Suppose $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue λ.
Then

$$A^2(\mathbf{v}) = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

So any eigenvector of A is also an eigenvector of A^2, corresponding to the square of the previous eigenvalue.

We can also define eigenvalues and eigenvectors for a linear transformation $T : V \to V$ on an abstract vector space V: a nonzero vector $\mathbf{v}$ in V is an eigenvector of T with corresponding eigenvalue λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.

Example: Consider $T : \mathbb{P}_3 \to \mathbb{P}_3$ given by $T(\mathbf{p}) = x\frac{d}{dx}\mathbf{p}$. Then $\mathbf{p}(x) = x^2$ is an eigenvector of T corresponding to the eigenvalue 2, because
$T(x^2) = x\frac{d}{dx}x^2 = x2x = 2x^2$.

§5.3: Diagonalisation

Remember that our motivation for finding eigenvectors is to find a basis relative to which a linear transformation is represented by a diagonal matrix.

Definition: (week 9 p17) Two square matrices A and B are *similar* if there is an invertible matrix P such that $A = PBP^{-1}$.

From the change-of-coordinates formula (week 9 p14)

$$\underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_{\mathcal{B}}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1},$$

similar matrices represent the same linear transformation relative to different bases.

Definition: A square matrix A is *diagonalisable* if it is similar to a diagonal matrix, i.e. if there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Theorem 5: Diagonalisation Theorem: An $n \times n$ matrix A is diagonalisable (i.e. $A = PDP^{-1}$) if and only if A has n linearly independent eigenvectors.

Proof: we prove a stronger theorem: An $n \times n$ matrix A satisfies $AP = PD$ for a $n \times k$ matrix P and a diagonal $k \times k$ matrix D if and only if the ith column of P is an eigenvector of A with eigenvalue d_{ii}, or is the zero vector. This comes from equating column by column the right hand sides of the following equations:

$$AP = A\begin{bmatrix} | & & | \\ \mathbf{p}_1 & \dots & \mathbf{p}_k \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\mathbf{p}_1 & \dots & A\mathbf{p}_k \\ | & & | \end{bmatrix}$$

$$PD = \begin{bmatrix} | & & | \\ \mathbf{p}_1 & \dots & \mathbf{p}_k \\ | & & | \end{bmatrix}\begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & & \\ \vdots & & \ddots & \\ 0 & & & d_{kk} \end{bmatrix} = \begin{bmatrix} | & & | \\ d_{11}\mathbf{p}_1 & \dots & d_{kk}\mathbf{p}_k \\ | & & | \end{bmatrix}$$

To deduce The Diagonalisation Theorem, note that $A = PDP^{-1}$ if and only if $AP = PD$ and P is invertible, i.e. (using Invertible Matrix Theorem) if and only if $AP = PD$ and the n columns of P are linearly independent.

iii.i: Diagonalise a matrix i.e. given A, find P and D with $A = PDP^{-1}$:

Example: Diagonalise $A = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix}$.

Answer:

Step 1 Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues.

From p11, $\det(A - \lambda I) = \lambda^2 - 8\lambda + 12$, eigenvalues are 2 and 6.

Step 2 For each eigenvalue λ, solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find a basis for the λ-eigenspace.

From p7, $\left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ is a basis for the 2-eigenspace,

You can check that $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the 6-eigenspace.

Notice that these two eigenvectors are linearly independent (this is automatic, p22).

If Step 2 gives fewer than n vectors, A is not diagonalisable (p26). Otherwise, continue:

Step 3 Put the eigenvectors from Step 2 as the columns of P. $P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}$.

Step 4 Put the corresponding eigenvalues as the diagonal entries of D. $D = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$.

Checking our answer: $PDP^{-1} = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 2 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} = A$.

The matrices P and D are not unique:

- In Step 2, we can choose a different basis for the eigenspaces:
  e.g. using $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ instead of $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ as a basis for the 2-eigenspace, we can take
  $P = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$, and then $PDP^{-1} = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} = A$.
- In Step 3, we can choose a different order for the columns of P, as long as we put the entries of D in the corresponding order:
  e.g. $P = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$ then
  $PDP^{-1} = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} = A$.

Example: Diagonalise $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$.

Answer:

Step 1 Solve the characteristic equation $\det(B - \lambda I) = 0$ to find the eigenvalues.

From p12, $\det(B - \lambda I) = (-3-\lambda)(\lambda+3)(\lambda+1)$, so the eigenvalues are -3 and -1.

Step 2 For each eigenvalue λ, solve $(B - \lambda I)\mathbf{x} = \mathbf{0}$ to find a basis for the λ-eigenspace.

From p8, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the -3-eigenspace; you can check that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the -1-eigenspace. You can check that these three eigenvectors are linearly independent (this is automatic, see p22).

If Step 2 gives fewer than n vectors, B is not diagonalisable (p26). Otherwise, continue:

Step 3 Put the eigenvectors from Step 2 as the columns of P.

Step 4 Put the corresponding eigenvalues as the diagonal entries of D.

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Remember that $B = PDP^{-1}$ if and only if $BP = PD$ and P is invertible. This allows us to check our answer without inverting P:

$$BP = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 \\ -3 & 0 & -1 \\ 0 & -3 & -1 \end{bmatrix},$$

$$PD = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 \\ -3 & 0 & -1 \\ 0 & -3 & -1 \end{bmatrix} = BP, \text{ and}$$

$$\det P = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -2 \neq 0.$$

We can use the matrices P and D to quickly calculate powers of B (see also week 9 p19):

$$\begin{aligned} B^3 &= (PDP^{-1})^3 \\ &= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \\ &= PD^3P^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}^3 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -27 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -27 & 0 & 0 \\ -13 & -14 & 13 \\ -13 & 13 & -14 \end{bmatrix}. \end{aligned}$$

(This sometimes works for“ fractional” and negative powers too, see Homework 5 Q3.)

At the end of Step 2, after finding a basis for each eigenspace, it is unnecessary to explicitly check that the eigenvectors in the different bases, together, are linearly independent:

Theorem 7c: Linear Independence of Eigenvectors: If $\mathcal{B}_1, \dots, \mathcal{B}_p$ are linearly independent sets of eigenvectors of a matrix A, corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_p$, then the total collection of vectors in the sets $\mathcal{B}_1, \dots, \mathcal{B}_p$ is linearly independent. (Proof idea: see practice problem 3 in §5.1 of textbook.)

Example: In the previous example, $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set in the -3-eigenspace, $\mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set in the -1-eigenspace, so the theorem says that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

An important special case of Theorem 7c is when each $\mathcal{B}_i$ contains a single vector:

THEOREM 2: Linear Independence of Eigenvectors: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors of a matrix A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.

To give you an idea of why this is true, and as an example of how to write proofs involving eigenvectors, we prove this in the simple case $p = 2$:

We are given that $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A corresponding to distinct eigenvalues λ_1, λ_2, i.e.

$$A\mathbf{v}_1 = \underline{\hspace{2cm}}; \qquad A\mathbf{v}_2 = \underline{\hspace{2cm}}$$

$$\mathbf{v}_1 \neq \underline{\hspace{2cm}}; \qquad \mathbf{v}_2 \neq \underline{\hspace{2cm}}; \qquad \lambda_1 \neq \underline{\hspace{2cm}}.$$

We want to show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, i.e. $c_1 = c_2 = 0$ is the only solution to

$$(*)$$

Multiply both sides by A:

$$c_1\lambda_1\mathbf{v}_1 +$$

Multiply equation $(*)$ by λ_1:

$$c_1\lambda_1\mathbf{v}_1 +$$

Subtract:

Because ______________ and ______________, we must have $c_2 = 0$.

Substituting into $(*)$ shows ______________, and because ______________, we must have $c_1 = 0$.

One proof for $p > 2$ is to repeat this (multiply by A, multiply by λ_i, subtract) $p-1$ times. P288 in the textbook phrases this differently, as a proof by contradiction.

iii.ii: Determine if a matrix is diagonalisable

From the Diagonalisation Theorem, we know that A is diagonalisable if and only if A has n linearly independent eigenvectors. Can we determine if A has enough eigenvectors without finding all those eigenvectors?

To do so, we need an extra idea:

Definition: The (algebraic) multiplicity of an eigenvalue λ_k is its multiplicity as a root of the characteristic equation, i.e. it is the number of times the linear factor $(\lambda - \lambda_k)$ occurs in $\det(A - \lambda I)$.

Example: Consider $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$. From p12, the characteristic polynomial of B is $\det(B - \lambda I) = (-3 - \lambda)(\lambda + 3)(\lambda + 1) = -(\lambda + 3)(\lambda + 3)(\lambda + 1)$. So -3 has multiplicity 2, and -1 has multiplicity 1.

Theorem 7b: Diagonalisability Criteria: An $n \times n$ matrix A is diagonalisable if and only if both the following conditions are true:

- i the characteristic polynomial $\det(A - \lambda I)$ factors completely into linear factors (i.e. it has n solutions counting with multiplicity);
- ii for each eigenvalue λ_k, the dimension of the λ_k-eigenspace is equal to the multiplicity of λ_k.

Example: (failure of i) Consider $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$, the standard matrix for rotation through $\frac{\pi}{6}$. Its characteristic polynomial is $\begin{vmatrix} \sqrt{3}/2 - \lambda & -1/2 \\ 1/2 & \sqrt{3}/2 - \lambda \end{vmatrix} = (\frac{\sqrt{3}}{2} - \lambda)^2 + \frac{1}{4}$. This polynomial cannot be written in the form $(\lambda - a)(\lambda - b)$ because it has no solutions, as its value is always $\geq \frac{1}{4}$. So this rotation matrix is not diagonalisable. (This makes sense because, after a rotation through $\frac{\pi}{6}$, no vector is in the same or opposite direction.)

The failure of i can be “fixed” by allowing eigenvalues to be complex numbers, so we concentrate on condition ii.

Theorem 7b: Diagonalisability Criteria: An $n \times n$ matrix A is diagonalisable if and only if both the following conditions are true:

i the characteristic polynomial $\det(A - \lambda I)$ factors completely into linear factors;

ii for each eigenvalue λ_k, the dimension of the λ_k-eigenspace is equal to the multiplicity of λ_k.

Example: (failure of ii) Consider $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is upper triangular, so its eigenvalues are its diagonal entries (with the same multiplicities), i.e. 0 with multiplicity 2. The eigenspace of eigenvalue 0 is the set of solutions to $\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 0I_2 \right) \mathbf{x} = \mathbf{0}$, which is Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. So the eigenspace has dimension $1 < 2$, and therefore $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalisable.

Fact: (theorem 7a in textbook): the dimension of the λ_k-eigenspace is at most the multiplicity of λ_k. (Proof on p30.) So failure of ii in the Diagonalisability Criteria happens only when the eigenspaces are "too small".

Example: Determine if $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$ is diagonalisable.

Answer:

Step 1 Solve $\det(B - \lambda I) = 0$ to find the eigenvalues and multiplicities.

From p12, $\det(B - \lambda I) = (-3 - \lambda)(\lambda + 3)(\lambda + 1)$, so the eigenvalues are -3 (with multiplicity 2) and -1 (with multiplicity 1).

Step 2 For each eigenvalue λ of multiplicity more than 1, find the dimension of the λ-eigenspace (e.g. by row-reducing $(B - \lambda I)$ to echelon form):

The dimensions of all eigenspaces are equal to their multiplicities → diagonalisable

The dimension of one eigenspace is less than its multiplicity → not diagonalisable

$\lambda = -1$ has multiplicity 1, so we don't need to study it (see p29 for the reason).

$\lambda = -3$ has multiplicity 2, so we need to examine it more closely:

$$B - (-3)I_3 = \begin{bmatrix} -3+3 & 0 & 0 \\ -1 & -2+3 & 1 \\ -1 & 1 & -2+3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row-reduction}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This has two free variables (x_2, x_3), so the dimension of the -3-eigenspace is two, which is equal to its multiplicity. So B is diagonalisable.

Example: Let $K = \begin{bmatrix} 6 & -4 & 4 & 9 \\ -9 & 9 & 8 & -17 \\ 0 & 0 & 5 & 0 \\ -5 & 4 & -4 & -8 \end{bmatrix}$. Given that $\det(K - \lambda I) = (\lambda - 1)^2(\lambda - 5)^2$, determine if K is diagonalisable.

Answer:

Step 1 Solve $\det(K - \lambda I) = 0$ to find the eigenvalues and multiplicities.

The eigenvalues are 1 (with multiplicity 2) and 5 (with multiplicity 2).

Step 2 For each eigenvalue λ of multiplicity more than 1, find the dimension of the λ-eigenspace (e.g. by row-reducing $(K - \lambda I)$ to echelon form):

The dimensions of all eigenspaces are equal to their multiplicities → diagonalisable

The dimension of one eigenspace is less than its multiplicity → not diagonalisable

$$\lambda = 1\text{: } K - 1I_4 = \begin{bmatrix} 5 & -4 & 4 & 9 \\ -9 & 8 & 8 & -17 \\ 0 & 0 & 4 & 0 \\ -5 & 4 & -4 & -9 \end{bmatrix} \to \begin{bmatrix} 5 & -4 & 4 & 9 \\ 0 & 4 & * & * \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ 5R_2 - 9R_1 \\ \\ R_4 - R_1 \end{matrix}.$$

x_4 is the only one free variable, so the dimension of the 1-eigenspace is one, which is less than its multiplicity. So K is not diagonalisable. (We don't need to also check $\lambda = 5$.)

In Step 2, why don't we need to look at eigenvalues with multiplicity 1?
Answer: because the dimension of an eigenspace is always at least 1. So if an eigenvalue has multiplicity 1, then the dimension of its eigenspace must be exactly 1.

In particular: suppose an $n \times n$ matrix has n different eigenvalues. The multiplicity of each eigenvalue is at least 1, and if any eigenvalue has multiplicity > 1, then χ will have more than n factors. So each eigenvalue must have multiplicity exactly 1.

Theorem 6: Distinct eigenvalues implies diagonalisable: If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalisable.

Example: Is $C = \begin{bmatrix} 3 & 6 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \end{bmatrix}$ diagonalisable?

Answer: C is upper triangular, so its eigenvalues are its diagonal entries (with the same multiplicities), i.e 3, 0 and 6. Since C is 3×3 and it has 3 different eigenvalues, C is diagonalisable.

Warning: an $n \times n$ matrix with fewer than n eigenvalues can still be diagonalisable!

To prove the Diagonalisability Criteria, we first need to prove
Fact: (theorem 7a in textbook): the dimension of the λ_k-eigenspace is at most the multiplicity of λ_k.

Proof: (sketch, hard) Let $\lambda_k = r$, and let d be the dimension of the r-eigenspace. We want to show that $(\lambda - r)^d$ divides $\det(A - \lambda I)$.

For simplicity, I show the case $d = 3$. Take a basis of the r-eigenspace: this gives 3 linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ corresponding to the eigenvalue r.
By the Linearly Independent Set theorem, we can extend $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to a basis of $\mathbb{R}^n$ called $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_4, \dots, \mathbf{w}_n\}$.
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation whose standard matrix is A.
Because $T(\mathbf{v}_i) = r\mathbf{v}_i$, we have

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = \begin{bmatrix} r & 0 & 0 & * & \dots & * \\ 0 & r & 0 & * & \dots & * \\ 0 & 0 & r & * & \dots & * \\ 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & * & \dots & * \end{bmatrix}.$$

So $\det(\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} - \lambda I_n)$ is: (expanding down the first column each time)

$$\begin{vmatrix} r-\lambda & 0 & 0 & * & \dots & * \\ 0 & r-\lambda & 0 & * & \dots & * \\ 0 & 0 & r-\lambda & * & \dots & * \\ 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & * & \dots & * \end{vmatrix} = r-\lambda \begin{vmatrix} r-\lambda & 0 & * & \dots & * \\ 0 & r-\lambda & * & \dots & * \\ 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \dots & * \end{vmatrix} = (r-\lambda)^2 \begin{vmatrix} r-\lambda & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \dots & * \end{vmatrix}$$
$$= (r-\lambda)^3 \times \text{ some polynomial}$$

So $(r-\lambda)^3$ divides $\det(\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} - \lambda I_n)$, which is the same as $\det(A - \lambda I_n)$ because similar matrices have the same characteristic polynomial (and therefore the same eigenvalues):

$$\begin{aligned} \det(PBP^{-1} - \lambda I) &= \det(PBP^{-1} - \lambda PP^{-1}) \\ &= \det(P(B - \lambda I)P^{-1}) \\ &= \det P \det(B - \lambda I) \det(P^{-1}) \\ &= \det P \det(B - \lambda I) \frac{1}{\det P} = \det(B - \lambda I). \end{aligned}$$

Write m_k for the multiplicity of λ_k. We proved on the previous page:

Fact: $\dim E_{\lambda_k} \leq m_k$.

We now use it to show **Theorem 7b: Diagonalisability Criteria**: An $n \times n$ matrix A is diagonalisable (i.e. A has n linearly independent eigenvectors) if and only if both the following conditions are true:

i the characteristic polynomial $\det(A - \lambda I)$ factors completely into linear factors, i.e. $m_1 + m_2 + \cdots = n$;

ii for each eigenvalue λ_k, we have $\dim E_{\lambda_k} = m_k$.

Proof: (sketch)

"if" part: This is the diagonalisation algorithm:

if ii holds, then a basis for each E_{λ_k} gives m_k linearly independent eigenvectors corresponding to the eigenvalue λ_k.

Putting these bases together gives $m_1 + m_2 + \cdots = n$ eigenvectors, which are linearly independent by the Linear Independence of Eigenvectors Theorem (p22).

Write m_k for the multiplicity of λ_k. We proved on the previous page:

Fact: $\dim E_{\lambda_k} \leq m_k$.

We now use it to show **Theorem 7b: Diagonalisability Criteria**: An $n \times n$ matrix A is diagonalisable (i.e. A has n linearly independent eigenvectors) if and only if both the following conditions are true:

i the characteristic polynomial $\det(A - \lambda I)$ factors completely into linear factors, i.e. $m_1 + m_2 + \cdots = n$;

ii for each eigenvalue λ_k, we have $\dim E_{\lambda_k} = m_k$.

Proof: (sketch)

"only if" part: Given a set of n linearly independent eigenvectors, suppose b_k of them correspond to λ_k (so $b_1 + b_2 + \cdots = n$). (1)

These b_k vectors are part of a larger linearly independent set, so they must themselves be independent.

So we have b_k linearly independent vectors in E_{λ_k}, thus $b_k \leq \dim E_{\lambda_k}$. (2)

Also $\dim E_{\lambda_k} \leq m_k$ from the Fact. (3)

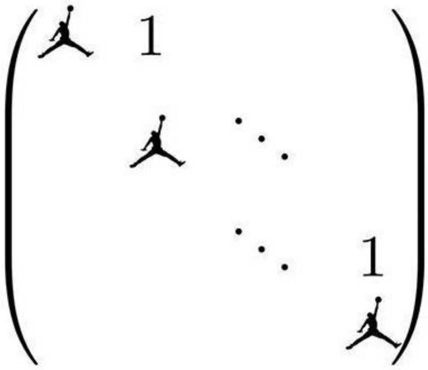
So $n \underset{(1)}{=} b_1 + b_2 + \cdots \underset{(2)}{\leq} \dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots \underset{(3)}{\leq} m_1 + m_2 + \cdots \leq n$,

so all our $\leq$ must be $=$.

Non-examinable: what to do when A is not diagonalisable:

We can still write A as PJP^{-1}, where J is "easy to understand and to compute with". Such a J is called a Jordan form.

For example, all non-diagonalisable 2×2 matrices are similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, where λ is the only eigenvalue (allowing complex eigenvalues).



A Jordan block of size-n with eigenvalue ,

(A Jordan form may contain more than one Jordan block, e.g. $\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$ contains two 2×2 Jordan blocks.)

Non-examinable: rectangular matrices (see §7.4 of textbook):

Any $m \times n$ matrix A can be decomposed as $A = QDP^{-1}$ where:

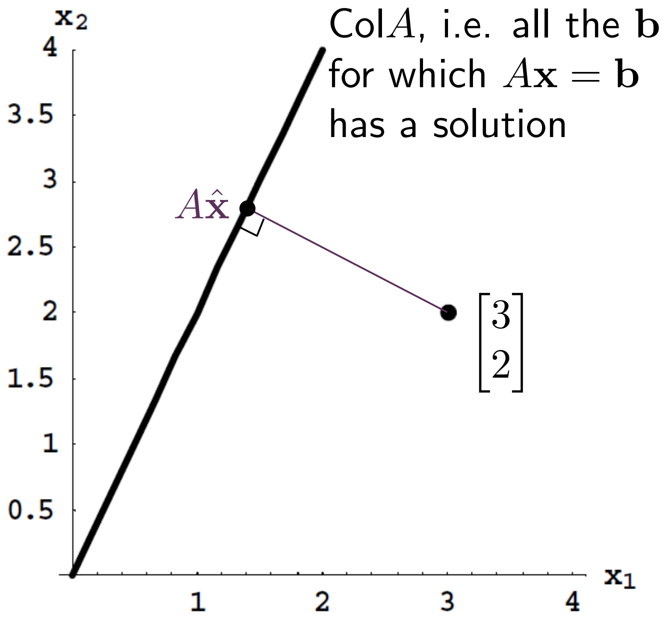
- P is an invertible $n \times n$ matrix with columns $\mathbf{p}_i$;
- Q is an invertible $m \times m$ matrix with columns $\mathbf{q}_i$;
- D is a "diagonal" $m \times n$ matrix with diagonal entries d_{ii}:

e.g. $\begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \\ 0 & 0 \end{bmatrix}$. So the maximal number of nonzero entries is the smaller of m and n.

Instead of $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$, this decomposition satisfies $A\mathbf{p}_i = d_{ii}\mathbf{q}_i$ for all $i \leq m, n$.

An important example is the singular value decomposition $A = U\Sigma V^T$. Each diagonal entry of Σ is a singular value of A, which is the squareroot of an eigenvalue of A^TA (a diagonalisable $n \times n$ matrix with non-negative eigenvalues). The singular values contain a lot of information about A, e.g. the largest singular value is the "maximal length scaling factor" of A. (Even for a square matrix, this is in general not true with the eigenvalues of A, so depending on the problem the SVD may be more useful than the diagonalisation of A.)

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. The linear system $A\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ does not have a solution, because $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is not in Col$A$ = Span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

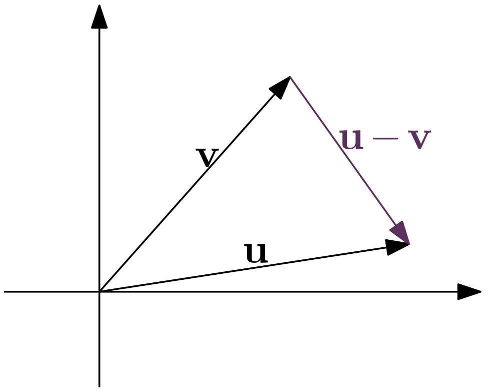


We wish to find a "closest approximate solution", i.e. a vector $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}}$ is the unique point in Col$A$ that is "closest" to $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. This is called a least-squares solution (p17).

To do this, we have to first define what we mean by "closest", i.e. define the idea of distance.

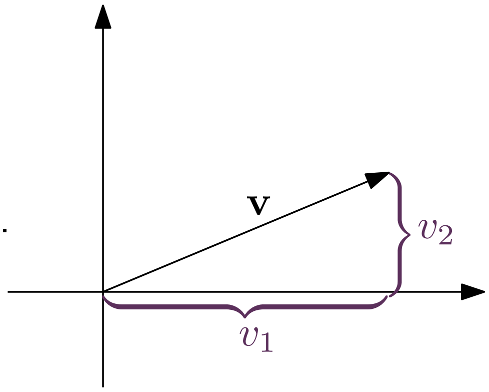
In $\mathbb{R}^2$, the distance between $\mathbf{u}$ and $\mathbf{v}$ is the length of their difference $\mathbf{u} - \mathbf{v}$.

So, to define distances in $\mathbb{R}^n$, it's enough to define the length of vectors.



In $\mathbb{R}^2$, the length of $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is $\sqrt{v_1^2 + v_2^2}$.

So we define the length of $\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is $\sqrt{v_1^2 + \cdots + v_n^2}$.



§6.1, p368: Length, Orthogonality, Best Approximation

It is more useful to define a more general idea:

Definition: The *dot product* of two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = u_1v_1 + \cdots + u_nv_n.$$

Warning: do not write $\mathbf{uv}$, which is an undefined matrix-vector product, or $\mathbf{u} \times \mathbf{v}$, which has a different meaning. Do not write $\mathbf{u}^2$, which is ambiguous.

Definition: The *length* or *norm* of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Definition: The *distance* between $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} - \mathbf{v}\|$.

$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = u_1v_1 + \cdots + u_nv_n.$
$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$
Distance between $\mathbf{u}$ and $\mathbf{v}$ is
$\|\mathbf{u} - \mathbf{v}\|.$

Example: $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}.$

$\mathbf{u} \cdot \mathbf{v} = 3 \cdot 8 + 0 \cdot 5 + -1 \cdot -6 = 24 + 0 + 6 = 30.$

The distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$\left\| \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix} \right\| = \sqrt{(-5)^2 + (-5)^2 + 5^2} = \sqrt{75} = 5\sqrt{3}.$$

Properties of the dot product:

Let **u**,**v** and **w** be vectors in $\mathbf{R}^n$, and let c be any scalar. Then

a. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ symmetry

b. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$ linearity in each input

c. $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$ separately

d. $\mathbf{u}\cdot\mathbf{u} \geq 0$, and $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$. positivity; and the only vector with length 0 is $\mathbf{0}$

Combining parts b and c, one can show

$$\boxed{(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p)\cdot\mathbf{w} = c_1(\mathbf{u}_1\cdot\mathbf{w}) + \cdots + c_p(\mathbf{u}_p\cdot\mathbf{w})}$$

So b and c together says that, for fixed $\mathbf{w}$, the function $\mathbf{x} \mapsto \mathbf{x}\cdot\mathbf{w}$ is linear - this is true because $\mathbf{x}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{x} = \mathbf{w}^T\mathbf{x}$ and matrix multiplication by $\mathbf{w}^T$ is linear.

From property c:

$$\|c\mathbf{v}\|^2 = (c\mathbf{v})\cdot(c\mathbf{v}) = c^2\mathbf{v}.\mathbf{v} = c^2\|\mathbf{v}\|^2,$$

so (squareroot both sides)

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|.$$

For many applications, we are interested in vectors of length 1.
Definition: A *unit vector* is a vector whose length is 1.

Given $\mathbf{v}$, to create a unit vector in the same direction as $\mathbf{v}$, we divide $\mathbf{v}$ by its length $\|\mathbf{v}\|$ (i.e. take $c = \frac{1}{\|\mathbf{v}\|}$ in the equation above). This process is called normalising.

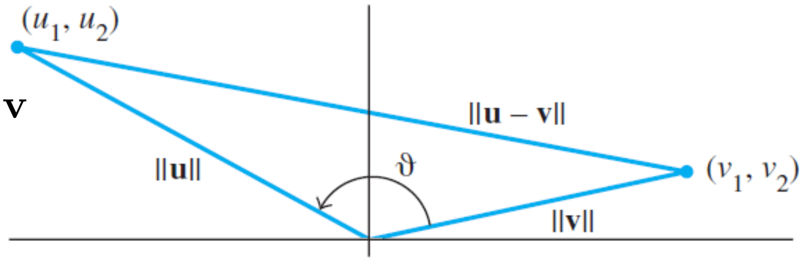
Example: Find a unit vector in the same direction as $\mathbf{v} = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$.

Answer: $\mathbf{v}\cdot\mathbf{v} = 8^2 + 5^2 + (-6)^2 = 125$.

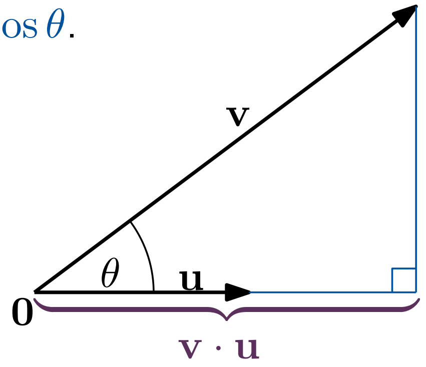
So a unit vector in the same direction as $\mathbf{v}$ is $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{125}}\begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$.

Visualising the dot product:
In $\mathbb{R}^2$, the cosine law says $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$.
We can "expand" the left hand side using dot products:

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2.\end{aligned}$$



Comparing with the cosine law, we see $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$.



In particular, if $\mathbf{u}$ is a unit vector, then $\mathbf{v}\cdot\mathbf{u} = \|\mathbf{v}\|\cos\theta$, as shown in the bottom picture.

Notice that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\theta = \frac{\pi}{2}$, i.e. when $\cos\theta = 0$. This is equivalent to $\mathbf{u}\cdot\mathbf{v} = 0$.

So, to generalise the idea of angles and perpendicularity to $\mathbb{R}^n$ for $n > 2$, we make the following definitions:

Definition: The *angle* between two vectors $\mathbf{u}$ and $\mathbf{v}$ is $\arccos\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$.

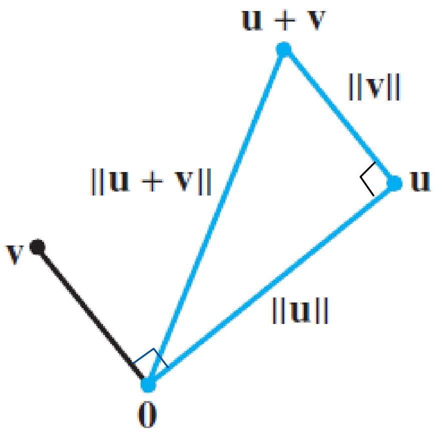
Definition: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u}\cdot\mathbf{v} = 0$.
We also say $\mathbf{u}$ *is orthogonal to* $\mathbf{v}$.

Another way to see that orthogonality generalises perpendicularity:

Theorem 2: Pythagorean Theorem: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Proof:

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2.\end{aligned}$$



So $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

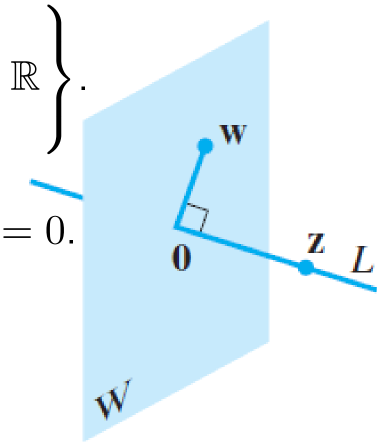
Instead of $\mathbf{v}$ being orthogonal to just a single vector $\mathbf{u}$, we can consider orthogonality to a set of vectors:

Definition: Let W be a subspace of $\mathbb{R}^n$ (or more generally a subset).
A vector $\mathbf{z}$ is *orthogonal to W* if it is orthogonal to every vector in W.
The *orthogonal complement* of W, written $W^\perp$, is the set of all vectors orthogonal to W. In other words, $\mathbf{z}$ is in $W^\perp$ means $\mathbf{z} \cdot \mathbf{w} = 0$ for all $\mathbf{w}$ in W.

Example: Let W be the x_1x_3-plane in $\mathbb{R}^3$, i.e. $W = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$.

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal to W, because $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = 0 \cdot a + 1 \cdot 0 + 0 \cdot b = 0.$

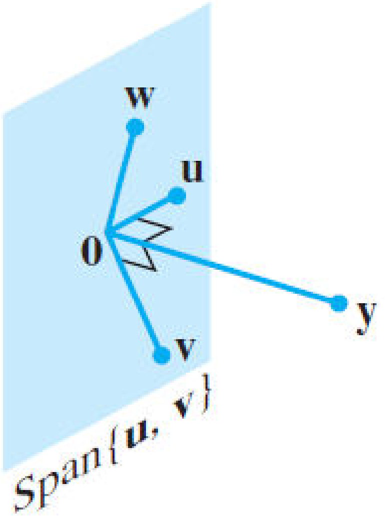
We show on p13 that $W^\perp$ is Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.



Key properties of $W^\perp$, for a subspace W of $\mathbb{R}^n$:

1. If $\mathbf{x}$ is in both W and $W^\perp$, then $\mathbf{x} = \mathbf{0}$ (ex. sheet #21 q2b).
2. If $W = \text{Span}\,\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, then $\mathbf{y}$ is in $W^\perp$ if and only if $\mathbf{y}$ is orthogonal to each $\mathbf{v}_i$ (same idea as ex. sheet q2a, see diagram).
3. $W^\perp$ is a subspace of $\mathbb{R}^n$ (checking the axioms directly is not hard, alternative proof p13).
4. $\dim W + \dim W^\perp = n$ (follows from alternative proof of 3, see p13).
5. If $W^\perp = U$, then $U^\perp = W$.
6. For a vector $\mathbf{y}$ in $\mathbb{R}^n$, the closest point in W to $\mathbf{y}$ is the unique point $\hat{\mathbf{y}}$ such that $\mathbf{y} - \hat{\mathbf{y}}$ is in $W^\perp$ (see p15-17).

(1 and 3 are true for any set W, even when W is not a subspace.)



Dot product and matrix multiplication:

Remember (week 2 p16, §1.4) the row-column method of matrix-vector multiplication:

Example: $$\begin{bmatrix} 4 & 3 \\ 2 & 6 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4(-2)+3(2) \\ 2(-2)+6(2) \\ 14(-2)+10(2) \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}.$$

This last entry is $\begin{bmatrix} 14 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

In general,

$$\begin{bmatrix} -- & \mathbf{r}_1 & -- \\ -- & \vdots & -- \\ -- & \mathbf{r}_m & -- \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix}. \qquad (*)$$

Now consider $\mathbf{x} \in \text{Nul}A$

By $(*)$, this is equivalent to $\mathbf{r}_i \cdot \mathbf{x} = 0$ for all i.

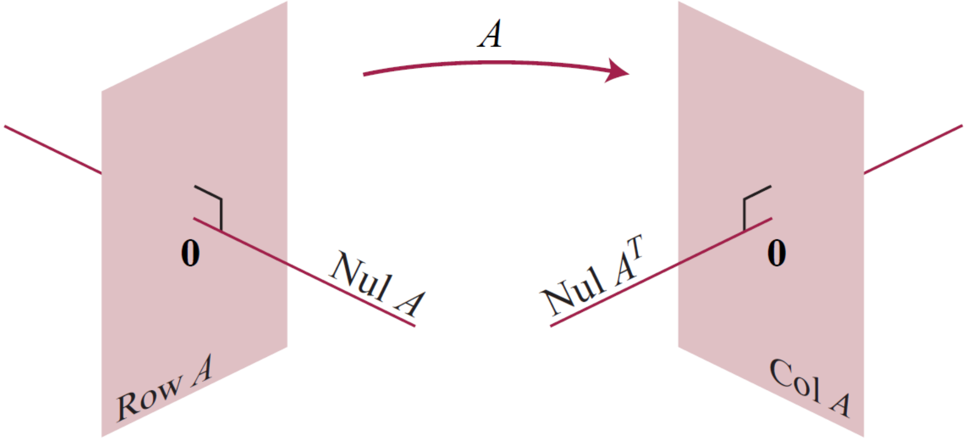
By property 2 on the previous page, this is equivalent to $\mathbf{r} \cdot \mathbf{x} = 0$ for all $\mathbf{r} \in \text{Span}\{\mathbf{r}_1, \dots, \mathbf{r}_m\} = \text{Row}A$.

By definition of orthogonal complement, this is equivalent to $\mathbf{x} \in (\text{Row}A)^\perp$

So $\mathbf{x} \in \text{Nul}A$ if and only if $\mathbf{x} \in (\text{Row}A)^\perp$. We have proved

Theorem 3: Orthogonality of Subspaces associated to Matrices: For a matrix A, $(\text{Row}A)^\perp = \text{Nul}A$ and $(\text{Col}A)^\perp = \text{Nul}A^T$.

The second assertion comes from applying the first statement to A^T instead of A, remembering that $\text{Row}A^T = \text{Col}A$.



Theorem 3: Orthogonality of Subspaces associated to Matrices: For a matrix A, $(\text{Row}A)^\perp = \text{Nul}A$ and $(\text{Col}A)^\perp = \text{Nul}A^T$.

We can use this theorem to prove that $W^\perp$ is a subspace: given a subspace W of $\mathbb{R}^n$, let A be the matrix whose rows is a basis for W, so $\text{Row}A = W$. Then $W^\perp = \text{Nul}A$, and null spaces are subspaces, so $W^\perp$ is a subspace.
Futhermore, the Rank Nullity Theorem says $\dim \text{Row}A + \dim \text{Nul}A = n$, so $\dim W + \dim W^\perp = n$.

The argument above also gives us a way to compute orthogonal complements:

Example: Let $W = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$. A basis for W is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $W = \text{Row}A$ so $W^\perp = \text{Nul}A$, i.e. the solutions to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

So $W^\perp = \left\{ s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$.

Notice $\dim W + \dim W^\perp = 2 + 1 = 3$.

On p11, we related the matrix-vector product to the dot product:

$$\begin{bmatrix} -- & \mathbf{r}_1 & -- \\ -- & \vdots & -- \\ -- & \mathbf{r}_m & -- \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix}.$$

Because each column of a matrix-matrix product is a matrix-vector product,

$$AB = A \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_p \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A\mathbf{b}_1 & \dots & A\mathbf{b}_p \\ | & | & | \end{bmatrix},$$

we can also express matrix-matrix products in terms of the dot product:
the (i, j)-entry of the product AB is (ith row of A) $\cdot$ (jth column of B)

$$\begin{bmatrix} -- & \mathbf{r}_1 & -- \\ -- & \vdots & -- \\ -- & \mathbf{r}_m & -- \end{bmatrix} \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_p \\ | & | & | \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b}_1 & \dots & \mathbf{r}_1 \cdot \mathbf{b}_p \\ \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{b}_1 & \dots & \mathbf{r}_m \cdot \mathbf{b}_p \end{bmatrix}.$$

Closest point to a subspace:

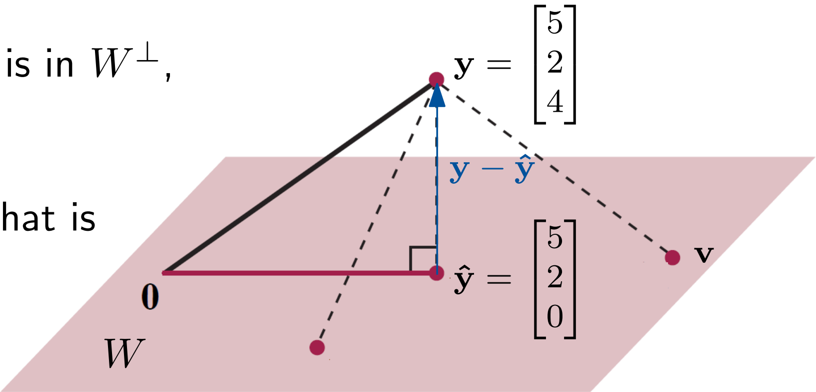
Theorem 9: Best Approximation Thoerem: Let W be a subspace of $\mathbb{R}^n$, and $\mathbf{y}$ a vector in $\mathbb{R}^n$. Then there is a unique point $\hat{\mathbf{y}}$ in W such that $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$, and this $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$ in the sense that $\|\mathbf{y}-\hat{\mathbf{y}}\| < \|\mathbf{y}-\mathbf{v}\|$ for all $\mathbf{v}$ in W with $\mathbf{v} \neq \hat{\mathbf{y}}$.

Example: Let $W = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$, so $W^\perp = \text{Span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Let $\mathbf{y} = \begin{bmatrix}5\\2\\4\end{bmatrix}$.

Take $\hat{\mathbf{y}} = \begin{bmatrix}5\\2\\0\end{bmatrix}$, then $\mathbf{y}-\hat{\mathbf{y}} = \begin{bmatrix}0\\0\\4\end{bmatrix}$ is in $W^\perp$,

so $\hat{\mathbf{y}} = \begin{bmatrix}5\\2\\0\end{bmatrix}$ is unique point in W that is

closest to $\begin{bmatrix}5\\2\\4\end{bmatrix}$.

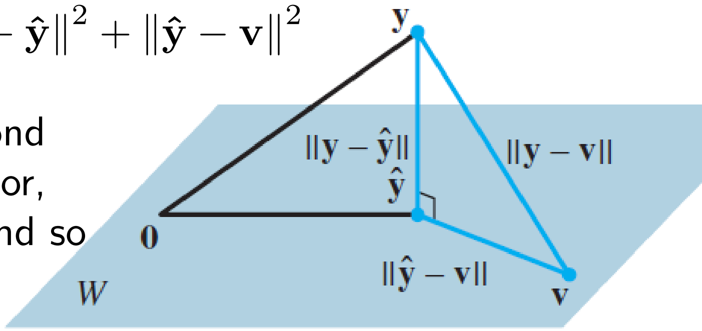


Theorem 9: Best Approximation Thoerem: Let W be a subspace of $\mathbb{R}^n$, and $\mathbf{y}$ a vector in $\mathbb{R}^n$. Then there is a unique point $\hat{\mathbf{y}}$ in W such that $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$, and this $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$ in the sense that $\|\mathbf{y}-\hat{\mathbf{y}}\| < \|\mathbf{y}-\mathbf{v}\|$ for all $\mathbf{v}$ in W with $\mathbf{v} \neq \hat{\mathbf{y}}$.

Partial Proof: We show here that, if $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$, then $\hat{\mathbf{y}}$ is the unique closest point (i.e. it satisfies the inequality). We will not show here that there is always a $\hat{\mathbf{y}}$ such that $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$. (See §6.3 on orthogonal projections, in Week 12 notes.)
We are assuming that $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$. (vertical blue edge)
$\hat{\mathbf{y}}-\mathbf{v}$ is a difference of vectors in W, so it is in W. (horizontal blue edge)
So $\mathbf{y}-\hat{\mathbf{y}}$ and $\hat{\mathbf{y}}-\mathbf{v}$ are orthogonal. Apply the Pythagorean Theorem (blue triangle):

$$\|(\mathbf{y}-\hat{\mathbf{y}})+(\hat{\mathbf{y}}-\mathbf{v})\|^2 = \|\mathbf{y}-\hat{\mathbf{y}}\|^2 + \|\hat{\mathbf{y}}-\mathbf{v}\|^2$$

The left hand side is $\|\mathbf{y}-\mathbf{v}\|^2$.
The right hand side: if $\mathbf{v} \neq \hat{\mathbf{y}}$, then the second term is the squared-length of a nonzero vector, so it is positive. So $\|\mathbf{y}-\mathbf{v}\|^2 > \|\mathbf{y}-\hat{\mathbf{y}}\|^2$ and so $\|\mathbf{y}-\mathbf{v}\| > \|\mathbf{y}-\hat{\mathbf{y}}\|$.

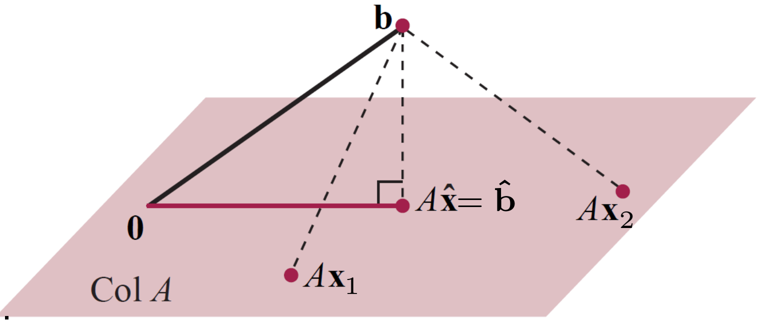


§6.5-6.6: Least Squares, Application to Regression

Remember our motivation: we have an inconsistent equation $A\mathbf{x} = \mathbf{b}$, and we want to find a "closest approximate solution" $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}}$ is the point in Col$A$ that is closest to $\mathbf{b}$.

Definition: If A is an $m \times n$ matrix and $\mathbf{b}$ is in $\mathbb{R}^m$, then a *least-squares solution* of $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ such that $\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Equivalently: we want to find a vector $\hat{\mathbf{b}}$ in Col$A$ that is closest to $\mathbf{b}$, and then solve $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$.



Because of the Best Approximation Theorem (p15-16): $\mathbf{b} - \hat{\mathbf{b}}$ is in $(\text{Col}A)^\perp$.
Because of Orthogonality of Subspaces associated to Matrices (p11-13): $(\text{Col}A)^\perp = \text{Nul}A^T$.
So we need $\hat{\mathbf{b}}$ so that $\mathbf{b} - \hat{\mathbf{b}}$ is in NulA^T.

The least-squares solutions to $A\mathbf{x} = \mathbf{b}$ are the solutions to $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is the unique vector such that $\mathbf{b} - \hat{\mathbf{b}}$ is in NulA^T.
Equivalently,

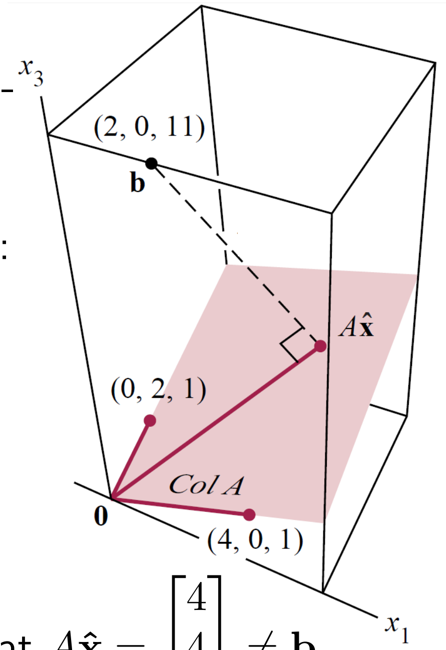
$$\begin{aligned} A^T(\mathbf{b} - \hat{\mathbf{b}}) &= \mathbf{0} \\ A^T\mathbf{b} - A^T\hat{\mathbf{b}} &= \mathbf{0} \\ A^T\mathbf{b} &= A^T\hat{\mathbf{b}} \\ A^T\mathbf{b} &= A^TA\hat{\mathbf{x}} \end{aligned}$$

So we have proved:
Theorem 13: Least-Squares Theorem: The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ is the set of solutions of the normal equations $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$.

Because of the existence part of the Best Approximation Theorem (that we will prove later), $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ is always consistent.
Warning: The terminology is confusing: a least-squares solution $\hat{\mathbf{x}}$, satisfying $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$, is in general not a solution to $A\mathbf{x} = \mathbf{b}$. That is, usually $A\hat{\mathbf{x}} \neq \mathbf{b}$.

Theorem 13: Least-Squares Theorem: The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ is the set of solutions of the normal equations $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$.

Example: Let $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$. Find a least-squares solution of the inconsistent equation $A\mathbf{x} = \mathbf{b}$.



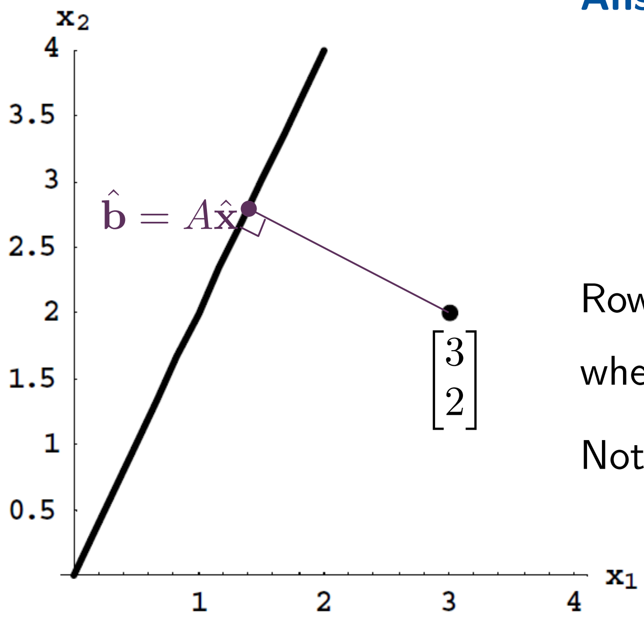
Answer: We solve the normal equations $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$:

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

By row-reducing $\left[\begin{array}{cc|c} 17 & 1 & 19 \\ 1 & 5 & 11 \end{array}\right]$, we find $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Note that $A\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} \neq \mathbf{b}$.

Example: (from p1) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Find the set of least-squares solutions of the inconsistent equation $A\mathbf{x} = \mathbf{b}$.



Answer: We solve $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

Row-reducing $\left[\begin{array}{cc|c} 5 & 10 & 7 \\ 10 & 20 & 14 \end{array}\right]$ gives $\hat{\mathbf{x}} = \begin{bmatrix} 7/5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ where s can take any value.

Note that $A\hat{\mathbf{x}} = A\left(\begin{bmatrix} 7/5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7/5 \\ 14/5 \end{bmatrix}$, independent of s: $A\hat{\mathbf{x}}$ is the closest point in Col$A$ to $\mathbf{b}$, which by the Best Approximation Theorem is unique.

Observations from the previous examples:

- A^TA is a square matrix and is symmetric. (Exercise: prove it!)
- The normal equations sometimes have a unique solution and sometimes have infinitely many solutions, but $A\hat{\mathbf{x}}$ is unique.

When is the least-squares solution unique?

Theorem 14: Uniqueness of Least-Squares Solutions: The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution if and only if the columns of A are linearly independent.

Consequences:

- The number of least-squares solutions to $A\mathbf{x} = \mathbf{b}$ does not depend on $\mathbf{b}$, only on A.
- Because A^TA is a square matrix, if the least-squares solution is unique, then it is $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$. This formula is useful theoretically (e.g. for deriving general expressions for regression coefficients).

Theorem 14: Uniqueness of Least-Squares Solutions: The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution if and only if the columns of A are linearly independent.

Proof 1: The least-squares solutions are the solutions to the normal equations $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$. So

- "unique least-squares solution" is equivalent to Nul$(A^TA) = \{\mathbf{0}\}$.
- "columns of A are linearly independent" is equivalent to Nul$A = \{\mathbf{0}\}$.

So the theorem will follow if we prove the stronger fact Nul$(A^TA) =$ Nul$A$; in other words, $A^TA\mathbf{x} = \mathbf{0}$ if and only if $A\mathbf{x} = \mathbf{0}$.

- If $A\mathbf{x} = \mathbf{0}$, then $A^TA\mathbf{x} = A^T(A\mathbf{x}) = A^T\mathbf{0} = \mathbf{0}$.
- If $A^TA\mathbf{x} = \mathbf{0}$, then $\|A\mathbf{x}\|^2 = (A\mathbf{x}) \cdot (A\mathbf{x}) = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x}$ $= \mathbf{x}^T(A^TA\mathbf{x}) = \mathbf{x}^T\mathbf{0} = 0$. So the length of $A\mathbf{x}$ is 0, which means it must be the zero vector.

Proof 2: The least-squares solutions are the solutions to $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is unique (the closest point in Col$A$ to $\mathbf{b}$). The equation $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ has a unique solution precisely when the columns of A are linearly independent.

Application: least-squares line

Suppose we have a model that relates two quantities x and y linearly, i.e. we expect $y = \beta_0 + \beta_1 x$, for some unknown numbers β_0, β_1.
To estimate β_0 and β_1, we do an experiment, whose results are $(x_1, y_1), \dots, (x_n, y_n)$.
Now we wish to solve (for β_0, β_1):

$$\begin{aligned} \beta_0 + \beta_1 x_1 &= y_1 \\ \beta_0 + \beta_1 x_2 &= y_2 \\ \vdots \quad & \quad \vdots \\ \beta_0 + \beta_1 x_n &= y_n \end{aligned} \qquad \text{i.e.} \quad \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

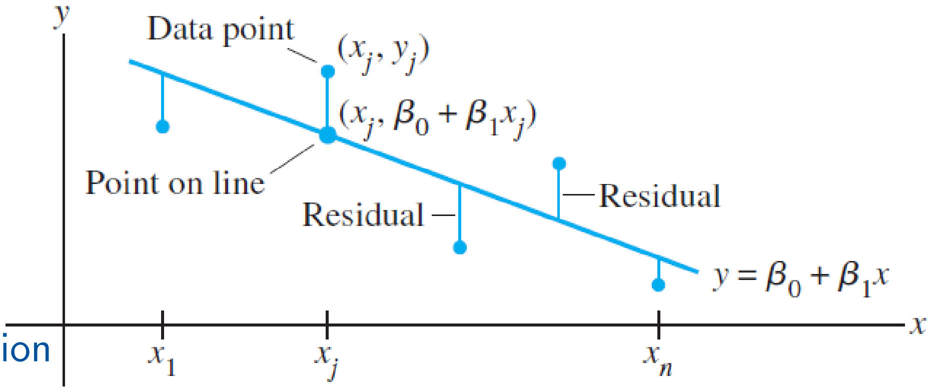
$A\mathbf{x} = \mathbf{b}$ with different notation $\longrightarrow$

$$\underset{\substack{/\!/ \\ \text{design} \\ \text{matrix}}}{X} \quad \underset{\substack{\| \\ \text{parameter} \\ \text{vector}}}{\boldsymbol{\beta}} \quad = \quad \underset{\substack{\backslash\!\backslash \\ \text{observation} \\ \text{vector}}}{\mathbf{y}}$$

We wish to solve (for β_0, β_1):

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\underset{\substack{/\!/ \\ \text{design} \\ \text{matrix}}}{X} \quad \underset{\substack{\| \\ \text{parameter} \\ \text{vector}}}{\boldsymbol{\beta}} \quad = \quad \underset{\substack{\backslash\!\backslash \\ \text{observation} \\ \text{vector}}}{\mathbf{y}}$$

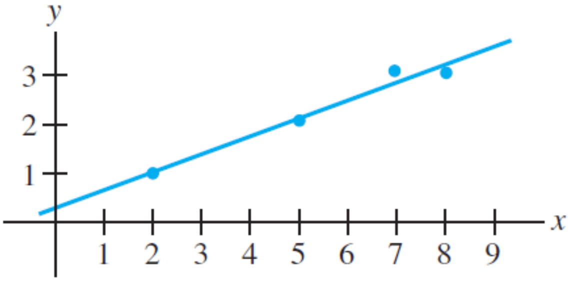


Because experiments are rarely perfect, our data points (x_i, y_i) probably don't all lie exactly on any line, i.e. this system probably doesn't have a solution. So we ask for a least-squares solution.

A least-squares solution minimises $\|\mathbf{y} - X\boldsymbol{\beta}\|$, which is equivalent to minimising $\|\mathbf{y} - X\boldsymbol{\beta}\|^2 = (y_1 - (\beta_0 + \beta_1 x_1))^2 + \cdots + (y_n - (\beta_0 + \beta_1 x_n))^2$, the sums of the squares of the residuals. (The residuals are the vertical distances between each data point and the line, as in the diagram above).

Example: Find the equation $y = \hat{\beta}_0 + \hat{\beta}_1 x$ for the least-squares line for the following data points:

x_i	2	5	7	8
y_i	1	2	3	3



Answer: The model equation $X\boldsymbol{\beta} = \mathbf{y}$ is

$$\begin{aligned} \hat{\beta}_0 + \hat{\beta}_1 2 &= 1 \\ \hat{\beta}_0 + \hat{\beta}_1 5 &= 2 \\ \hat{\beta}_0 + \hat{\beta}_1 7 &= 3 \\ \hat{\beta}_0 + \hat{\beta}_1 8 &= 3 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \boldsymbol{\beta} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

The normal equations $X^T X \hat{\boldsymbol{\beta}} = X^T \mathbf{y}$ are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \hat{\boldsymbol{\beta}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix} \hat{\boldsymbol{\beta}} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}.$$

Row-reducing gives $\hat{\boldsymbol{\beta}} = \begin{bmatrix} 2/7 \\ 5/14 \end{bmatrix}$, so the equation of the least-squares line is $y = 2/7 + 5/14x$.

Application: least-squares fitting of other curves

Suppose we model y as a more complicated function of x, i.e. $y = \beta_0 f_0(x) + \beta_1 f_1(x) + \cdots + \beta_k f_k(x)$, where $f_0, \ldots, f_k$ are known functions, and $\beta_0, \ldots, \beta_k$ are unknown parameters that we will estimate from experimental data. Such a model is still called a “linear model”, because it is linear in the parameters $\beta_0, \ldots, \beta_k$.

Example: Estimate the parameters $\beta_1, \beta_2, \beta_3$ in the model $y = \beta_1 x + \beta_2 x^2 + \beta_3 x^3$, given the data

x_i	2	3	4	6	7
y_i	1.6	2.0	2.5	3.1	3.4

Answer: The model equations are $\beta_1 2 + \beta_2 2^2 + \beta_3 2^3 = 1.6$
$\beta_1 3 + \beta_2 3^2 + \beta_3 3^3 = 2.0$, and so on.

In matrix form: $\begin{bmatrix} 2 & 4 & 8 \\ 3 & 9 & 27 \\ 4 & 16 & 64 \\ 6 & 36 & 216 \\ 7 & 49 & 343 \end{bmatrix} \boldsymbol{\beta} = \begin{bmatrix} 1.6 \\ 2.0 \\ 2.5 \\ 3.1 \\ 3.4 \end{bmatrix}$. Then we solve the normal equations etc...

So in general, to estimate the parameters $\beta_0, \ldots, \beta_k$ in a linear model $y = \beta_0 f_0(x) + \beta_1 f_1(x) + \cdots + \beta_k f_k(x)$, we find the least-squares solution to

$$\begin{aligned} \beta_0 f_0(x_1) + \beta_1 f_1(x_1) + \cdots + \beta_k f_k(x_1) &= y_1 \\ \beta_0 f_0(x_2) + \beta_1 f_1(x_2) + \cdots + \beta_k f_k(x_2) &= y_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \end{aligned}$$

more general design matrix

parameter vector with more rows

i.e. $$\begin{bmatrix} f_0(x_1) & f_1(x_1) & \ldots & f_k(x_1) \\ f_0(x_2) & f_1(x_2) & \ldots & f_k(x_2) \\ \vdots & \vdots & & \vdots \\ f_0(x_n) & f_1(x_n) & \ldots & f_k(x_n) \end{bmatrix} \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

same observation vector

(Least-squares lines correspond to the case $f_0(x) = 1, f_1(x) = x$.)

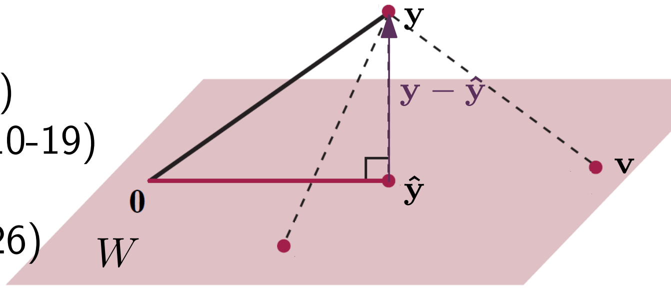
Least-squares techniques can also be used to fit a surface to experimental data, for linear models with more than one input variable (e.g. $y = \beta_0 + \beta_1 x + \beta_2 xw$, for input variables x and w) - this is called multiple regression.

Remember from last week:

Theorem 9: Best Approximation Thoerem: Let W be a subspace of $\mathbb{R}^n$, and $\mathbf{y}$ a vector in $\mathbb{R}^n$. Then there is a unique point $\hat{\mathbf{y}}$ in W such that $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$, and this $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$ in the sense that $\|\mathbf{y}-\hat{\mathbf{y}}\| < \|\mathbf{y}-\mathbf{v}\|$ for all $\mathbf{v}$ in W with $\mathbf{v} \neq \hat{\mathbf{y}}$.

We proved last week that, if $\hat{\mathbf{y}}$ is in W, and $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$, then $\hat{\mathbf{y}}$ is the unique closest point in W to $\mathbf{y}$. But we did not prove that a $\hat{\mathbf{y}}$ satisfying these conditions always exist.

We will show that the function $\mathbf{y} \mapsto \hat{\mathbf{y}}$ is a linear transformation, called the orthogonal projection onto W, and calculate it using an orthogonal basis for W.



Our remaining goals:
- §6.2 The properties of orthogonal bases (p2-9)
- §6.3 Calculating the orthogonal projection (p10-19)
- §6.4 Constructing orthogonal bases (p20-22)
- §6.2 Matrices with orthogonal columns (p23-26)

§6.2: Orthogonal Bases

Definition:
- A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, i.e. if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.
- A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an *orthonormal set* if it is an orthogonal set and each $\mathbf{u}_i$ is a unit vector.

Example: $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\5\\-3\end{bmatrix}, \begin{bmatrix}3\\0\\-1\end{bmatrix} \right\}$ is an orthogonal set, because

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}-1\\5\\-3\end{bmatrix} = -1+10-9 = 0, \quad \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}3\\0\\-1\end{bmatrix} = 3+0-3 = 0, \quad \begin{bmatrix}-1\\5\\-3\end{bmatrix} \cdot \begin{bmatrix}3\\0\\-1\end{bmatrix} = -3+0+3 = 0.$$

To obtain an orthonormal set, we normalise each vector in the set:

$$\left\{ \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}, \begin{bmatrix}-1/\sqrt{35}\\5/\sqrt{35}\\-3/\sqrt{35}\end{bmatrix}, \begin{bmatrix}3/\sqrt{10}\\0\\-1/\sqrt{10}\end{bmatrix} \right\} \text{ is an orthonormal set.}$$

EXAMPLE: In $\mathbb{R}^6$, the set $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_6, \mathbf{0}\}$ is an orthogonal set, because $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for all $i \neq j$, and $\mathbf{e}_i \cdot \mathbf{0} = 0$.

So an orthogonal set **may contain the zero vector**. But when it doesn't:

THEOREM 4 If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set of nonzero vectors, then it is linearly independent.

PROOF We need to show that ____________________ is the only solution to

$$(*)$$

Take the dot product of both sides with $\mathbf{v}_1$:

$$c_1\underline{\qquad} + c_2\underline{\qquad} + \cdots + c_p\underline{\qquad} = \underline{\qquad}$$

If $j \neq 1$, then $\mathbf{v}_j \cdot \mathbf{v}_1 = \underline{\qquad}$, so

$$c_1\underline{\qquad} + c_2\underline{\qquad} + \cdots + c_p\underline{\qquad} = \underline{\qquad}$$

Because _______________ , we have $\mathbf{v}_1 \cdot \mathbf{v}_1$ is nonzero, so it must be that $c_1 = 0$.

By taking the dot product of $(*)$ with each of the other $\mathbf{v}_i$s and using this argument, each c_i must be 0.

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set of nonzero vectors, as before, and use the same idea with

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p. \qquad (*)$$

Take the dot product of both sides with $\mathbf{v}_1$:

$$\mathbf{y} \cdot \mathbf{v}_1 = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) \cdot \mathbf{v}_1$$

$$\mathbf{y} \cdot \mathbf{v}_1 = c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + c_2\mathbf{v}_2 \cdot \mathbf{v}_1 + \cdots + c_p\mathbf{v}_p \cdot \mathbf{v}_1.$$

Using that $\mathbf{v}_j \cdot \mathbf{v}_1 = 0$ whenever $j \neq 1$:

$$\mathbf{y} \cdot \mathbf{v}_1 = c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + c_2 0 \qquad + \cdots + c_p 0$$

Since $\mathbf{v}_1$ is nonzero, $\mathbf{v}_1 \cdot \mathbf{v}_1$ is nonzero, we can divide both sides by $\mathbf{v}_1 \cdot \mathbf{v}_1$:

$$\frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} = c_1$$

By taking the dot product of $(*)$ with each of the other $\mathbf{v}_j$s and using this argument, we obtain $c_j = \dfrac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}$.

So finding the weights in a linear combination of orthogonal vectors is much easier than for arbitrary vectors (see the example on p6).

Definition:

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an *orthogonal basis* for a subspace W if it is both an orthogonal set and a basis for W.
- A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an *orthonormal basis* for a subspace W if it is both an orthonormal set and a basis for W.

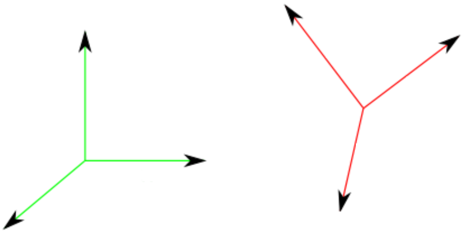
Example: The standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.
By the previous theorem, if S is a orthogonal set of nonzero vectors, then S is an orthogonal basis for the subspace Span(S).

As proved on the previous page, a big advantage of orthogonal bases is:
Theorem 5: Weights for Orthogonal Bases: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W, then, for each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}.$$



In particular, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis, then the weights are $c_j = \mathbf{y} \cdot \mathbf{u}_j$.

Example: Express $\begin{bmatrix}10\\9\\0\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\5\\-3\end{bmatrix}, \begin{bmatrix}3\\0\\-1\end{bmatrix}$.

Slow Answer: (works for any basis)

$$\left[\begin{array}{ccc|c}1&-1&3&10\\2&5&0&9\\3&-3&-1&0\end{array}\right]$$

$$\begin{array}{l}R_2-2R_1\\R_3-3R_1\end{array}\left[\begin{array}{ccc|c}1&-1&3&10\\0&7&-6&-11\\0&0&-10&-30\end{array}\right]$$

$$R_3/-10\quad\left[\begin{array}{ccc|c}1&-1&3&10\\0&7&-6&-11\\0&0&1&3\end{array}\right]$$

$$\begin{array}{l}R_1-3R_3\\R_2+6R_3\end{array}\left[\begin{array}{ccc|c}1&-1&0&1\\0&7&0&7\\0&0&1&3\end{array}\right]$$

$$R_2/7\quad\left[\begin{array}{ccc|c}1&-1&0&1\\0&1&0&1\\0&0&1&3\end{array}\right]$$

$$R_1+R_2\quad\left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&1\\0&0&1&3\end{array}\right]$$

So $\begin{bmatrix}10\\9\\0\end{bmatrix}=2\begin{bmatrix}1\\2\\3\end{bmatrix}+1\begin{bmatrix}-1\\5\\-3\end{bmatrix}+3\begin{bmatrix}3\\0\\-1\end{bmatrix}$.

Example: Express $\begin{bmatrix}10\\9\\0\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\5\\-3\end{bmatrix}, \begin{bmatrix}3\\0\\-1\end{bmatrix}$.

Fast Answer: (for an orthogonal basis) We showed on p2 that these three vectors form an orthogonal set. Since the vectors are nonzero, the set is linearly independent, and is therefore a basis for $\mathbb{R}^3$. Now use the formula $c_j=\dfrac{\mathbf{y}\cdot\mathbf{v}_j}{\mathbf{v}_j\cdot\mathbf{v}_j}$:

$$c_1=\frac{\begin{bmatrix}10\\9\\0\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}}{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}}=\frac{10+18+0}{1^2+2^2+3^2}=2,\qquad c_2=\frac{\begin{bmatrix}10\\9\\0\end{bmatrix}\cdot\begin{bmatrix}-1\\5\\3\end{bmatrix}}{\begin{bmatrix}-1\\5\\3\end{bmatrix}\cdot\begin{bmatrix}-1\\5\\3\end{bmatrix}}=\frac{-10+45+0}{(-1)^2+5^2+(-3)^2}=1,$$

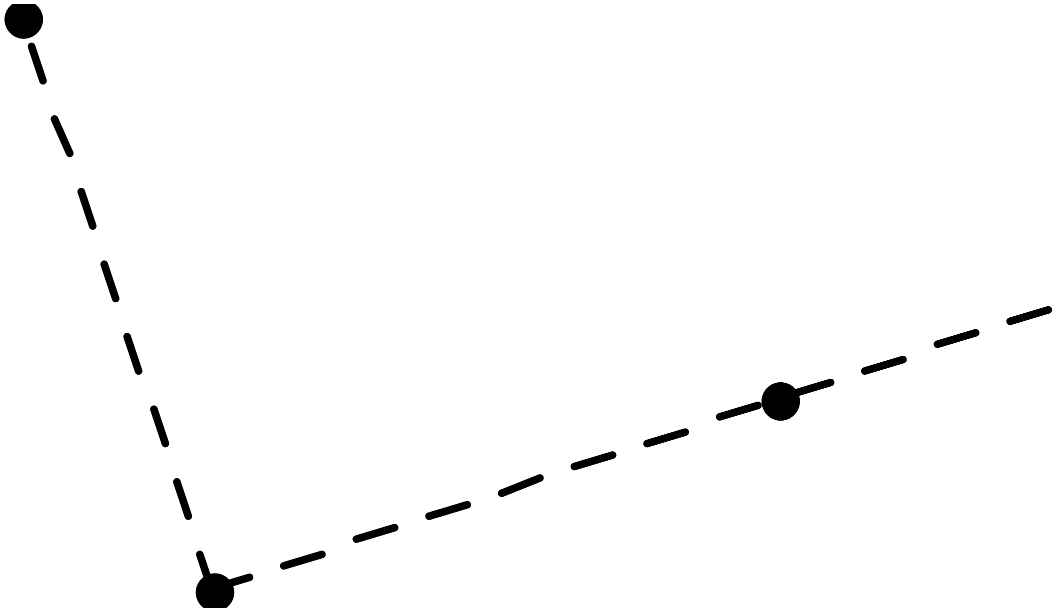
$$c_3=\frac{\begin{bmatrix}10\\9\\0\end{bmatrix}\cdot\begin{bmatrix}3\\0\\-1\end{bmatrix}}{\begin{bmatrix}3\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}3\\0\\-1\end{bmatrix}}=\frac{30+0+0}{3^2+0+(-1)^2}=3,$$

So $\begin{bmatrix}10\\9\\0\end{bmatrix}=2\begin{bmatrix}1\\2\\3\end{bmatrix}+1\begin{bmatrix}-1\\5\\-3\end{bmatrix}+3\begin{bmatrix}3\\0\\-1\end{bmatrix}$.

From the Weights for Orthogonal Bases Theorem: if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W in $\mathbb{R}^n$, then each $\mathbf{y}$ in W is

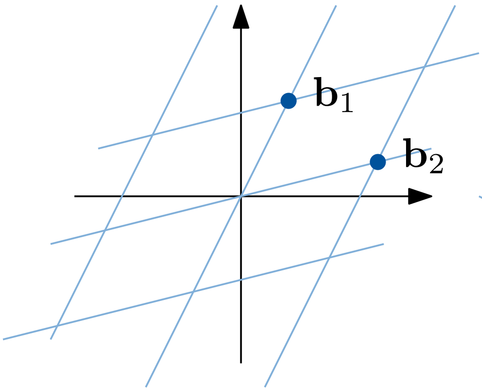
$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p.$$

A geometric interpretation of this decomposition in $\mathbb{R}^2$:

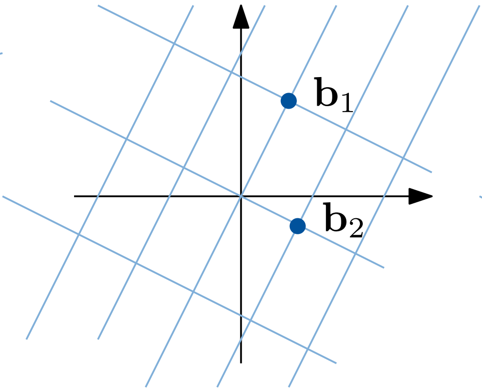


A geometric comparison of bases with different properties:

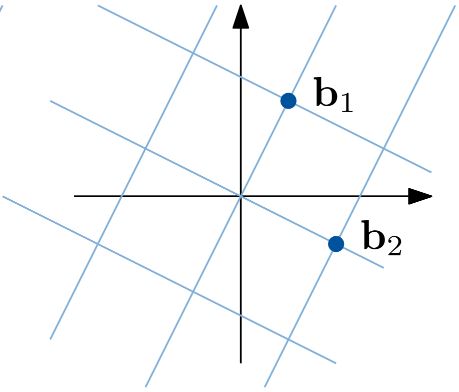
arbitrary basis -
parallelogram grid



orthogonal basis -
rectangular grid



orthonormal basis -
square grid



§6.3: Orthogonal Projections

Recall that our motivation for defining orthogonal bases is to calculate the unique closest point in a subspace.

Let W be a subspace, and $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be an orthogonal basis for W. Let $\mathbf{y}$ be any vector, and $\hat{\mathbf{y}}$ be the vector in W that is closest to $\mathbf{y}$.

Since $\hat{\mathbf{y}}$ is in W, and $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for W, we must have $\hat{\mathbf{y}} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$ for some weights $c_1, \dots, c_p$.

We know from the Best Approximation Theorem that $\mathbf{y} - \hat{\mathbf{y}}$ is in $W^\perp$. By the properties of $W^\perp$, it's enough to show that $(\mathbf{y} - \hat{\mathbf{y}}) \cdot \mathbf{v}_i = 0$ for each i. We can use this condition to solve for c_i:

$$\begin{aligned}(\mathbf{y} - \hat{\mathbf{y}}) \cdot \mathbf{v}_1 &= 0\\ (\mathbf{y} - c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - \dots - c_p\mathbf{v}_p) \cdot \mathbf{v}_1 &= 0\\ \mathbf{y}\cdot\mathbf{v}_1 - c_1\mathbf{v}_1\cdot\mathbf{v}_1 - c_2\mathbf{v}_2\cdot\mathbf{v}_1 - \dots - c_p\mathbf{v}_p\cdot\mathbf{v}_1 &= 0\\ \mathbf{y}\cdot\mathbf{v}_1 - c_1\mathbf{v}_1\cdot\mathbf{v}_1 - c_2\,0 \qquad - \dots - c_p0 \qquad &= 0\end{aligned}$$

so $c_1 = \dfrac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}$. Similarly, $c_i = \dfrac{\mathbf{y}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}$.

So we have proved (using the Best Approximation Theorem to deduce the uniqueness of $\hat{\mathbf{y}}$):

Theorem 8: Orthogonal Decomposition Theorem: Let W be a subspace of $\mathbb{R}^n$. Then every $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ with $\hat{\mathbf{y}}$ in W and $\mathbf{z}$ in $W^\perp$. In fact, if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is any orthogonal basis for W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \dots + \frac{\mathbf{y}\cdot\mathbf{v}_p}{\mathbf{v}_p\cdot\mathbf{v}_p}\mathbf{v}_p \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

(Technically, to complete the proof, we need to show that every subspace has an orthogonal basis - see p20-22 for an explicit construction.)

Definition: The *orthogonal projection onto W* is the function $\text{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ such that $\text{proj}_W(\mathbf{y})$ is the unique $\hat{\mathbf{y}}$ in the above theorem. The image vector $\text{proj}_W(\mathbf{y})$ is the *orthogonal projection of $\mathbf{y}$ onto W*.

The uniqueness part of the theorem means that the $\text{proj}_W(\mathbf{y})$ does not depend on the orthogonal basis used to calculate it.

Example: Let $\mathbf{y} = \begin{bmatrix} 6 \\ 7 \\ -2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$ and let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Find the point in W closest to $\mathbf{y}$ and the distance from $\mathbf{y}$ to W.

Answer: $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W. So the point in W closest to $\mathbf{y}$ is

$$\text{Proj}_W(\mathbf{y}) = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{\begin{bmatrix} 6 \\ 7 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{\begin{bmatrix} 6 \\ 7 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}}{\begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}} \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$$

$$= \frac{6+14-6}{1^2+2^2+3^2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{-6+35+6}{(-1)^2+5^2+(-3)^2} \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}.$$

So the distance from $\mathbf{y}$ to W is $\|\mathbf{y} - \text{Proj}_W(\mathbf{y})\| = \left\| \begin{bmatrix} 6 \\ 7 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix} \right\| = \sqrt{40}$.

Theorem 8: Orthogonal Decomposition Theorem: Let W be a subspace of $\mathbb{R}^n$. Then every $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ with $\hat{\mathbf{y}}$ in W and $\mathbf{z}$ in $W^\perp$. In fact, if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is any orthogonal basis for W, then

$$\hat{\mathbf{y}} = \text{Proj}_W(\mathbf{y}) = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{v}_p}{\mathbf{v}_p \cdot \mathbf{v}_p}\mathbf{v}_p \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

The Best Approximation Theorem tells us that $\hat{\mathbf{y}}$ and $\mathbf{z}$ are unique, but here is an alternative proof that does not use the distance between $\hat{\mathbf{y}}$ and $\mathbf{y}$.

Suppose $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ and $\mathbf{y} = \hat{\mathbf{y}}_1 + \mathbf{z}_1$ are two such decompositions, so $\hat{\mathbf{y}}, \hat{\mathbf{y}}_1$ are in W, and $\mathbf{z}, \mathbf{z}_1$ are in $W^\perp$, and

$$\hat{\mathbf{y}} + \mathbf{z} = \hat{\mathbf{y}}_1 + \mathbf{z}_1$$
$$\hat{\mathbf{y}} - \hat{\mathbf{y}}_1 = \mathbf{z}_1 - \mathbf{z}.$$

LHS: Because $\hat{\mathbf{y}}, \hat{\mathbf{y}}_1$ are in W and W is a subspace, the difference $\hat{\mathbf{y}} - \hat{\mathbf{y}}_1$ is in W.
RHS: Because $\mathbf{z}, \mathbf{z}_1$ are in $W^\perp$ and $W^\perp$ is a subspace, the difference $\mathbf{z}_1 - \mathbf{z}$ is in $W^\perp$.
So the vector $\hat{\mathbf{y}} - \hat{\mathbf{y}}_1 = \mathbf{z}_1 - \mathbf{z}$ is in both W and $W^\perp$, this vector is the zero vector (property 1 on week 11, p10). So $\hat{\mathbf{y}} = \hat{\mathbf{y}}_1$ and $\mathbf{z}_1 = \mathbf{z}$.

Theorem 8: Orthogonal Decomposition Theorem: Let W be a subspace of $\mathbb{R}^n$. Then every $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ with $\hat{\mathbf{y}}$ in W and $\mathbf{z}$ in $W^\perp$. In fact, if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is any orthogonal basis for W, then

$$\hat{\mathbf{y}} = \text{Proj}_W(\mathbf{y}) = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{v}_p}{\mathbf{v}_p \cdot \mathbf{v}_p}\mathbf{v}_p \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

The formula for $\text{Proj}_W(\mathbf{y})$ above is similar to the Weights for Orthogonal Bases Theorem (p5). Let's look at how they are related.

For a vector $\mathbf{y}$ in W, the Weights for Orthogonal Bases Theorem says that $\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{v}_p}{\mathbf{v}_p \cdot \mathbf{v}_p}\mathbf{v}_p = \text{Proj}_W(\mathbf{y})$. This makes sense because, if $\mathbf{y}$ is already in W, then the closest point in W to $\mathbf{y}$ must be $\mathbf{y}$ itself.

If $\mathbf{y}$ is not in W, then suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is part of a larger orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}, \dots, \mathbf{v}_n\}$ for $\mathbb{R}^n$. So the Weights for Orthogonal Bases Theorem says that $\mathbf{y} = \underbrace{\frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{v}_p}{\mathbf{v}_p \cdot \mathbf{v}_p}\mathbf{v}_p}_{\text{Proj}_W \mathbf{y}} + \underbrace{\frac{\mathbf{y} \cdot \mathbf{v}_{p+1}}{\mathbf{v}_{p+1} \cdot \mathbf{v}_{p+1}}\mathbf{v}_{p+1} + \cdots + \frac{\mathbf{y} \cdot \mathbf{v}_n}{\mathbf{v}_n \cdot \mathbf{v}_n}\mathbf{v}_n}_{\mathbf{z}}.$

If an orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ for W is part of a larger orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}, \dots, \mathbf{v}_n\}$ for $\mathbb{R}^n$, then

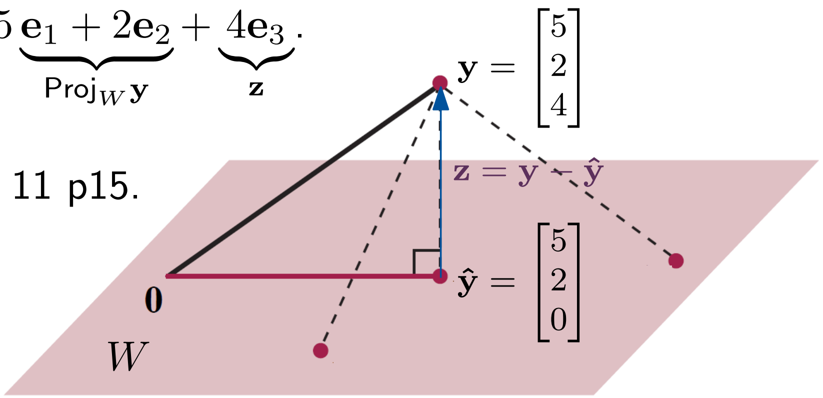
$$\mathbf{y} = \underbrace{\frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{v}_p}{\mathbf{v}_p \cdot \mathbf{v}_p}\mathbf{v}_p}_{\text{Proj}_W \mathbf{y}} + \underbrace{\frac{\mathbf{y} \cdot \mathbf{v}_{p+1}}{\mathbf{v}_{p+1} \cdot \mathbf{v}_{p+1}}\mathbf{v}_{p+1} + \cdots + \frac{\mathbf{y} \cdot \mathbf{v}_n}{\mathbf{v}_n \cdot \mathbf{v}_n}\mathbf{v}_n}_{\mathbf{z}}.$$

Example: Consider the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$. Let

$W = \text{Span}\{\mathbf{e}_1, \mathbf{e}_2\}$, and $\mathbf{y} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = \underbrace{5\,\mathbf{e}_1 + 2\mathbf{e}_2}_{\text{Proj}_W \mathbf{y}} + \underbrace{4\mathbf{e}_3}_{\mathbf{z}}$.

So $\text{Proj}_W(\mathbf{y}) = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$, as we saw week 11 p15.

So, informally, the orthogonal projection "changes the coordinates outside W to 0".



If an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for W is part of a larger orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$, then

$$\mathbf{y} = \underbrace{\frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{v}_p}{\mathbf{v}_p\cdot\mathbf{v}_p}\mathbf{v}_p}_{\mathrm{Proj}_W\mathbf{y}} + \underbrace{\frac{\mathbf{y}\cdot\mathbf{v}_{p+1}}{\mathbf{v}_{p+1}\cdot\mathbf{v}_{p+1}}\mathbf{v}_{p+1} + \cdots + \frac{\mathbf{y}\cdot\mathbf{v}_n}{\mathbf{v}_n\cdot\mathbf{v}_n}\mathbf{v}_n}_{\mathbf{z}}.$$

What is $\mathbf{z}$?

a) $\{\mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$ is an orthogonal set because

b) $\{\mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$ is linearly independent because

c) $\{\mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$ is in $W^\perp$ because

d) $\{\mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$ is a basis for $W^\perp$ because

So $\{\mathbf{v}_{p+1}, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $W^\perp$, and so

$$\mathbf{z} = \frac{\mathbf{y}\cdot\mathbf{v}_{p+1}}{\mathbf{v}_{p+1}\cdot\mathbf{v}_{p+1}}\mathbf{v}_{p+1} + \cdots + \frac{\mathbf{y}\cdot\mathbf{v}_n}{\mathbf{v}_n\cdot\mathbf{v}_n}\mathbf{v}_n = \mathrm{Proj}_{W^\perp}\mathbf{y}.$$

Another way to phrase this:

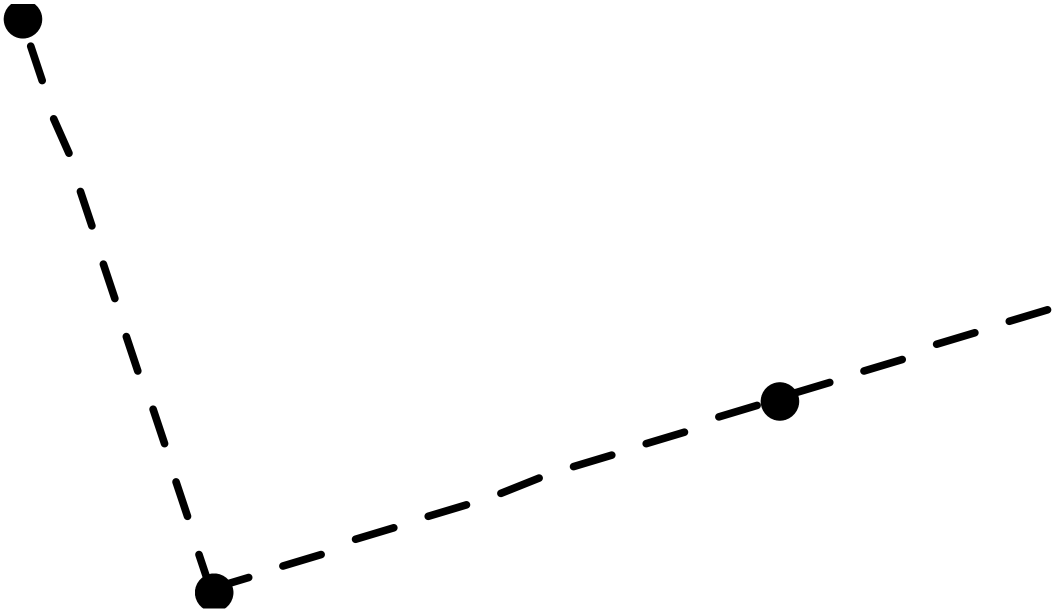
$$\mathbf{y} = \mathrm{Proj}_W\mathbf{y} + \mathrm{Proj}_{W^\perp}\mathbf{y}.$$

In other words, $\mathrm{Proj}_W\mathbf{y} = \mathbf{y} - \mathrm{Proj}_{W^\perp}\mathbf{y}$, and this is sometimes useful in computations, e.g. if you already have an orthogonal basis for $W^\perp$ but not for W, then $\mathrm{Proj}_{W^\perp}\mathbf{y}$ is easier to find than $\mathrm{Proj}_W\mathbf{y}$ (see Homework 6 Q3).

Let W be a subspace of $\mathbb{R}^n$. If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal basis for W, then, for every $\mathbf{y}$ in $\mathbb{R}^n$,

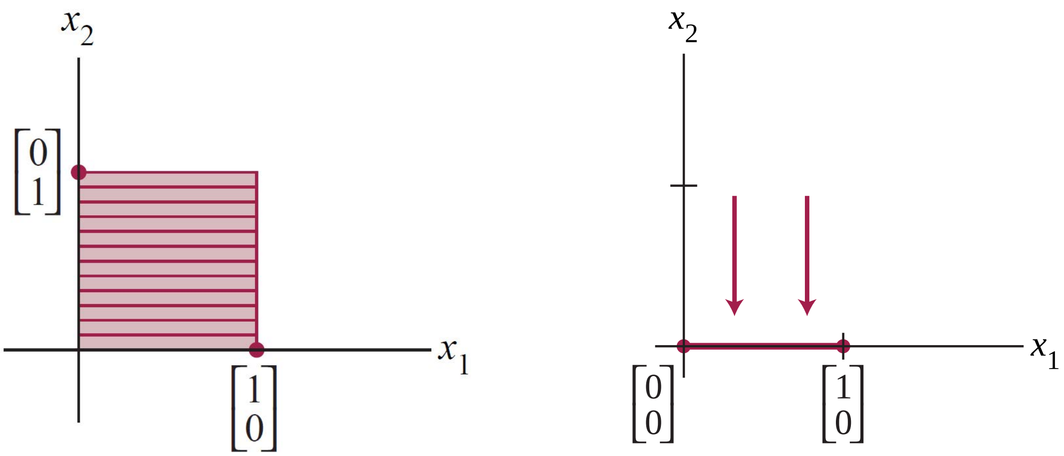
$$\mathrm{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p.$$

Thinking about $\mathrm{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ as a function:



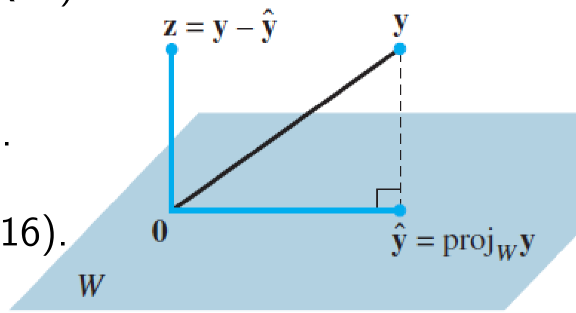
We saw a special case in Week 4 §1.8-1.9:

Projection onto the x_1-axis



Properties of $\text{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ from the picture (exercise: prove them algebraically):

a. proj_W is a linear transformation (ex. sheet #25 Q1a).
b. $\text{proj}_W(\mathbf{y}) = \mathbf{y}$ if and only if $\mathbf{y}$ is in W.
c. The range of proj_W is W.
d. The kernel of proj_W is $W^\perp$ (ex. sheet #25 Q1b).
e. $\text{proj}_W^2 = \text{proj}_W$ (ex. sheet #25 Q1c).
f. $\text{proj}_W + \text{proj}_{W^\perp}$ is the identity transformation (p16).



Non-examinable: instead of using the formula for proj_W, we can prove these properties from the existence and uniqueness of the orthogonal decomposition, e.g. for a: if we have orthogonal decompositions $\mathbf{y}_1 = \text{proj}_W(\mathbf{y}_1) + \mathbf{z}_1$ and $\mathbf{y}_2 = \text{proj}_W(\mathbf{y}_2) + \mathbf{z}_2$, then

$$\begin{aligned} c\mathbf{y}_1 + d\mathbf{y}_2 &= c(\text{proj}_W(\mathbf{y}_1) + \mathbf{z}_1) + d(\text{proj}_W(\mathbf{y}_2) + \mathbf{z}_2) \\ &= \underbrace{c\text{proj}_W(\mathbf{y}_1) + d\text{proj}_W(\mathbf{y}_2)}_{\text{in } W} + \underbrace{c\mathbf{z}_1 + d\mathbf{z}_2}_{\text{in } W^\perp} \end{aligned}$$

Since the orthogonal decomposition is unique, this shows $\text{proj}_W(c\mathbf{y}_1 + d\mathbf{y}_2) = c\text{proj}_W(\mathbf{y}_1) + d\text{proj}_W(\mathbf{y}_2)$.

The orthogonal projection is a linear transformation, so we can ask for its standard matrix. (It is faster to compute orthogonal projections by taking dot products (formula on p11) than using the standard matrix, but this result is useful theoretically.)

Theorem 10: Matrix for Orthogonal Projection: Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthonormal basis for a subspace W, and U be the matrix $U = \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_p \\ | & & | \end{bmatrix}$.

Then the standard matrix for proj_W is $[\text{proj}_W]_{\mathcal{E}} = UU^T$.

Proof:

$$UU^T\mathbf{y} = \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_p \\ | & & | \end{bmatrix} \begin{bmatrix} -- & \mathbf{u}_1 & -- \\ -- & \vdots & -- \\ -- & \mathbf{u}_p & -- \end{bmatrix} \mathbf{y} = \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_p \\ | & & | \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{y} \\ \vdots \\ \mathbf{u}_p \cdot \mathbf{y} \end{bmatrix}$$

$$= (\mathbf{u}_1 \cdot \mathbf{y})\mathbf{u}_1 + \dots + (\mathbf{u}_p \cdot \mathbf{y})\mathbf{u}_p.$$

Tip: to remember that $[\text{proj}_W]_{\mathcal{E}} = UU^T$ and not U^TU (which is important too, see p23), make sure this matrix is $n \times n$.

§6.4: The Gram-Schmidt Process

This is an algorithm to make an orthogonal basis out of an arbitrary basis.

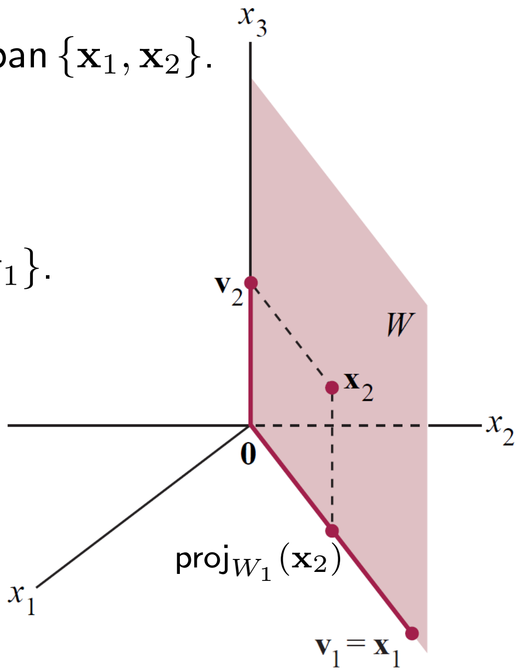
Example: Let $\mathbf{x}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for W.

Answer: Let $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$, and let $W_1 = \text{Span}\{\mathbf{v}_1\}$.

By the Orthogonal Decomposition Theorem,
$\mathbf{x}_2 - \text{proj}_{W_1}(\mathbf{x}_2)$ is orthogonal to W_1.
So let $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1}(\mathbf{x}_2) = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$

$$= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \frac{8+2+0}{4^2+2^2+0}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$



For subspaces of dimension $p>2$, we repeat this idea p times, like this:

EXAMPLE Let $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -6 \\ 7 \\ 2 \\ 1 \end{bmatrix}$, and suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for a subspace W of $\mathbf{R}^4$. Construct an orthogonal basis for W.

Solution:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{W}_1 = \mathrm{Span}\{\mathbf{v}_1\}.$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \mathrm{Proj}_{\mathbf{W}_1}(\mathbf{x}_2) = \underline{\hspace{4cm}} = \begin{bmatrix} 8 \\ 5 \\ -6 \\ 0 \end{bmatrix} - \underline{\hspace{3cm}} \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} =$$

$\underline{\hspace{4cm}}$

Check our answer so far:

Let $\mathbf{W}_2 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$

$$\mathbf{v}_3 = \mathbf{x}_3 - \mathrm{Proj}_{\mathbf{W}_2}(\mathbf{x}_3) =$$

$$= \begin{bmatrix} -6 \\ 7 \\ 2 \\ 1 \end{bmatrix} - \left(\underline{\hspace{3cm}} \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \underline{\hspace{3cm}} \begin{bmatrix} -1 \\ 5 \\ -3 \\ 0 \end{bmatrix} \right) =$$

Check our answer:

In general:

Theorem 11: Gram-Schmidt: Given a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ for a subspace W of $\mathbb{R}^n$, define

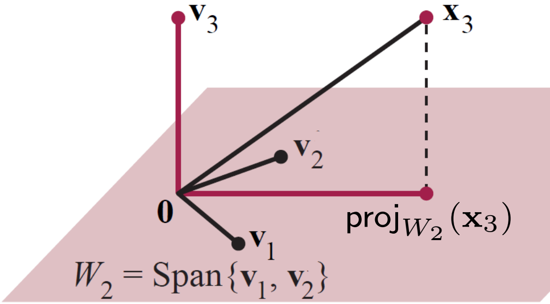
$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2\right)$$

$$\vdots$$

$$\mathbf{v}_p = \mathbf{x}_p - \left(\frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}\right)$$



Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W, and Span $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ = Span $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for each k between 1 and p.

In fact, you can apply this algorithm to any spanning set (not necessarily linearly independent). Then some $\mathbf{v}_k$s might be zero, and you simply remove them.

pp361-362: Matrices with orthonormal columns

This is an important class of matrices.

Theorem 6: Matrices with Orthonormal Columns: A matrix U has orthonormal columns (i.e. the columns of U are an orthonormal set) if and only if $U^TU = I$.

Proof: Let $\mathbf{u}_i$ denote the ith column of U. From the row-column rule of matrix multiplication (week 11 p14):

$$\begin{bmatrix} -- & \mathbf{u}_1 & -- \\ -- & \vdots & -- \\ -- & \mathbf{u}_p & -- \end{bmatrix} \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_p \\ | & & | \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \dots & \mathbf{u}_1 \cdot \mathbf{u}_p \\ \vdots & & \vdots \\ \mathbf{u}_p \cdot \mathbf{u}_1 & \dots & \mathbf{u}_p \cdot \mathbf{u}_p \end{bmatrix}.$$

so $U^TU = I$ if and only if $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ for each i (diagonal entries), and
$\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for each pair $i \neq j$ (non-diagonal entries).

Theorem 7: Matrices with Orthonormal Columns represent Length-Preserving Linear Transformations: Let U be an $m \times n$ matrix with orthonormal columns. Then, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}.$$

In particular, $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$, and $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof:

$$(U\mathbf{x}) \cdot (U\mathbf{y}) = (U\mathbf{x})^T(U\mathbf{y}) = \mathbf{x}^TU^TU\mathbf{y} \underset{\uparrow}{=} \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

because $U^TU = I_n$, by the previous theorem

Length-preserving linear transformations are sometimes called isometries.
Exercise: prove that an isometry also preserves angles; that is, if A is any matrix such that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$, then $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$. (Hint: think about $\mathbf{x} + \mathbf{y}$.)

An important special case:

Definition: A matrix U is *orthogonal* if it is a square matrix with orthonormal columns. Equivalently, $U^{-1} = U^T$.

Warning: An orthogonal matrix has orthonormal columns, not simply orthogonal columns.

Example: The standard matrix of a rotation in $\mathbb{R}^2$ is $U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, and this is an orthogonal matrix because

$$U^T U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It can be shown that every orthogonal 2×2 matrix U represents either a rotation (if $\det U = 1$) or a reflection (if $\det U = -1$). (Exercise: why are these the only possible values of $\det U$?) An orthogonal $n \times n$ matrix with determinant 1 is a high-dimensional generalisation of a rotation.

Recall (week 9 p7, §4.4) that, if $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$, then the change-of-coordinates matrix from $\mathcal{B}$-coordinates to standard coordinates is

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} | & | & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_n \\ | & | & | \end{bmatrix}.$$

So an orthogonal matrix can also be viewed as a change-of-coordinates matrix from an orthonormal basis to the standard basis.

Question: Given $\mathbf{x}$, how can we find $[\mathbf{x}]_\mathcal{B}$?

Answer 1:

$$[\mathbf{x}]_\mathcal{B} = \underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}\mathbf{x} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}\mathbf{x} = U^{-1}\mathbf{x} = U^T\mathbf{x} = \begin{bmatrix} -- & \mathbf{u}_1 & -- \\ -- & \vdots & -- \\ -- & \mathbf{u}_n & -- \end{bmatrix}\mathbf{x} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{x} \\ \vdots \\ \mathbf{u}_n \cdot \mathbf{x} \end{bmatrix}.$$

Answer 2: By the Weights for Orthogonal Bases Theorem, $\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + \dots + (\mathbf{x} \cdot \mathbf{u}_n)\mathbf{u}_n$, so, by the definition of coordinates, $[\mathbf{x}]_\mathcal{B} = (\mathbf{x} \cdot \mathbf{u}_1, \dots, \mathbf{x} \cdot \mathbf{u}_n)$.

Non-examinable: distances for abstract vector spaces

On an abstract vector space, a function that takes two vectors to a scalar satisfying the symmetry, linearity and positivity properties (week 11 p5) is called an inner product. The inner product of $\mathbf{u}$ and $\mathbf{v}$ is often written $\langle \mathbf{u}, \mathbf{v} \rangle$ or $\langle \mathbf{u} | \mathbf{v} \rangle$. (So the dot product is one example of an inner product on $\mathbb{R}^n$, but other useful inner products exist; these can be used to compute weighted regression lines, see §6.8 of the textbook)

Many common inner products on $C([0,1])$, the vector space of continuous functions, have the form

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(t)g(t)w(t)\,dt$$

for some non-negative weight function $w(t)$. Orthogonal projections using these inner products compute polynomial approximations and Fourier approximations to functions, see §6.7-6.8 of the textbook.

Applying Gram-Schmidt to $\{1, t, t^2, \dots\}$ produces various families of orthogonal polynomials, which is a big field of study.

With the concept of inner product, we can understand what the transpose means for linear transformations:

First notice: if A is an $m \times n$ matrix, then, for all $\mathbf{v}$ in $\mathbb{R}^n$ and all $\mathbf{u}$ in $\mathbb{R}^m$:

$$\underbrace{(A^T\mathbf{u}) \cdot \mathbf{v}}_{\text{dot product in } \mathbb{R}^n} = (A^T\mathbf{u})^T\mathbf{v} = \mathbf{u}^T A\mathbf{v} = \underbrace{\mathbf{u} \cdot (A\mathbf{v})}_{\text{dot product in } \mathbb{R}^m}.$$

So, if A is the standard matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, then A^T is the standard matrix of its adjoint $T^* : \mathbb{R}^m \to \mathbb{R}^n$, which satisfies

$$(T^*\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (T\mathbf{v}).$$

or, for abstract vector space with an inner product:

$$\langle T^*\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, T\mathbf{v} \rangle.$$

So symmetric matrices ($A^T = A$) represent self-adjoint linear transformations ($T^* = T$). For example, on $C([0,1])$ with any integral inner product, the multiplication-by-x function $\mathbf{f} \mapsto x\mathbf{f}$ is self-adjoint.

§7.1: Diagonalisation of Symmetric Matrices

Symmetric matrices ($A = A^T$) arise naturally in many contexts, when a_{ij} depends on i and j but not on their order (e.g. the friendship matrix from Homework 3 Q7, the Hessian matrix of second partial derivatives from Multivariate Calculus). The goal of this section is to observe some very nice properties about the eigenvectors of a symmetric matrix.

Example: $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$ is a symmetric matrix.

$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is a -1-eigenvector.

$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a 4-eigenvector.

Notice that the eigenvectors are orthogonal: $\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$. This is not a coincidence...

Theorem 1: Eigenvectors of Symmetric Matrices: If A is a symmetric matrix, then eigenvectors corresponding to distinct eigenvalues are orthogonal.
Compare: for an arbitrary matrix, eigenvectors corresponding to distinct eigenvalues are linearly independent (week 10 p22).

Proof: Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2. Then

$$(A\mathbf{v}_1) \cdot \mathbf{v}_2 = (\lambda_1 \mathbf{v}_1) \cdot \mathbf{v}_2 = \lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2),$$

and

$$\mathbf{v}_1 \cdot (A\mathbf{v}_2) = \mathbf{v}_1 \cdot (\lambda_2 \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2).$$

But the two left hand sides above are equal, because (see also week 12 p28)

$$(A\mathbf{v}_1) \cdot \mathbf{v}_2 = (A\mathbf{v}_1)^T \mathbf{v}_2 = \mathbf{v}_1^T \underbrace{A^T \mathbf{v}_2 = \mathbf{v}_1^T A}_{A \text{ is symmetric}} \mathbf{v}_2 = \mathbf{v}_1 \cdot (A\mathbf{v}_2).$$

So the two right hand sides are equal: $\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2)$. Since $\lambda_1 \neq \lambda_2$, it must be that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

Remember from week 10 §5:
Definition: A square matrix A is *diagonalisable* if there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
Diagonalisation Theorem: An $n \times n$ matrix A is diagonalisable if and only if A has n linearly independent eigenvectors. Those eigenvectors are the columns of P.

Given our previous observation, we are interested in when a matrix has n orthogonal eigenvectors. Because any scalar multiple of an eigenvector is also an eigenvector, this is the same as asking, when does a matrix have n orthonormal eigenvectors, i.e. when is the matrix P in the Diagonalisation Theorem an orthogonal matrix?
Definition: A square matrix A is *orthogonally diagonalisable* if there is an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$, or equivalently, $A = PDP^T$.

We can extend the previous theorem (being careful about eigenvectors with the same eigenvalue) to show that any diagonalisable symmetric matrix is orthogonally diagonalisable, see the example on the next page.

Example: Orthogonally diagonalise $B = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$, i.e. find an orthogonal P and diagonal D with $B = PDP^{-1}$:

Answer:

Step 1 Solve the characteristic equation $\det(B - \lambda I) = 0$ to find the eigenvalues.

Eigenvalues are 2 and 5.

Step 2 For each eigenvalue λ, solve $(B - \lambda I)\mathbf{x} = \mathbf{0}$ to find a basis for the λ-eigenspace.

This gives $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ as a basis for the 2-eigenspace, and $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ as a basis for the 5-eigenspace. Notice the 2-eigenvector is orthogonal to both the 5-eigenvectors, but the two 5-eigenvectors are not orthogonal.

Step 2A For each eigenspace of dimension > 1, find an orthogonal basis (e.g. by Gram-Schmidt) Applying Gram-Schmidt to the above basis for the 5-eigenspace gives $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$. To avoid fractions, let's use $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$, which is still an orthogonal set.

Step 2B Normalise all the eigenvectors

$\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$ is an orthonormal basis for the 2-eigenspace, and $\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for the 5-eigenspace.

Step 3 Put the normalised eigenvectors from Step 2B as the columns of P.

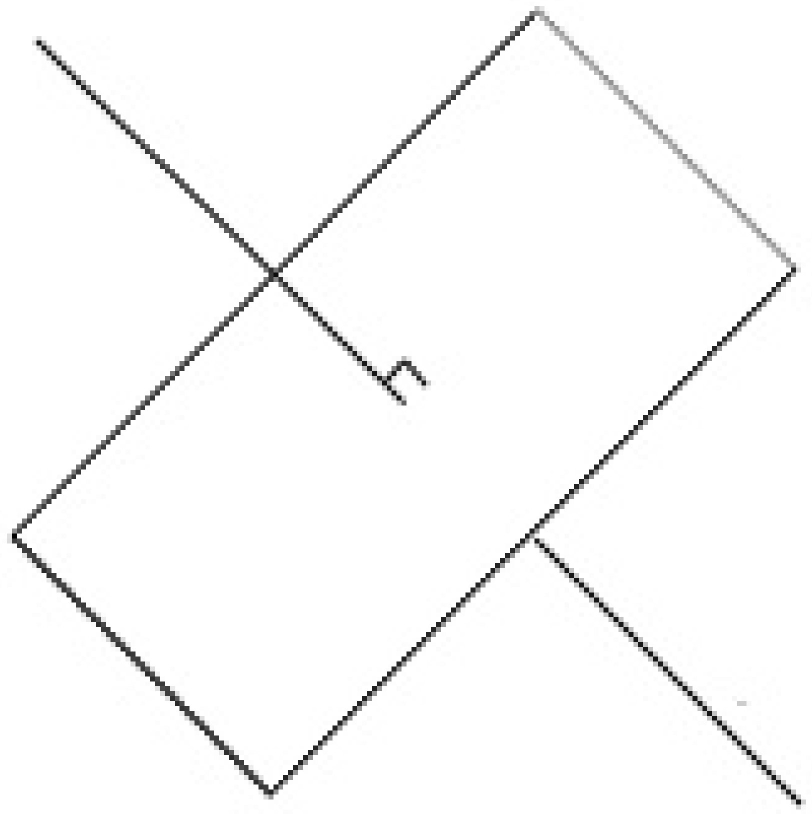
Step 4 Put the corresponding eigenvalues as the diagonal entries of D.

$$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Check our answer:

$$PDP^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}.$$

A geometric illustration of "orthonormalising" the eigenvectors:



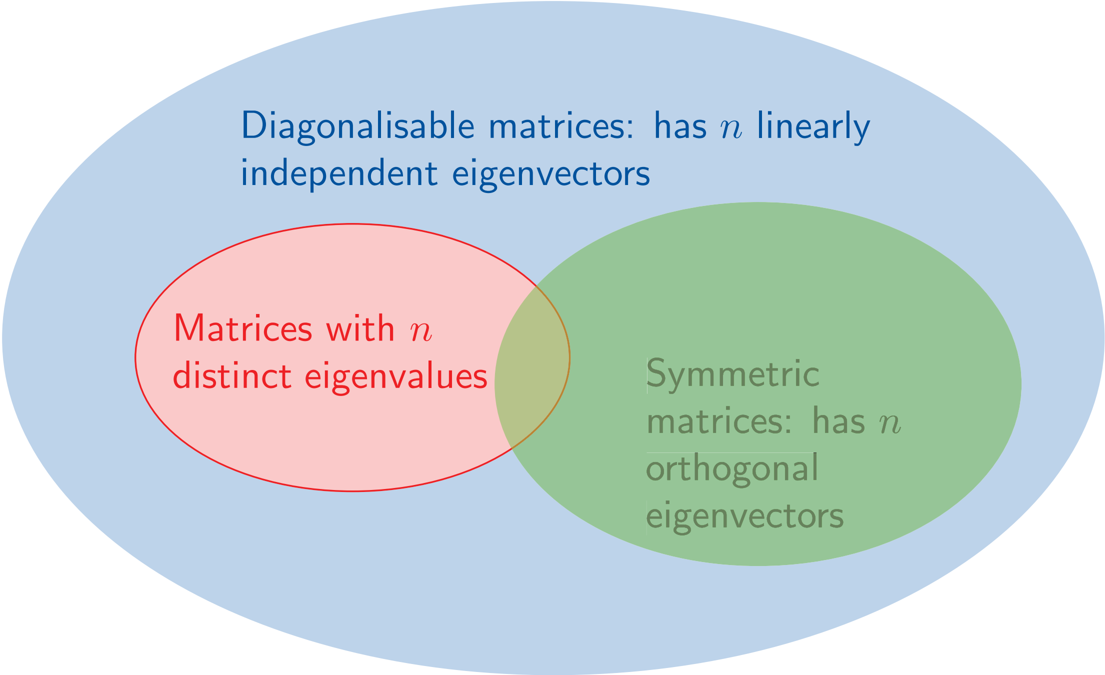
This algorithm shows that any diagonalisable symmetric matrix is orthogonally diagonalisable.
Amazingly, every symmetric matrix is diagonalisable:
Theorem 3: Spectral Theorem for Symmetric Matrices: A symmetric matrix is orthogonally diagonalisable, i.e. it has a orthonormal basis of eigenvectors.
(The name of the theorem is because the **set** of eigenvalues and multiplicities of a matrix is called its spectrum. There are spectral theorems for many types of linear transformations.)

The reverse direction is also true, and easy:
Theorem 2: Orthogonally diagonalisable matrices are symmetric: If $A = PDP^{-1}$ and P is orthogonal and D is diagonal, then A is symmetric.
Proof:

$$A^T = (PD\boxed{P^{-1}})^T = (PD\boxed{P^T})^T = (P^T)^T \boxed{D^T} P^T = P\boxed{D}P^T = A.$$

P is orthogonal $\qquad$ D is diagonal

A diagram to summarise what we know about diagonalisation:



Diagonalisable matrices: has n linearly independent eigenvectors

Matrices with n distinct eigenvalues

Symmetric matrices: has n orthogonal eigenvectors

Non-examinable: ideas behind the proof of the spectral theorem

Because we need to work on subspaces of $\mathbb{R}^n$ in the proof, we consider self-adjoint linear transformations ($(T\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (T\mathbf{v})$) instead of symmetric matrices. So we want to show: a self-adjoint linear transformation has an orthogonal basis of eigenvectors.

The key ideas are:

1. Every linear transformation (on any vector space) has a complex eigenvector.
   Proof: Every polynomial has a solution if we allow complex numbers. Apply this to the characteristic polynomial.
2. Any complex eigenvector of a (real) self-adjoint linear transformation is a real eigenvector corresponding to a real eigenvalue. (We won't comment on the proof.)
3. Let $\mathbf{v}$ be an eigenvector of a self-adjoint linear transformation T, and $\mathbf{w}$ be any vector orthogonal to $\mathbf{v}$. Then $T(\mathbf{w})$ is still orthogonal to $\mathbf{v}$.
   Proof: $\mathbf{v} \cdot (T(\mathbf{w})) = (T(\mathbf{v})) \cdot \mathbf{w} = \lambda \mathbf{v} \cdot \mathbf{w} = \lambda 0 = 0$.

Putting these together: if $T : \mathbb{R}^n \to \mathbb{R}^n$ is self-adjoint, then by 1 and 2 it has a real eigenvector $\mathbf{v}$. Let $W = (\text{Span}\,\{\mathbf{v}\})^{\perp}$, the subspace of vectors orthogonal to $\mathbf{v}$. By 3, any vector in W stays in W after applying T (i.e. W is an invariant subspace under T), so we can consider the restriction $T : W \to W$, which is self-adjoint. So repeat this argument on W (i.e. use induction on the dimension of the domain of T).